Gravitational action for a massive Majorana fermion in $2d$ quantum gravity – notes

Harold Erbin*[1,2,3]

[1]Université Paris-Saclay, CEA, LIST, F-91120 Palaiseau, France
[2]Center for Theoretical Physics, Massachusetts Institute of Technology
Cambridge, MA 02139, USA
[3]NSF AI Institute for Artificial Intelligence and Fundamental Interactions

May 2021

Contents

*erbin@mit.edu

1 Introduction

The goal of these notes is to compute the gravitational action for the simplest example – the massive Majorana fermion – where it is possible to compare the DDK ansatz with an explicit computation.

The following cases are treated in the literature (see also [1, 2]):

- Weyl [3–5];
- Majorana [6];
- Majorana–Weyl [7, 8].

The paper [9] summarizes the effective action for all the particles from spin 0 to 3/2 and ghosts. For computations see [8, 10]. Very useful papers for the definition of the path integrals (with zero-modes, Majorana condition, etc.) are [11–15] (see also [16–18]).

The action for Majorana–Weyl fermion can be written covariantly by using the operator $\partial_\mu \pm i\epsilon_{\mu\nu}\partial^\nu$ [9]. Something similar may be possible for Euclidean by looking only at one of the two holomorphic components.

2 Classical action

For formulas with Dirac matrices and fermions see appendices A and B.

The different representations of the gamma matrices described in this section are given in the local frame. As a consequence, the action is also written in the local frame (in particular, ∂_0 and ∂_1 denote derivatives with respect to the local frame coordinates).

2.1 Lorentzian signature

The action for a Majorana fermion on a curved space with Lorentzian signature is

$$\boxed{S = -\frac{\mathrm{i}}{4\pi} \int \mathrm{d}^2 x \sqrt{g}\, \bar{\Psi} (\nabla\!\!\!/ - m) \Psi,} \tag{2.1}$$

where $\bar{\Psi} := \Psi^\dagger \gamma^0$. Note the additional 1/2 factor with respect to the Dirac action, and the normalization is taken to be $1/2\pi$. Since the fermion is Majorana, the flip relation (B.35) implies that the connection term vanishes. Nonetheless, there are some subtleties that we describe below.

2.1.1 Dirac matrices

The Clifford algebra is

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{2.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1)$. The matrices γ^0 and γ^1 are respectively anti-Hermitian and Hermitian

$$(\gamma^\mu)^\dagger = -\gamma^0 \gamma^\mu \gamma^0. \tag{2.3}$$

These matrices can be found from the Euclidean ones by multiplying the 0-th one by i. The chirality matrix is defined by

$$\gamma_* = \gamma^0 \gamma^1. \tag{2.4}$$

using $\eta_* = -1$.

Different relations are:

$$\hat{\gamma}^{-1} = \hat{\gamma}^\dagger = -\hat{\gamma}, \tag{2.5a}$$

$$\hat{\gamma}^t = -\eta\, C_\eta \hat{\gamma} C_\eta^{-1}, \qquad \hat{\gamma}^* = -\hat{\gamma}^t, \tag{2.5b}$$

$$\gamma^{\mu\nu} = -\eta_*\, \epsilon^{\mu\nu} \gamma_*, \tag{2.5c}$$

$$B_\zeta = -\xi \epsilon \eta\, C_\eta \hat{\gamma}, \tag{2.5d}$$

$$B_\zeta B_\zeta^* = \epsilon\, \eta. \tag{2.5e}$$

The Majorana condition reads:

$$\Psi^* = B\Psi, \qquad \epsilon\, \eta = 1, \qquad |\alpha| = 1, \tag{2.6}$$

and setting $\alpha = 1$.

2.1.2 Weyl and Majorana basis

The Majorana basis and Weyl basis coincide [19, sec. 7.5, 20, sec. 3, 21, 22, sec. 4.1] (and [23, sec. 2.1.2] which uses $(\gamma^1, -\gamma^0)$ as a basis).

Gamma matrices The simplest representation reads:

$$\boxed{\gamma^0 = \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_* = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.} \tag{2.7}$$

This is logical since one can impose both Majorana and Weyl conditions. Note that all matrices are real.

The charge conjugation matrix:

$$\boxed{C_+ = \hat{\gamma} = \gamma^0, \qquad \eta = +1.} \tag{2.8}$$

Indeed, we have

$$C_+ \gamma^0 C_+^{-1} = \gamma^0 = -(\gamma^0)^t, \qquad C_+ \gamma^1 C_+^{-1} = -\gamma^1 = -(\gamma^1)^t. \tag{2.9}$$

This implies:

$$\epsilon = 1, \tag{2.10}$$

and

$$\boxed{B_+ = -\xi\, C_+ \hat{\gamma} = 1, \qquad \zeta = 1,} \tag{2.11}$$

using $\xi = 1$ for the normalization. Indeed, we have

$$(\gamma^0)^* = \gamma^0, \qquad (\gamma^1)^* = \gamma^1. \tag{2.12}$$

Note that with the mostly minus signature, all matrices are rescaled by i such that C_+ is mapped to C_- [15].

There is a second representation with $C_- = \gamma^1$, but we will not use it.

Spinors We write a Dirac spinor as

$$\boxed{\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.} \tag{2.13}$$

The Majorana condition reads:

$$\boxed{\psi_\pm^* = \psi_\pm,} \tag{2.14}$$

such that the (anticommuting) components of the Majorana fermion are real.

We also find that:

$$P_+ \Psi = 2 \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} \tag{2.15}$$

which shows that ψ_+ has a positive chirality.

We have:

$$\bar{\Psi} = \begin{pmatrix} -\psi_-^* & \psi_+^* \end{pmatrix}. \tag{2.16}$$

Action It will be convenient to introduce light-cone coordinates

$$x^{\pm} = x^0 \pm x^1, \qquad \partial_{\pm} = \frac{1}{2}(\partial_0 \pm \partial_1) \tag{2.17}$$

such that

$$\gamma^+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad \gamma^- = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}. \tag{2.18}$$

We have:

$$\gamma^\mu \partial_\mu = 2 \begin{pmatrix} 0 & \partial_+ \\ -\partial_- & 0 \end{pmatrix}. \tag{2.19}$$

This follows from:

$$\gamma^\mu \partial_\mu = \partial_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \partial_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial_0 + \partial_1 \\ -\partial_0 + \partial_1 & 0 \end{pmatrix}. \tag{2.20}$$

The Dirac bilinears and kinetic operator read [19, sec. 7.5]:

$$\boxed{\begin{gathered} \bar{\Psi}\Psi = \psi_+ \psi_-^* + \psi_+^* \psi_-, \qquad \bar{\Psi}\gamma_* \Psi = \psi_+ \psi_-^* - \psi_+^* \psi_-, \\ \bar{\Psi}\partial\!\!\!/ \Psi = -2(\psi_-^* \partial_+ \psi_- + \psi_+^* \partial_- \psi_+) \end{gathered}} \tag{2.21}$$

since we have:

$$\begin{aligned} \bar{\Psi}\Psi &= \begin{pmatrix} -\psi_-^* & \psi_+^* \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \\ \bar{\Psi}\partial\!\!\!/ \Psi &= 2 \begin{pmatrix} -\psi_-^* & \psi_+^* \end{pmatrix} \begin{pmatrix} 0 & \partial_+ \\ -\partial_- & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \end{aligned}$$

The kinetic term is anti-Hermitian:

$$(\bar{\Psi}\partial\!\!\!/ \Psi)^\dagger = -2(\partial_+ \psi_-^* \, \psi_- + \partial_- \psi_+^* \, \psi_+) \overset{\text{IPP}}{=} 2(\psi_-^* \, \partial_+ \psi_- + \psi_+^* \, \partial_- \psi_+) = -\bar{\Psi}\partial\!\!\!/ \Psi \tag{2.22}$$

and a factor of i is needed in the action. [In [24] (see Extra, ch. 3), complex conjugation also exchanges the order of fermions (or equivalently it adds a minus sign). Does this change this property?] ⇐ 1

For a Majorana fermion, we have:

$$\begin{gathered} \bar{\Psi}\Psi = 2\psi_+ \psi_-, \qquad \bar{\Psi}\gamma_* \Psi = 0, \\ \bar{\Psi}\partial\!\!\!/ \Psi = -2(\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+). \end{gathered} \tag{2.23}$$

The action (2.1) can be rewritten as

$$\boxed{S = \frac{\mathrm{i}}{2\pi} \int \mathrm{d}^2 x \sqrt{g}\, (\psi_+ \partial_- \psi_+ + \psi_- \partial_+ \psi_- + m \psi_- \psi_+).} \tag{2.24}$$

2.1.3 Variations and equations of motion

Handling this action presents some subtleties due to the fact that $\bar{\Psi}$ and Ψ cannot be treated as independent variables. First, it can be rewritten as

$$\boxed{S = -\frac{\mathrm{i}}{8\pi} \int \mathrm{d}^2\sigma \sqrt{g} \left(\bar{\Psi}(\nabla\!\!\!/ - m)\Psi - \bar{\Psi}(\overleftarrow{\nabla\!\!\!/} - m)\Psi \right).} \tag{2.25}$$

by integrating by part (2.1). Writing explicitly the covariant derivative and using the relation (B.35) for $n = 3$, one finds that the connection term vanishes and this leads to the action [6, 19]:

$$S = -\frac{\mathrm{i}}{4\pi} \int \mathrm{d}^2\sigma \sqrt{g}\, \bar{\Psi}(\partial\!\!\!/ - m)\Psi. \tag{2.26}$$

which is purely imaginary. On the other hand, canceling the connection in (2.25) leads to

$$S = -\frac{\mathrm{i}}{8\pi} \int \mathrm{d}^2\sigma \sqrt{g} \left(\bar{\Psi}(\partial\!\!\!/ - m)\Psi - \bar{\Psi}(\overleftarrow{\partial\!\!\!/} - m)\Psi \right) \tag{2.27}$$

which is real. Now it looks like (2.27) is different from (2.26), but we can show by integrating by part and using again the flip relation (B.35) that they agree:

$$\begin{aligned} \int \mathrm{d}^2x \sqrt{g}\, \partial_\mu \bar{\Psi}\gamma^\mu \Psi &= \int \mathrm{d}^2x \left(\bar{\Psi}\partial_\mu(\sqrt{g}\gamma^\mu)\Psi + \sqrt{g}\, \bar{\Psi}\partial\!\!\!/\Psi \right) \\ &= \int \mathrm{d}^2x \sqrt{g} \left(\cancel{\frac{1}{2}\, \bar{\Psi}\omega_\mu\gamma^\mu\gamma_*\Psi} + \bar{\Psi}\partial\!\!\!/\Psi \right), \end{aligned}$$

using (A.46) and the flip relation in the last step. Note that no boundary term can appear since the latter would be the derivative of $\bar{\Psi}\gamma^\mu\Psi$ (and of other factors) which vanish due to the flip relation.

The equation of motion for the imaginary action (2.26) is

$$(\nabla\!\!\!/ - m)\Psi = 0 \tag{2.28}$$

since the variation of the action is [25]

$$\begin{aligned} \delta S &= -\frac{\mathrm{i}}{4\pi} \int \mathrm{d}^2\sigma \sqrt{g}\, (\delta\bar{\Psi}(\partial\!\!\!/ - m)\Psi + \bar{\Psi}(\partial\!\!\!/ - m)\delta\Psi) \\ &= -\frac{\mathrm{i}}{4\pi} \int \mathrm{d}^2\sigma \sqrt{g} \left(\delta\bar{\Psi}(\partial\!\!\!/ - m)\Psi - \bar{\Psi}(\overleftarrow{\partial\!\!\!/} + m)\delta\Psi - \frac{1}{\sqrt{g}}\, \bar{\Psi}\partial_\mu(\sqrt{g}\gamma^\mu)\delta\Psi \right) \\ &= -\frac{\mathrm{i}}{2\pi} \int \mathrm{d}^2\sigma \sqrt{g} \left(\delta\bar{\Psi}(\partial\!\!\!/ - m)\Psi - \bar{\Psi}(\overleftarrow{\partial\!\!\!/} + m)\delta\Psi - \frac{1}{4}\, \bar{\Psi}\omega\!\!\!/\gamma_*\delta\Psi \right), \end{aligned}$$

where we have integrated by part in the second line, and then used the flip property to exchange the position of Ψ and its variation

$$\bar{\Psi}\delta\Psi = \delta\bar{\Psi}\Psi, \qquad \partial_\mu\bar{\Psi}\gamma^\mu\delta\Psi = -\delta\bar{\Psi}\gamma^\mu\partial_\mu\Psi, \qquad \bar{\Psi}\gamma^\mu\gamma_*\delta\Psi = \delta\bar{\Psi}\gamma_*\gamma^\mu\Psi, \tag{2.29}$$

and the last line follows from (A.46). One should not forget to vary both Ψ and $\bar{\Psi}$ since they are not independent for a Majorana field (recall that $\bar{\Psi} \sim \Psi^t$): varying $\bar{\Psi}$ alone would lead to the incorrect equation $(\partial\!\!\!/ - m)\Psi = 0$. One recovers the same equation from the real action (2.27), but in this case the derivation is simpler since one can treat $\bar{\Psi}$ as an independent variable since the action is symmetric in Ψ and $\bar{\Psi}$.

2.1.4 Energy–momentum tensor

[HE: check these formulas.] ⇐ 2

The (symmetric) energy–momentum tensor reads

$$T_{\mu\nu} = -\frac{\mathrm{i}}{2}\,\bar{\Psi}\gamma_{(\mu}\partial_{\nu)}\Psi + \frac{\mathrm{i}}{2}\,g_{\mu\nu}\bar{\Psi}(\partial\!\!\!/ - m)\Psi, \qquad T = \mathrm{i}m\,\bar{\Psi}\Psi. \tag{2.30}$$

2.2 Euclidean signature

The (massive) Majorana fermion Ψ in two dimensions is equivalent to the (massive) Ising model. The massless theory is a CFT with $c = 1/2$. The path integral of this model on flat space can be found in [26].

The action for a Majorana fermion on a curved space with Euclidean signature is [6]:

$$\boxed{S = \frac{1}{4\pi}\int \mathrm{d}^2x\sqrt{g}\,\bar{\Psi}(\mathrm{i}\nabla\!\!\!\!/ + m\gamma_*)\Psi,} \tag{2.31}$$

where $\bar{\Psi} := \Psi^\dagger$. Note the additional 1/2 factor with respect to the Dirac action, and the normalization is taken to be $1/2\pi$. The i is necessary because:

$$(\bar{\Psi}\gamma^\mu\partial_\mu\Psi)^\dagger = \partial_\mu\bar{\Psi}\gamma^\mu\Psi \overset{\text{IPP}}{=} -\bar{\Psi}\gamma^\mu\partial_\mu\Psi.$$

Since the fermion is Majorana, the flip relation implies that the connection term vanishes.

Weyl transformation Under a Weyl transformation of the metric

$$g_{\mu\nu} = \mathrm{e}^{2\phi}\hat{g}_{\mu\nu}, \tag{2.32}$$

the action (2.31) transforms as:

$$S = \frac{1}{4\pi}\int \mathrm{d}^2x\sqrt{\hat{g}}\,\mathrm{e}^{2\phi}\bar{\Psi}(\mathrm{e}^{-\frac{3}{2}\phi}\,\mathrm{i}\hat{\nabla\!\!\!\!/}\,\mathrm{e}^{\frac{\phi}{2}} + m\gamma_*)\Psi, \tag{2.33}$$

using the formulas from Appendix A.5. This shows that the action is invariant only if Ψ transforms and if the mass vanishes:

$$\boxed{\Psi = \mathrm{e}^{-\frac{\phi}{2}}\hat{\Psi}, \qquad m = 0.} \tag{2.34}$$

Note that it is necessary to take into account the transformation of the connection in order to get the appropriate transformation of the action, see (A.50e).

2.2.1 Dirac matrices

The Clifford algebra is

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} \tag{2.35}$$

where $\delta_{\mu\nu} = \mathrm{diag}(1, 1)$. The matrices γ^0 and γ^1 are Hermitian:

$$(\gamma^\mu)^\dagger = \gamma^\mu \tag{2.36}$$

and $\hat{\gamma}$ is the identity matrix:

$$\boxed{\hat{\gamma} = 1.} \tag{2.37}$$

The chirality matrix is defined by

$$\boxed{\gamma_* = \mathrm{i}\gamma^0\gamma^1.} \tag{2.38}$$

using $\eta_* = 1$.

Different relations are:

$$B_\zeta = -\epsilon\, C_\eta, \qquad \xi = 1, \tag{2.39a}$$
$$\gamma^{\mu\nu} = -\mathrm{i}\,\epsilon^{\mu\nu}\gamma_*, \tag{2.39b}$$
$$B_\zeta B_\zeta^* = -\epsilon, \tag{2.39c}$$
$$\zeta = -\eta. \tag{2.39d}$$

For each basis, we first consider the Majorana case:

$$t_0 = -1, \quad t_1 = -1 \quad \Longrightarrow \quad \epsilon = -1, \quad \eta = -1, \tag{2.40}$$

which selects C_- and sets $\zeta = +1$ such that:

$$\boxed{B_+ = C_-,} \tag{2.41}$$

Then, we consider the pseudo-Majorana case:

$$t_0 = 1, \quad t_1 = -1, \quad \Longrightarrow \quad \epsilon = 1, \quad \eta = 1, \tag{2.42}$$

which selects C_+ and sets $\zeta = -1$ such that:

$$B_- = C_+, \tag{2.43}$$

The flip relation implies:

$$\widetilde{\Psi}_1\gamma^{\mu\nu}\Psi_2 = -\widetilde{\Psi}_2\gamma^{\mu\nu}\Psi_1, \tag{2.44}$$

and the latter vanishes for $\Psi_1 = \Psi_2$.

2.2.2 Weyl basis, Majorana signs

Gamma matrices In the Weyl basis (Majorana signs), the Dirac matrices are [27, sec. 12.6.1]:

$$\boxed{\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_* = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.} \tag{2.45}$$

The matrices γ^1 and γ_* are symmetric and real, γ^0 is anti-symmetric and imaginary.

The charge conjugation matrix is then

$$\boxed{C_- = B_+ = \gamma^1.} \tag{2.46}$$

Indeed, γ^1 is symmetric as it should and:

$$C_-\gamma^0 C_-^{-1} = -\gamma^0 = (\gamma^0)^t, \qquad C_-\gamma^1 C_-^{-1} = \gamma^1 = (\gamma^1)^t. \tag{2.47}$$

Spinors Writing a spinor as

$$\boxed{\Psi = \begin{pmatrix} \bar\psi \\ \psi \end{pmatrix},} \tag{2.48}$$

its Dirac conjugate is:

$$\bar\Psi = \begin{pmatrix} \bar\psi^* & \psi^* \end{pmatrix}. \tag{2.49}$$

We find that ψ and $\bar\psi$ have respectively negative and positive chiralities:

$$P_+ \Psi = 2 \begin{pmatrix} \bar\psi \\ 0 \end{pmatrix}. \tag{2.50}$$

The Majorana condition states that the components are conjugate to each others:

$$\boxed{\psi^* = \bar\psi.} \tag{2.51}$$

Action It will be convenient to introduce complex coordinates:

$$z = x^0 + \mathrm{i}x^1, \qquad \bar z = x^0 - \mathrm{i}x^1, \qquad \partial = \frac{1}{2}(\partial_0 - \mathrm{i}\partial_1), \qquad \bar\partial = \frac{1}{2}(\partial_0 + \mathrm{i}\partial_1). \tag{2.52}$$

We have:

$$\gamma^\mu \partial_\mu = 2\mathrm{i} \begin{pmatrix} 0 & -\bar\partial \\ \partial & 0 \end{pmatrix}. \tag{2.53}$$

This follows from:

$$\gamma^\mu \partial_\mu = \partial_0 \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} + \partial_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}\partial_0 + \partial_1 \\ \mathrm{i}\partial_0 + \partial_1 & 0 \end{pmatrix}. \tag{2.54}$$

The Dirac bilinears and kinetic operator read:

$$\boxed{\begin{gathered} \bar\Psi\Psi = \psi^*\psi + \bar\psi^*\bar\psi, \qquad \bar\Psi\gamma_*\Psi = -\psi^*\psi + \bar\psi^*\bar\psi, \\ \bar\Psi \partial\!\!\!/ \Psi = -2\mathrm{i}\,\bar\psi^*\bar\partial\psi + 2\mathrm{i}\,\psi^*\partial\bar\psi \end{gathered}} \tag{2.55}$$

since we have:

$$\begin{aligned} \bar\Psi\Psi &= \begin{pmatrix} \bar\psi^* & \psi^* \end{pmatrix} \begin{pmatrix} \bar\psi \\ \psi \end{pmatrix} \\ \bar\Psi \partial\!\!\!/ \Psi &= 2\mathrm{i} \begin{pmatrix} \bar\psi^* & \psi^* \end{pmatrix} \begin{pmatrix} 0 & -\bar\partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \bar\psi \\ \psi \end{pmatrix} \end{aligned}$$

For a Majorana fermion, we have:

$$\begin{gathered} \bar\Psi\Psi = 0, \qquad \bar\Psi\gamma_*\Psi = 2\psi\bar\psi, \\ \bar\Psi \partial\!\!\!/ \Psi = 2\mathrm{i}\,\psi\bar\partial\psi - 2\mathrm{i}\,\bar\psi\partial\bar\psi \end{gathered} \tag{2.56}$$

The action (2.31) for a Majorana fermion can be rewritten as [28, sec. 9.2.2, eq. (171), sec. 10.5.2, eq. (326b)][sec. 4]DHoker:1986:LoopAmplitudesFermionic:

$$\boxed{S = \frac{1}{2\pi} \int \mathrm{d}^2x \sqrt{g}\, (\psi\bar\partial\psi - \bar\psi\partial\bar\psi + m\psi\bar\psi).} \tag{2.57}$$

The second term comes with a minus sign which is not the usual normalization [29, app. A.1, 30, 23, sec. 5.3.2, 31]. However, this is just a consequence of choosing the representation C_-: using the inequivalent representation C_+ changes the relative sign between the two terms, showing that the difference is not physical.

2.2.3 Weyl basis, Majorana signs, alternative basis

Gamma matrices We consider again the Weyl basis (Majorana signs) but exchange γ^0 and γ^1 [16, app. A]:

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \gamma_* = \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.58}$$

The matrices γ^0 and γ_* are symmetric and real, γ^1 is anti-symmetric and imaginary.

The charge conjugation matrix is then

$$C_- = B_+ = \gamma^0. \tag{2.59}$$

Indeed, γ^1 is symmetric as it should and:

$$C_-\gamma^0 C_-^{-1} = \gamma^0 = (\gamma^0)^t, \qquad C_-\gamma^1 C_-^{-1} = -\gamma^1 = (\gamma^1)^t. \tag{2.60}$$

Spinors Writing a spinor as

$$\Psi = \begin{pmatrix} \psi \\ \bar\psi \end{pmatrix}, \tag{2.61}$$

its Dirac conjugate is:

$$\bar\Psi = \begin{pmatrix} \psi^* & \bar\psi^* \end{pmatrix}. \tag{2.62}$$

We find that ψ and $\bar\psi$ have respectively negative and positive chiralities:

$$P_+\Psi = 2\begin{pmatrix} 0 \\ \bar\psi \end{pmatrix}. \tag{2.63}$$

The Majorana condition states that the components are conjugate to each others:

$$\psi^* = \bar\psi. \tag{2.64}$$

Action We have:

$$\gamma^\mu\partial_\mu = 2\begin{pmatrix} 0 & \partial \\ \bar\partial & 0 \end{pmatrix}. \tag{2.65}$$

This follows from:

$$\gamma^\mu\partial_\mu = \partial_0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \partial_1\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{2.66}$$

The Dirac bilinears and kinetic operator read:

$$\begin{gathered} \bar\Psi\Psi = \psi^*\psi + \bar\psi^*\bar\psi, \qquad \bar\Psi\gamma_*\Psi = -\psi^*\psi + \bar\psi^*\bar\psi, \\ \bar\Psi\partial\!\!\!/\Psi = 2\bar\psi^*\bar\partial\psi + 2\psi^*\partial\bar\psi \end{gathered} \tag{2.67}$$

since we have:

$$\begin{gathered} \bar\Psi\Psi = \begin{pmatrix} \psi^* & \bar\psi^* \end{pmatrix}\begin{pmatrix} \psi \\ \bar\psi \end{pmatrix} \\ \bar\Psi\partial\!\!\!/\Psi = 2\begin{pmatrix} \psi^* & \bar\psi^* \end{pmatrix}\begin{pmatrix} 0 & \partial \\ \bar\partial & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \bar\psi \end{pmatrix} \end{gathered}$$

For a Majorana fermion, we have:

$$\begin{gathered}\bar\Psi\Psi = 0, \qquad \bar\Psi\gamma_*\Psi = 2\psi\bar\psi, \\ \bar\Psi\partial\!\!\!/\Psi = 2\psi\bar\partial\psi + 2\bar\psi\partial\bar\psi\end{gathered} \tag{2.68}$$

The action (2.31) for a Majorana fermion can be rewritten as:

$$S = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\,\big(\mathrm{i}\,\psi\bar\partial\psi + \mathrm{i}\,\bar\psi\partial\bar\psi + m\psi\bar\psi\big). \tag{2.69}$$

The second term comes with a minus sign which is not the usual normalization. However, this is just a consequence of choosing the representation C_-: using the inequivalent representation C_+ changes the relative sign between the two terms, showing that the difference is not physical.

2.2.4 Weyl basis, pseudo-Majorana signs

Gamma matrices In the Weyl basis (pseudo-Majorana signs), the Dirac matrices are still (2.45) but we take the charge conjugation matrix to be:

$$C_+ = B_- = \gamma^0. \tag{2.70}$$

Indeed, γ^0 is anti-symmetric as it should and:

$$C_+\gamma^0 C_+^{-1} = \gamma^0 = -(\gamma^0)^t, \qquad C_+\gamma^1 C_+^{-1} = -\gamma^1 = -(\gamma^1)^t. \tag{2.71}$$

Spinors Writing a spinor as

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{2.72}$$

the Majorana condition gives:

$$\psi_1^* = -\mathrm{i}\psi_2. \tag{2.73}$$

Hence, it is useful to define $\psi_2 := \psi$ and $\psi_1 := \mathrm{i}\bar\psi$ such that

$$\Psi = \begin{pmatrix}\mathrm{i}\bar\psi\\ \psi\end{pmatrix}, \tag{2.74}$$

where ψ and $\bar\psi$ are independent for a general Dirac spinor.

Action The Dirac bilinears and kinetic operator read:

$$\begin{gathered}\bar\Psi\Psi = \psi^*\psi + \bar\psi^*\bar\psi, \qquad \bar\Psi\gamma_*\Psi = -\psi^*\psi + \bar\psi^*\bar\psi, \\ \bar\Psi\partial\!\!\!/\Psi = -2\bar\psi^*\bar\partial\psi - 2\psi^*\partial\bar\psi.\end{gathered} \tag{2.75}$$

For a Majorana fermion, we have:

$$\begin{gathered}\bar\Psi\Psi = 0, \qquad \bar\Psi\gamma_*\Psi = 2\psi\bar\psi, \\ \bar\Psi\partial\!\!\!/\Psi = -2\psi\bar\partial\psi - 2\bar\psi\partial\bar\psi.\end{gathered} \tag{2.76}$$

The action (2.31) reads in components:

$$S = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\,\big(-\mathrm{i}\,\psi\bar\partial\psi - \mathrm{i}\,\bar\psi\partial\bar\psi + m\psi\bar\psi\big). \tag{2.77}$$

2.2.5 Majorana basis, Majorana signs

Gamma matrices In the Majorana basis (Majorana signs), the Dirac matrices are [32] (and [21] up to permutation of γ^0 and γ^1)

$$\boxed{\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_* = \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.} \tag{2.78}$$

The matrices γ^0 and γ^1 are symmetric and real, γ_* is anti-symmetric and imaginary. The charge conjugation matrix is then the identity:

$$\boxed{C_- = B_+ = 1.} \tag{2.79}$$

Spinors Writing a spinor as

$$\boxed{\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},} \tag{2.80}$$

its Dirac conjugate is:

$$\bar\Psi = \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix}. \tag{2.81}$$

The Majorana condition states that the components are real:

$$\boxed{\psi_1^* = \psi_1, \qquad \psi_2^* = \psi_2.} \tag{2.82}$$

This implies that $\bar\Psi = \Psi^\dagger = \Psi^t$.

It is straightforward to see that the components in each basis are related by [32]:

$$\boxed{\begin{aligned} \psi &= \frac{1}{\sqrt{2}}(\psi_1 + \mathrm{i}\psi_2), \qquad & \bar\psi &= \frac{1}{\sqrt{2}}(\psi_1 - \mathrm{i}\psi_2), \\ \psi_1 &= \frac{1}{\sqrt{2}}(\psi + \bar\psi), \qquad & \psi_2 &= -\frac{\mathrm{i}}{\sqrt{2}}(\psi - \bar\psi). \end{aligned}} \tag{2.83}$$

The corresponding change of basis matrix is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}, \tag{2.84}$$

and remember that the Dirac matrices transform according to (B.13).

Action We have:

$$\gamma^\mu \partial_\mu = 2 \begin{pmatrix} \partial_1 & \partial_0 \\ \partial_0 & -\partial_1 \end{pmatrix}. \tag{2.85}$$

This follows from:

$$\gamma^\mu \partial_\mu = \partial_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \partial_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \partial_1 & \partial_0 \\ \partial_0 & -\partial_1 \end{pmatrix}. \tag{2.86}$$

The Dirac bilinears and kinetic operator read:

$$\boxed{\begin{gathered} \bar\Psi\Psi = \psi_1^*\psi_1 + \psi_2^*\psi_2, \qquad \bar\Psi\gamma_*\Psi = -\mathrm{i}\psi_1^*\psi_2 - \mathrm{i}\psi_1\psi_2^*, \\ \bar\Psi \partial\!\!\!/ \Psi = \psi_1^*(\partial_1\psi_1 + \partial_0\psi_2) + \psi_2^*(\partial_0\psi_1 - \partial_1\psi_2) \end{gathered}} \tag{2.87}$$

since we have:

$$\bar{\Psi}\Psi = \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\bar{\Psi}\gamma_*\Psi = \mathrm{i} \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\bar{\Psi}\not{\partial}\Psi = \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} \begin{pmatrix} \partial_1 & \partial_0 \\ \partial_0 & -\partial_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} \begin{pmatrix} \partial_1\psi_1 + \partial_0\psi_2 \\ \partial_0\psi_1 - \partial_1\psi_2 \end{pmatrix}$$

For a Majorana fermion, we have:

$$\bar{\Psi}\Psi = 0, \qquad \bar{\Psi}\gamma_*\Psi = -2\mathrm{i}\psi_1\psi_2, \\ \bar{\Psi}\not{\partial}\Psi = \psi_1(\partial_1\psi_1 + \partial_0\psi_2) + \psi_2(\partial_0\psi_1 - \partial_1\psi_2). \tag{2.88}$$

2.2.6 Analytically continued action

Some authors consider the analytic continuation from the Lorentzian action (2.1), in which case the Dirac conjugation matrix is the same as in Lorentzian signature, $\hat{\gamma} = \gamma^0$ [23, sec. 5.3.2, 32, sec. 9.6, 9.7, 12.3], but it does not fit the proper representation theory of Euclidean Clifford algebra. See [33] for a discussion for the Wick rotation with spinors. For comparison, we reproduce the computation here.

After Wick rotation of the coordinates, the action (2.1) reads:

$$S = -\frac{1}{4\pi} \int \mathrm{d}^2x \sqrt{g}\, \Psi^\dagger \gamma_M^0 (\gamma_M^\mu \nabla_\mu - m)\Psi. \tag{2.89}$$

where we display explicitly the Lorentzian gamma matrices γ_M^μ. The basis (2.45) is obtained from the basis (2.7) by Wick rotating $\gamma_M^0 = -\mathrm{i}\gamma_E^0$ and then flipping the sign of γ^0 in order to remove the sign of γ_*. This gives:

$$S = \frac{1}{4\pi} \int \mathrm{d}^2x \sqrt{g}\, \Psi^\dagger \gamma_E^0 (\gamma_E^\mu \nabla_\mu + m)\Psi. \tag{2.90}$$

[HE: Need to reverse time or space to account properly for $\not{\partial}$?] From now on, we omit the index E. ⇐ 3

We recall the basis (2.45):

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_* = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.91}$$

and that

$$\not{\partial} = 2\mathrm{i} \begin{pmatrix} 0 & -\bar{\partial} \\ \partial & 0 \end{pmatrix} \tag{2.92}$$

such that[1]

$$\gamma^0 \not{\partial} = 2 \begin{pmatrix} \partial & 0 \\ 0 & \bar{\partial} \end{pmatrix}. \tag{2.93}$$

Writing the spinor as

$$\Psi = \begin{pmatrix} \bar{\psi} \\ \psi \end{pmatrix}, \tag{2.94}$$

[1]The elements on the diagonal are exchanged in [23]. This may arise from choosing to reverse space or time when continuing analytically.

we find the action in components:

$$S=\frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\,(\psi\bar{\partial}\psi+\bar{\psi}\partial\bar{\psi}+\mathrm{i}m\bar{\psi}\psi). \tag{2.95}$$

This form can be recovered by direct analytic continuation of the action (2.24) (see Appendix A.1.2) and changing the sign of m:

$$S=\frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\,(\psi_+\bar{\partial}\psi_+ +\psi_-\partial\psi_- +\mathrm{i}m\psi_-\psi_+), \tag{2.96}$$

upon identifying:

$$\psi_- := \bar{\psi}, \qquad \psi_+ := \psi. \tag{2.97}$$

3 Functional integral

We want to setup everything needed to compute the gravitational action of the massive Majorana fermion in Euclidean signature.

The effective action of the metric is obtained by integrating out the fermion:

$$\boxed{\mathrm{e}^{-S_{\text{eff}}[g]} := \int \mathrm{d}\psi\, \mathrm{e}^{-S[g,\psi]}.} \tag{3.1}$$

Since the action (2.31) is quadratic, the result is the determinant of the kinetic operator D

$$\boxed{S_{\text{eff}}=-\frac{1}{2}\ln\det D} \tag{3.2}$$

where the 1/2 factor comes from the fact that the fermion is Majorana and the operator D is

$$\boxed{D := \mathrm{i}\nabla\!\!\!/ + m\gamma_*.} \tag{3.3}$$

Then, one can use the formula (A.37) to get

$$\boxed{S_{\text{eff}}=-\frac{1}{4}\ln\det D^2} \tag{3.4}$$

with

$$\boxed{D^2=-\nabla\!\!\!/^2+m^2=-\Delta+\frac{R}{4}+m^2,} \tag{3.5}$$

the latter formula following from the fact that the chirality matrix commutes with all other matrices and $(\gamma_*)^2=1$. This is different from what happens with a Dirac fermion, which requires to introduce the dual operator $\tilde{D}=\gamma_* D\gamma_*$.

We are interested in computing the gravitational action defined as the difference of the effective action evaluated in two different metrics g and $\hat{g}$:

$$\boxed{S_{\text{grav}}[g,\hat{g}] := S_{\text{eff}}[g]-S_{\text{eff}}[\hat{g}]=-\frac{1}{4}\ln\frac{\det D^2}{\det \hat{D}^2}.} \tag{3.6}$$

Since the connection term vanishes, one may be tempted to compute the determinant of the operator $D'=\mathrm{i}\partial\!\!\!/+m\gamma_*$. It seems more logical to use D since the latter is covariant and this is necessary to define a covariant measure on the field space.

In particular, this measure is defined in terms of the modes which are eigenvectors of this operator, and this amounts to solve the classical equation of motion, which involves the covariant operator. Moreover, $\partial\!\!\!/$ does not have a nice transformation under Weyl transformations. In the literature, it is always the operator D which is used [9] (but note that [6] does not write it in the action). This seems also more logical since it gives correctly half of the determinant of that of a Dirac fermion.

3.1 Definition of the functional integral

In order to define the functional integral and the effective action (3.2) rigorously, we need to expand the field in terms of eigenmodes of the operator D. However, this procedure is more complicated for Majorana fermions than for Dirac fermions. We follow mostly [15, sec. 13.3] (see also [16]), except that we don't pick a specific basis.

3.1.1 Mode expansion

There is no solution to the equation

$$D\Psi = (\mathrm{i}\nabla\!\!\!/ + m\gamma_*)\Psi \overset{?}{=} \lambda\Psi \tag{3.7}$$

such that the eigenvalue is real (which is necessary for defining the heat kernel properly) and that Ψ satisfies the Majorana condition:

$$\lambda \in \mathbb{R}, \qquad \Psi^* = C\Psi. \tag{3.8}$$

Indeed, by taking the conjugate of the equation and inserting the Majorana condition one finds

$$D\Psi = (\mathrm{i}\nabla\!\!\!/ + m\gamma_*)\Psi = -\lambda^*\Psi \tag{3.9}$$

from

$$\begin{aligned} \big(- \mathrm{i}\nabla\!\!\!/^* + m(\gamma_*)^*\big)\Psi^* = \lambda^*\Psi^* \\ -C(\mathrm{i}\,\nabla\!\!\!/ + m\gamma_*)C^{-1}C\Psi = \lambda^* C\Psi. \end{aligned}$$

This problem can be solved by looking for complex eigenvectors to be decomposed into their real and imaginary parts (under the Majorana conjugation). Hence, we are looking for complex eigenfunctions $\Psi_n \in \mathbb{C}$ of D with real eigenvalues λ_n ($n \in \mathbb{Z}$) [15, sec. 13.3]:[2]

$$\boxed{D\Psi_n = (\mathrm{i}\nabla\!\!\!/ + m\gamma_*)\Psi_n = \lambda_n\,\Psi_n, \qquad \lambda_n \in \mathbb{R}.} \tag{3.10}$$

The inner-product between two spinors ψ_1 and ψ_2 is defined as:

$$\boxed{\langle\psi_1|\psi_2\rangle := \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\,\psi_1(x)^\dagger\psi_2(x).} \tag{3.11}$$

Note that it vanishes for anti-commuting Majorana spinors (for example, in the Majorana basis: $\psi^\dagger\psi = \psi^t\psi = 0$); however, this is not a problem since the eigenmodes

[2]Note that we could work with real eigenmodes by inserting the matrix γ_* on the RHS. However, this complicates all expressions since this matrix would appear in the definition of the inner-product, Green functions, etc.

are commuting and complex.[3] The eigenfunctions form a complete set and are taken to be orthonormal:[4]

$$\boxed{\langle \Psi_m | \Psi_n \rangle = \delta_{mn}.} \tag{3.12}$$

By computing the inner-product with an insertion of D, we can easily check that the eigenvalues are real:

$$\begin{aligned} \langle \Psi_n | D \Psi_n \rangle &= \lambda_n \\ &= \langle D \Psi_n | \Psi_n \rangle = \lambda_n^*. \end{aligned} \tag{3.13}$$

One can check that if λ_n is the eigenvalue associated to ψ_n, then $-\lambda_n$ is the eigenvalue of $C^{-1} \Psi_n^*$ for $n \neq 0$:

$$D(C^{-1} \Psi_n^*) = -\lambda_n \, (C^{-1} \Psi_n^*) \tag{3.14}$$

as can be seen from:

$$\begin{aligned} \big(- \mathrm{i} \nabla\!\!\!\!/^{\,*} + m (\gamma_*)^* \big) \Psi_n^* &= \lambda_n \, \Psi_n^* \\ -C (\mathrm{i} \, \nabla\!\!\!\!/ + m \gamma_*) C^{-1} \Psi_n^* &= -\lambda_n \, \Psi_n^*. \end{aligned}$$

As a consequence, we define

$$\boxed{\forall n \in \mathbb{N}^* : \qquad \Psi_{-n} := C^{-1} \Psi_n^*, \qquad \lambda_{-n} := -\lambda_n.} \tag{3.15}$$

The Majorana field is expanded on the modes Ψ_n as:

$$\boxed{\Psi := \sum_{n \geq 0} (a_n \Psi_n + a_{-n} \Psi_{-n}),} \tag{3.16}$$

where the a_n are complex Grassmann variables and satisfy:

$$\boxed{a_{-n} = a_n^\dagger.} \tag{3.17}$$

Note that in this decomposition the coefficients are taken to be Grassmann numbers while the eigenfunctions are commuting functions. As a consequence, the normalization (3.12) would be non-trivial even without inserting γ_*. The coefficient a_n can be recovered by taking the inner-product with Ψ_n:

$$\boxed{a_n = \langle \Psi_n | \Psi \rangle \, .} \tag{3.18}$$

The Dirac conjugate is:

$$\bar{\Psi} = \sum_{n \geq 0} (a_n \Psi_n^t C + a_{-n} \Psi_{-n}^t C) = \sum_{n \geq 0} \, (a_n \bar{\Psi}_{-n} + a_{-n} \bar{\Psi}_n), \tag{3.19}$$

using that $\bar{\Psi} = \bar{\Psi}^c = \Psi^t C$ since $(C^{-1})^\dagger = (C^*)^\dagger = C^t = C$ (for $\epsilon = -1$), and the second equality follows from (3.15). Since $\bar{\Psi} = \Psi^\dagger$, one can also recover the expression (3.17) from the coefficient of $\bar{\Psi}_n$.

[3] It would be necessary to add γ_* in the definition of the inner-product if it also appears in the RHS of the eigenvalue equation. But, as pointed in the previous footnote, this makes all expressions much more complicated. One particular problem is that D is not self-adjoint for this product, instead: $\langle \psi_1 | D \psi_2 \rangle = \langle \widetilde{D} \psi_1 | \psi_2 \rangle$, where $\widetilde{D} := \gamma_* D \gamma_*$.

[4] In fact, modes with $\lambda_n = m$ (where m is the mass) are degenerate and generically not orthonormal, see Section 3.3. However, ignoring this subtlety does not change the computation in general and the fact that zero-modes are not orthonormal will be taken care of when needed.

Then, we can define real modes (under the Majorana conjugate) for $n \geq 0$:

$$\boxed{\chi_n := \frac{1}{\sqrt{2}}(\Psi_n + \Psi_{-n}), \qquad \phi_n := -\frac{\mathrm{i}}{\sqrt{2}}(\Psi_n - \Psi_{-n}),} \tag{3.20}$$

such that

$$\boxed{\chi_n^* = C\chi_n, \qquad \phi_n^* = C\phi_n.} \tag{3.21}$$

These modes form two orthonormal sets:

$$\boxed{\langle \chi_m | \chi_n \rangle = \langle \phi_m | \phi_n \rangle = \delta_{mn}, \qquad \langle \chi_m | \phi_n \rangle = 0.} \tag{3.22}$$

It is then straightforward to check that these modes satisfy the equations:

$$\boxed{D\chi_n = \mathrm{i}\lambda_n \, \phi_n, \qquad D\phi_n = -\mathrm{i}\lambda_n \, \chi_n} \tag{3.23}$$

since

$$D\chi_n = \frac{1}{\sqrt{2}} D(\Psi_n + \Psi_{-n}) = \frac{\lambda_n}{\sqrt{2}} \gamma_*(\Psi_n - \Psi_{-n}) = \mathrm{i}\lambda_n \, \phi_n.$$

Squaring this equation gives:

$$\boxed{D^2\chi_n = \Lambda_n \chi_n, \qquad D^2\phi_n = \Lambda_n \phi_n, \qquad \Lambda_n := \lambda_n^2.} \tag{3.24}$$

This also implies:

$$\boxed{D^2\Psi_n = \Lambda_n \Psi_n.} \tag{3.25}$$

This means that the (χ_n, ϕ_n) are eigenfunctions of the second-order (Laplace-type) kinetic operators, but not of the Dirac operator. Note that it should be related with the decomposition of a Majorana spinor into a Weyl spinor and its complex conjugate [24, sec. 3.4].

The eigenvalues are indexed by $n \in \mathbb{N}$ and sorted by ascending order:

$$0 < m^2 \leq \Lambda_0 \leq \Lambda_1 \leq \cdots \tag{3.26}$$

In particular, there is no zero-mode if $m^2 > 0$ (Sections 3.2 and 3.3).

The Majorana field is expanded on the real modes as

$$\boxed{\Psi = \sum_{n \geq 0} (b_n \chi_n + c_n \phi_n)} \tag{3.27}$$

where (b_n, c_n) are real Grassmann variables such that

$$\boxed{a_n = \frac{1}{\sqrt{2}}(b_n + \mathrm{i}c_n), \qquad a_n^\dagger = \frac{1}{\sqrt{2}}(b_n - \mathrm{i}c_n),} \tag{3.28}$$

and we have the relation

$$a_n^\dagger a_n = \frac{\mathrm{i}}{2}(b_n c_n - c_n b_n) = \mathrm{i}\, b_n c_n. \tag{3.29}$$

Note also that the Dirac conjugate is

$$\bar{\Psi} = \sum_{n \geq 0} (b_n \chi_n^t + c_n \phi_n^t) C. \tag{3.30}$$

3.1.2 Evaluation of the functional integral

We want to compute the generating functional with source η:

$$\boxed{Z[\eta] := \int \mathrm{d}\Psi \exp\big(- S[\Psi] + \mathrm{i}\, \langle \bar{\eta} | \Psi \rangle \big)} \tag{3.31}$$

where the action can be written in terms of the inner-product (3.11) as:

$$S[\Psi] = \langle \Psi | \, D \, | \Psi \rangle \, . \tag{3.32}$$

The source is decomposed as

$$\boxed{\eta = \sum_{n \geq 0} (u_n \chi_n + v_n \phi_n) = \sum_{n \geq 0} (s_n \Psi_n + s_{-n} \Psi_{-n}), \qquad s_{-n} = s_n^\dagger,} \tag{3.33}$$

where

$$\boxed{s_n = \frac{1}{\sqrt{2}} (u_n + \mathrm{i} v_n), \qquad s_n^\dagger = \frac{1}{\sqrt{2}} (u_n - \mathrm{i} v_n).} \tag{3.34}$$

We have the relation:

$$s_n^\dagger s_n = \mathrm{i} u_n v_n. \tag{3.35}$$

We can evaluate the inner-product which appears in the path integral:

$$\boxed{\langle \Psi | \, D \, | \Psi \rangle = 2\mathrm{i} \sum_n \lambda_n b_n c_n = 2 \sum_n \lambda_n a_n^\dagger a_n} \tag{3.36}$$

which comes from

$$\begin{aligned}
\langle \Psi | \, D \, | \Psi \rangle &= \sum_{m,n} \langle b_m \chi_m + c_m \phi_m | \, D \, | b_n \chi_n + c_n \phi_n \rangle \\
&= \mathrm{i} \sum_{m,n} \lambda_n \, \langle b_m \chi_m + c_m \phi_m | b_n \phi_n - c_n \chi_n \rangle \\
&= \mathrm{i} \sum_n \lambda_n (b_n c_n - c_n b_n).
\end{aligned}$$

We also have:

$$\boxed{\langle \eta | \Psi \rangle = \sum_{n \geq 0} (u_n b_n + v_n c_n) = \sum_{n \geq 0} (s_n a_n^\dagger + s_n^\dagger a_n).} \tag{3.37}$$

The functional integral reads:

$$Z[\eta] = \int \prod_{n \geq 0} \mathrm{d}b_n \mathrm{d}c_n \exp\left(-\mathrm{i} \sum_{n \geq 0} \left[\lambda_n b_n c_n + u_n b_n + v_n c_n \right] \right) \tag{3.38a}$$

$$= \int \prod_{n \geq 0} \mathrm{d}a_n \mathrm{d}a_n^\dagger \exp\left(\sum_{n \geq 0} \left[- \lambda_n a_n^\dagger a_n + \mathrm{i}\, s_n a_n^\dagger + \mathrm{i}\, s_n^\dagger a_n \right] \right) . \tag{3.38b}$$

The next step consists in shifting the variables a_n:

$$\bar{a}_n = a_n + \frac{\mathrm{i}}{\lambda_n} \, s_n, \qquad \bar{a}_n^\dagger = a_n^\dagger - \frac{\mathrm{i}}{\lambda_n} \, s_n^\dagger \tag{3.39}$$

such that

$$Z[\eta] = \exp\left(\sum_{n\geq 0} \frac{1}{\lambda_n} s_n^\dagger s_n\right) \int \prod_{n\geq 0} \mathrm{d}\bar{a}_n \mathrm{d}\bar{a}_n^\dagger \exp\left(-\sum_{n\geq 0} \lambda_n \bar{a}_n^\dagger \bar{a}_n\right). \tag{3.40}$$

The integral is a simple Gaussian integral of complex Grassmann variables:

$$\boxed{Z[\eta] = \exp\left(\sum_{n\geq 0} \frac{1}{\lambda_n} s_n^\dagger s_n\right) \prod_{n\geq 0} \lambda_n.} \tag{3.41}$$

Note that only half of the eigenvalues are included because we had to combine the real functions into complex functions, which lifts the double degeneracy that one has with a Dirac fermion. The product of the positive eigenvalues give the squareroot of the determinant:

$$\prod_{n\geq 0} \lambda_n = \sqrt{\prod_{n\geq 0} \lambda_n^2} = \left(\prod_{n\in\mathbb{Z}} \lambda_n^2\right)^{\frac{1}{4}} = \left(\det D^2\right)^{1/4} = \det\left(-\Delta + \frac{R}{4} + m^2\right)^{1/4}. \tag{3.42}$$

The first equality allows to write squares of eigenvalues, such that one can extend the range to negative n after the second equality since $\lambda_{-n} = -\lambda_n$. Using formal manipulations of determinants, we can rewrite $\sqrt{\det D^2} = \det D$ such that:

$$\prod_{n\geq 0} \lambda_n = \sqrt{\det D} \overset{1/2}{=} \det\left(\mathrm{i}\nabla\!\!\!/ + m\gamma_*\right). \tag{3.43}$$

Taking the logarithm reproduces (3.2) and (3.4). Note that the fact that one can take the squareroot without ambiguity (up to a sign) is a consequence of the self-adjointness of the operator [11, p. 1470].

The Green function corresponds to

$$\boxed{\begin{aligned} S(x,y) &:= \langle x| \frac{1}{D} |y\rangle := \langle x| \frac{1}{\mathrm{i}\nabla\!\!\!/ + m\gamma_*} |y\rangle \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{\lambda_n} \Psi_n(x)\Psi_n(y)^\dagger = \mathrm{i}\sum_{n\geq 0} \frac{1}{\lambda_n} \left(\phi(x)\chi(y)^t - \chi(x)\phi(y)^t\right). \end{aligned}} \tag{3.44}$$

It follows from:

$$\langle x| \frac{1}{D} |y\rangle = \sum_{n\in\mathbb{Z}} \langle x| \frac{1}{D} |\Psi_n\rangle \langle\Psi_n|y\rangle = \sum_{n\in\mathbb{Z}} \frac{1}{\lambda_n} \langle x|\Psi_n\rangle \langle\Psi_n|y\rangle. \tag{3.45}$$

The Green function is antisymmetric and purely imaginary:

$$S_{\alpha\beta}(x,y) = -S_{\beta\alpha}(y,x). \tag{3.46}$$

In full similarity, we obtain the Green function of D^2:

$$\boxed{G(x,y) := \langle x| \frac{1}{D^2} |y\rangle = \sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n} \Psi_n(x)\Psi_n(y)^\dagger.} \tag{3.47}$$

Note that S and G are 2-dimensional matrices in terms of Dirac indices. The trace over Dirac indices is denoted by tr_D.

Finally, we have that

$$\boxed{\langle\eta|\,S\,|\eta\rangle = \sum_{n\in\mathbb{Z}} \frac{1}{\lambda_n}\, s_n^\dagger s_n = 2\mathrm{i} \sum_{n\geq 0} \frac{1}{\lambda_n}\, u_n v_n,} \tag{3.48}$$

since

$$\begin{aligned}
\langle\eta|\,\frac{1}{\mathrm{i}\nabla\!\!\!/ + m\gamma_*}\,|\eta\rangle &= \sum_{m,n\geq 0} \langle u_m\chi_m + v_m\phi_m|\,\frac{1}{D}\,|u_n\chi_n + v_n\phi_n\rangle \\
&= -\sum_{m,n\geq 0} \frac{\mathrm{i}}{\lambda_n}\,\langle u_m\chi_m + v_m\phi_m|\phi_n\rangle\, u_n + \sum_{m,n\geq 0} \frac{\mathrm{i}}{\lambda_n}\,\langle u_m\chi_m + v_m\phi_m|\chi_n\rangle\, v_n \\
&= \mathrm{i}\sum_{n\geq 0} \frac{1}{\lambda_n}\big(-v_n u_n + u_n v_n\big) = \sum_{n\geq 0} \frac{1}{\lambda_n}\big(-s_n s_n^\dagger + s_n^\dagger s_n\big) \\
&= \sum_{n\geq 0} \frac{s_n s_n^\dagger}{-\lambda_n} s_n s_n^\dagger + \sum_{n\geq 0} \frac{s_n^\dagger s_n}{\lambda_n} = \sum_{n\leq 0} \frac{s_n^\dagger s_n}{\lambda_n} s_n s_n^\dagger + \sum_{n\geq 0} \frac{s_n^\dagger s_n}{\lambda_n}.
\end{aligned}$$

As a conclusion, we find:

$$\boxed{Z[\eta] = \exp\left(\frac{1}{2}\langle\eta|\,S\,|\eta\rangle\right) \det\left(-\Delta + \frac{R}{4} + m^2\right)^{1/4}.} \tag{3.49}$$

The factor of $1/2$ arises because the Green function has a sum $n \in \mathbb{Z}$ but the functional integral gives only $n \geq 0$ in the exponential. This proves (3.4).

Let us pause to comment on the case where there are zero-modes (vanishing eigenvalues). In this case, $Z[\eta]$ in (3.41) looks ill-defined because the first term diverges and the product of eigenvalues vanishes. However, the eigenvalues $\lambda_n = 0$ do not appear in the sum in $\langle\Psi|D\Psi\rangle$ such that the product would be only over strictly positive eigenvalues, $n > 0$. Similarly, (3.49) has instead $\det' D$ and $\widetilde{S}$, the determinant and Green functions without zero-modes. Moreover, the integrals over zero-modes would remain to be done: since they are fermionic, it looks like the result would vanish. This is solved by inserting zero-modes in the functional integral and carefully normalizing [11, 34].

3.2 Eigenmodes and eigenvalues

In Section 3.1, we have introduced the operator D and D^2 which we recall here for convenience:

$$D := \mathrm{i}\nabla\!\!\!/ + m\gamma_*, \qquad D^2 = -\Delta + \frac{R}{4} + m^2. \tag{3.50}$$

The eigenvalue equations for D and D^2 are:

$$D\,|\Psi_n\rangle = \lambda_n\,|\Psi_n\rangle\,, \tag{3.51a}$$
$$D^2\,|\Psi_n\rangle = \Lambda_n\,|\Psi_n\rangle \tag{3.51b}$$

with the modes normalized as:

$$\langle\Psi_m|\Psi_n\rangle = \int \mathrm{d}^2x\sqrt{g}\,\Psi_m(x)^\dagger\Psi_n(x) = \delta_{mn}. \tag{3.52}$$

The eigenvalues are related as:

$$\Lambda_n = \lambda_n^2 \tag{3.53}$$

and are conventionally sorted by ascending order:

$$n \geq 0: \qquad \Lambda_n < \Lambda_{n+1}. \tag{3.54}$$

Eigenvalues can be degenerate: in this case, the mode Ψ_n associated with the degenerate eigenvalue Λ_n is a general linear combination of eigenmodes with eigenvalue Λ_n. In general, we will assume that there is no degeneracy for $n \neq 0$, the general result following by continuity [35, p. 15]. The only caveat is for the lowest eigenvalue Λ_0 associated with zero-modes (Section 3.3): in most cases, the degeneracy does not matter and we will take it into account only when necessary.

The modes are dimensionless, while λ_n and Λ_n have the dimensions of a mass and mass-squared respectively:

$$[\Psi_n] = 0, \qquad [\lambda_n] = M, \qquad [\Lambda_n] = M^2. \tag{3.55}$$

We have also obtained the expressions for the Green functions in terms of modes:

$$\boxed{S(x,y) = \sum_{n\in\mathbb{Z}} \frac{1}{\lambda_n} \Psi_n(x)\Psi_n(y)^\dagger,} \tag{3.56a}$$

$$\boxed{G(x,y) = \sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n} \Psi_n(x)\Psi_n(y)^\dagger,} \tag{3.56b}$$

We note that $\nabla\!\!\!/ \Psi_n$ is also an eigenmode of of D^2:

$$D^2(\nabla\!\!\!/ \Psi_n) = \Lambda_n(\nabla\!\!\!/ \Psi_n). \tag{3.57}$$

However, they are not normalized:

$$\boxed{\int \mathrm{d}^2x\sqrt{g}\,\big(\nabla\!\!\!/ \Psi_n(x)\big)^\dagger \gamma_*\big(\nabla\!\!\!/ \Psi_n(x)\big) = \Lambda_n - m^2.} \tag{3.58}$$

which follows by integrating by part.

3.3 Zero-modes

The problem of the zero modes of the Dirac is treated in [11] (see also [12, 14, 16]).

3.3.1 Massless fermion and properties of eigenmodes

We first consider the massless case because it allows to derive many properties on the eigenmodes, even when $m \neq 0$. Since the operator $D^{(0)}$ is simpler than D, it is possible to characterize more precisely its eigenmodes. The first objective is to show that D, D^2 and $D^{(0)2}$ have a common basis of eigenmodes. Next, we will show that we can use the same basis for the eigenmodes associated with the lowest (absolute) eigenvalue of each operator (zero-modes).

Quantities corresponding to the massless case $m = 0$ are indicated by a superscript (0). For example, the massless Dirac operator and its square are:

$$\boxed{D^{(0)} := \mathrm{i}\nabla\!\!\!/, \qquad (D^{(0)})^2 := -\Delta + \frac{R}{4},} \tag{3.59}$$

The eigenvalues are denoted as $\lambda_n^{(0)}$ and $\Lambda_n^{(0)}$ and there is a common basis of eigenmodes $\Psi_n^{(0)}$ for $D^{(0)}$ and $(D^{(0)})^2$:

$$\boxed{D^{(0)}\Psi_n^{(0)} = \lambda_n^{(0)}\,\Psi_n^{(0)}, \qquad (D^{(0)})^2\Psi_n^{(0)} = \Lambda_n^{(0)}\Psi_n^{(0)}, \qquad \Lambda_n^{(0)} := (\lambda_n^{(0)})^2.} \tag{3.60}$$

It can be shown that $D^{(0)2}$ is a positive-definite operator:

$$0 \le \Lambda_0^{(0)} < \Lambda_1^{(0)} < \cdots \tag{3.61}$$

It is obvious that an eigenmode Ψ_n of D^2 with eigenvalue Λ_n is also an eigenmode of $(D^{(0)})^2$ with eigenvalue $\Lambda_n^{(0)}$ such that:

$$\boxed{\Lambda_n = \Lambda_n^{(0)} + m^2.} \tag{3.62}$$

This is particularly convenient since several useful properties can be deduced in the massless case. For general properties of $D^{(0)}$, see [36]. However, we *cannot* conclude that Ψ_n is also an eigenmode of $D^{(0)}$: this can be understood from the fact that Ψ_n can be eigenmode of both D and $D^{(0)}$ only if it is an eigenmode of γ_*, which is not possible (γ_* and $D^{(0)}$ do not commute so they cannot be diagonalized simultaneously).

Since $(D^{(0)})^2$ commutes with γ_*, it is possible to find eigenmodes $\Psi_{n,\pm}$ which diagonalize both simultaneously:

$$(D^{(0)})^2\Psi_{n,\pm} = \Lambda_n^{(0)}\Psi_{n,\pm}, \qquad \gamma_*\Psi_{n,\pm} = \pm\Psi_{n,\pm}, \tag{3.63}$$

implying that $\Psi_{n,\pm}$ are Weyl spinors.[5] However, this basis must be different from the basis $\Psi_n^{(0)}$ introduced above because $\Psi_{n,\pm}$ are not eigenmodes of $D^{(0)}$ as we show below. The modes are normalized as:

$$\langle\Psi_{m,+}|\Psi_{n,+}\rangle = \delta_{m,n}, \qquad \langle\Psi_{m,-}|\Psi_{n,-}\rangle = \delta_{m,n}, \qquad \langle\Psi_{m,+}|\Psi_{n,-}\rangle = 0. \tag{3.64}$$

Hence, we can write the identity as:

$$1 = \sum_{n\in\mathbb{Z}} \big(|\Psi_{n,+}\rangle\langle\Psi_{n,+}| + |\Psi_{n,-}\rangle\langle\Psi_{n,-}|\big). \tag{3.65}$$

The RHS is correctly invariant when conjugating with γ_*.[6]

Since $\{\gamma_*, D^{(0)}\} = 0$, this implies first that $D^{(0)}\Psi_{n,\pm}$ has chirality $\mp$ since γ_* anticommutes with $D^{(0)}$. Moreover, it also implies that $\gamma_*\Psi_n^{(0)}$ is an eigenmode of $D^{(0)}$ with eigenvalue $-\lambda_n^{(0)}$:

$$D^{(0)}(\gamma_*\Psi_n^{(0)}) = -\lambda_n(\gamma_*\Psi_n^{(0)}). \tag{3.66}$$

[5] Spinors in $2d$ Euclidean space cannot satisfy the Weyl and Majorana conditions. However, we have seen that the eigenmodes cannot be real, such that it makes sense to impose the Weyl condition.

[6] One has to be careful with the zero-modes, which are not orthonormal in general. We will see below that everything works as expected.

If $\lambda_n \neq 0$, this implies that it is orthogonal to $\Psi_n^{(0)}$ and proportional to $C^{-1}\Psi_n^{(0)*}$:

$$\langle \Psi_m^{(0)} | \, \gamma_* \, | \Psi_n^{(0)} \rangle \propto \delta_{m,-n}. \tag{3.67}$$

On the other hand, if $\Psi_n^{(0)}$ is a zero-mode, $\lambda_n^{(0)} = 0$, then $\gamma_* \Psi_n^{(0)}$ is also a zero-mode.

We can now prove that $\Psi_{n,\pm}$ are not eigenmodes of $D^{(0)}$ if $\lambda_n^{(0)} \neq 0$. Writing (3.66) for $\Psi_{n,\pm}$, we get:

$$\begin{aligned} D^{(0)}(\gamma_* \Psi_{n,\pm}) &= -\lambda_n (\gamma_* \Psi_{n,\pm}) = \mp \lambda_n \Psi_{n,\pm} \\ &= \pm D^{(0)} \Psi_{n,\pm} = \pm \lambda_n^{(0)} \Psi_{n,\pm} \end{aligned} \tag{3.68}$$

such that

$$\lambda_n^{(0)} \Psi_{n,\pm} = 0. \tag{3.69}$$

We see that $\Psi_{n,\pm}$ are eigenmodes of $D^{(0)}$ only if $\lambda_n^{(0)} = 0$.

A *zero-mode* (or harmonic spinor) of $D^{(0)}$ is an eigenmode Ψ_0 with zero eigenvalue $\lambda_0^{(0)}$:

$$D^{(0)} \Psi_0 = 0, \qquad \lambda_0^{(0)} = 0. \tag{3.70}$$

It is also a zero-mode of $(D^{(0)})^2$ because $D^{(0)}$ is self-adjoint [37]:

$$(D^{(0)})^2 \Psi_0 = 0, \qquad \Lambda_0^{(0)} = 0. \tag{3.71}$$

We have omitted the superscript, i.e. we write Ψ_0 instead of $\Psi_0^{(0)}$ because we will find that we can use the same basis for the modes associated to the lowest (absolute) eigenvalue of all operators. It is possible to show that $D^{(0)}$ admits up to g zero-modes on a Riemann surface of genus g [38] We denote by N_0 the degeneracy of eigenvalue $\lambda_0^{(0)} = 0$ and the associated zero-modes by $\psi_{0,i}$ $(i = 1, \ldots, N_0)$:

$$D^{(0)} \psi_{0,i} = \mathrm{i} \nabla\!\!\!/ \, \psi_{0,i} = 0. \tag{3.72}$$

As explained in Section 3.2, Ψ_0 should be understood as a general linear combination of the zero-modes $\psi_{0,i}$.

Massive modes The minimal value of the lowest eigenvalue Λ_0 in the massive case is achieved when Ψ_0 is a zero-mode of $(D^{(0)})^2$:

$$D^2 \Psi_0 = m^2 \Psi_0 = \Lambda_0 \Psi_0. \tag{3.73}$$

Hence, D^2 itself has no zero-mode and one has:

$$0 < m^2 \leq \Lambda_0 < \Lambda_1 < \cdots \tag{3.74}$$

This follows directly from (3.62).

Consider chiral zero-modes $\Psi_{0,\pm}$ of $D^{(0)}$

$$D^{(0)} \Psi_{0,\pm} = 0. \tag{3.75}$$

Then, they are also eigenmodes of D:

$$D \Psi_{0,\pm} = m \gamma_* \Psi_{0,\pm} = \pm m \Psi_{0,\pm}. \tag{3.76}$$

For illustration, let's solve the equation

$$m\gamma_* \Psi_0 = \lambda_0 \Psi_0. \tag{3.77}$$

in the Majorana basis (2.78):

$$\gamma_* = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{3.78}$$

such that writing the field in components $\Psi_0 = (\psi_1, \psi_2)$ in the above system gives:

$$-\mathrm{i}m\psi_2 = \lambda_0 \psi_1, \qquad \mathrm{i}m\psi_1 = \lambda_0 \psi_2. \tag{3.79}$$

Solving for ψ_2 and putting back in the first equation, we find:

$$\lambda_0^2 = m^2. \tag{3.80}$$

For $\lambda_0 = \pm m$, we have the solutions $\psi_2 = \pm \mathrm{i}\psi_1$. As expected, the spinor does not satisfy the Majorana condition (2.82).

Given the form (A.38) of D^2, we see that it commutes with γ_* such that the $\Psi_{n,\pm}$ also provide a basis of eigenmodes for D^2. However, we will find it more useful to work with the basis Ψ_n of eigenmodes for both D and D^2. The only exception if for zero-modes Ψ_0, for which we will need to use a base of definite chiralities to be able to prove some properties.

In the rest of these notes, we will call "zero-modes" the modes $\psi_{0,i}$, with the understanding that the term "zero-modes" refers to the operator $D^{(0)}$ and not D.

3.3.2 Inner-product matrix and projector

We consider the general case where the zero-modes do not form an orthonormal basis, and we define the inner-product matrix:

$$\boxed{\kappa_{ij}[g] := \langle \psi_i | \psi_j \rangle = \int \mathrm{d}^2 x \sqrt{g}\, \psi_{0,i}(x)^\dagger \psi_{0,j}(x).} \tag{3.81}$$

For an orthonormal basis, one has $\kappa_{ij} = \delta_{ij}$. The reason for not considering such as basis is that Weyl transformations can mix the zero-modes such that the new basis is not orthonormal. Its inverse is denoted by κ^{ij} such that the projector $P(x, y)$ on the zero-modes reads:

$$\boxed{P(x, y) = \sum_{i,j=1}^{N_0} \psi_{0,i}(x) \kappa^{ij} \psi_{0,j}(y)^\dagger.} \tag{3.82}$$

We also define:

$$P(x) := P(x, x). \tag{3.83}$$

We obviously have:

$$\int \mathrm{d}^2 z \sqrt{g}\, P(x, z) P(z, y) = P(x, y) \tag{3.84}$$

and

$$\nabla\!\!\!/_x P(x, y) = 0, \qquad \Delta_x P(x, y) = 0 \tag{3.85}$$

as the derivative and Laplacian act on a zero-modes.

The number of zero-modes can be rewritten as the trace of $P(x,y)$:

$$\boxed{N_0 = \int \mathrm{d}^2x \sqrt{g}\; \mathrm{tr}_D\, P(x).} \tag{3.86}$$

This follows from:

$$\begin{aligned}
\int \mathrm{d}^2x \sqrt{g}\; \mathrm{tr}_D\, P(x) &= \sum_{i,j} \int \mathrm{d}^2x \sqrt{g}\; \mathrm{tr}_D\, \psi_{0,i}(x) \kappa^{ij}[g] \psi_{0,j}(x)^\dagger \\
&= \sum_{i,j} \kappa^{ij}[g] \int \mathrm{d}^2x \sqrt{g}\, \psi_{0,j}(x)^\dagger \psi_{0,i}(x) \\
&= \sum_{i,j} \kappa^{ij}[g] \kappa_{ij}[g] = \sum_{i=1}^{N_0} 1.
\end{aligned}$$

A prime on a operation (such as $\mathrm{tr}'\, D^{(0)}$ or $\det'\, D^{(0)}$) indicates that one removes the zero-modes before performing the computation.

We have seen in Section 3.3.1 that modes are also eigenmodes of γ_*. In this case, the matrix κ defined in (3.3.1) is block diagonal and the projector splits in two separate sums over modes of positive and negative chiralities. As a consequence, the projector (3.82) is invariant under conjugation by γ_*:

$$\boxed{\gamma_* P(x,y) \gamma_* = P(x,y).} \tag{3.87}$$

3.3.3 Zero-modes on Riemann surfaces

In this subsection, we discuss explicitly the zero-modes of the operator $D^2 = -\Delta + R/4$ (3.5) for the different genus-g Riemann surfaces [38, sec. VI.F]. The theory of Riemann surfaces implies that there can be at most $g+1$ zero-modes [37, 39]. The number of zero-modes depend on the spin structure for $g \geq 1$, and on the moduli for $g \geq 3$ [37, prop. 1.3, 38, p. 927]. Generically, there is no zero-mode for even spin structure and one zero-mode for odd spin structure [38, p. 1018].

Sphere Let's consider $g_{\mu\nu}$ to be the round metric for the sphere, $g = 0$. Because it has a positive constant curvature and since $-\Delta$ is positive-definite, the operator D^2 is strictly positive and has no zero-mode:

$$N_0 = 0, \qquad P(x,y) = 0. \tag{3.88}$$

Torus Let's consider the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$ on the torus, $g = 1$. Zero-modes are solutions of the equation

$$\partial\!\!\!/ \psi_0 = 0. \tag{3.89}$$

There are two obvious constant solutions:

$$\psi_0 = \frac{1}{\sqrt{A}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \psi_0' = \frac{1}{\sqrt{A}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.90}$$

It is possible to prove that there are no other solutions. The corresponding projector and κ-matrix are:

$$\boxed{\kappa = 1_2, \qquad P(x,y) = \frac{1}{A}\, 1_2.} \tag{3.91}$$

Next, one needs to impose boundary conditions which depend on the spin structure. The solution survives only for the odd spin structure which has periodic boundary conditions in both directions. If at least one boundary has anti-periodic conditions, then there is no solution. As a consequence, there is one (complex) zero-mode for the odd spin structure, and no zero-mode for the three even spin structures:

$$N_0 = 0, 1. \tag{3.92}$$

Higher-genus surfaces For a surface with genus $g \geq 2$, there is generically one (resp. no) zero-mode when the spin structure is odd (resp. even). However, there can be up to g zero-modes as stated above [38]. Zero-modes can be computed by taking g to be the metric such that $R = -1$.

3.4 Green functions

We define the Green functions S and G for the operators D and D^2:

$$\boxed{D_x S(x,y) = (\mathrm{i}\nabla\!\!\!/_x + m\gamma_*) S(x,y) = \frac{\delta(x-y)}{\sqrt{g}},} \tag{3.93a}$$

$$\boxed{D_x^2 G(x,y) = \Big(-\Delta_x + \frac{R(x)}{4} + m^2 \Big) G(x,y) = \frac{\delta(x-y)}{\sqrt{g}},} \tag{3.93b}$$

and we recall that

$$-\nabla\!\!\!/^2 = -\Delta + \frac{R}{4} = D^2 - m^2. \tag{3.94}$$

We obviously have the relation

$$\boxed{S(x,y) = D_x G(x,y) = (\mathrm{i}\nabla\!\!\!/_x + m\gamma_*) G(x,y).} \tag{3.95}$$

We can also find another relation

$$\boxed{G(x,y) = \int \mathrm{d}^2 z \sqrt{g}\, S(x,z) S(z,y)} \tag{3.96}$$

from

$$\begin{aligned}\int \mathrm{d}^2 z \sqrt{g}\, S(x,z) S(z,y) &= \int \mathrm{d}^2 z \sqrt{g}\, (D_z G(z,x))^\dagger D_z G(z,y) \\ &= \int \mathrm{d}^2 z \sqrt{g}\, G(x,z) D_z^2 G(z,y),\end{aligned}$$

and then one can use (3.93b) (there is no sign because the derivative term contains also a factor i, and the mass term is not integrated). In fact, one can also obtain (3.96) directly by solving the equation (3.95) through convolution.

Another identity satisfied by G is:

$$\boxed{G(x,y). = -\int \mathrm{d}^2 z \sqrt{g}\, G(x,z) \overleftarrow{D}_z D_z G(z,y)} \tag{3.97}$$

This follows by integrating by part the LHS and using (3.93b).

The adjoint of (3.93a) is

$$\boxed{S(y,x)(-\mathrm{i}\overleftarrow{\nabla\!\!\!/}_x + m\gamma_*) = \frac{\delta(x-y)}{\sqrt{g}}.} \tag{3.98}$$

since the gamma matrices are Hermitian and since

$$S(x,y)^\dagger = S(y,x). \tag{3.99}$$

We also have

$$S(y,x) = G(y,x)(-\mathrm{i}\overleftarrow{\nabla\!\!\!/}_x + m\gamma_*). \tag{3.100}$$

3.5 Green functions without zero-modes

If the operator contains zero-modes, then it is not invertible: one needs subtract their contribution from any equation, and the corresponding projector is $P(x,y)$ defined in (3.82). As we have seen, the operators for $m \neq 0$ have no zero-modes, but the limits of the associated Green functions (and similar quantities) as $m \to 0$ are ill-defined. For this reason, it is useful to introduce Green functions $\widetilde{S}$ and $\widetilde{G}$ which have a well-defined $m \to 0$ limit by subtracting the would-be zero-modes:

$$\boxed{D_x\widetilde{S}(x,y) = (\mathrm{i}\nabla\!\!\!/_x + m\gamma_*)\widetilde{S}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} - P(x,y),} \tag{3.101a}$$

$$\boxed{D_x^2\widetilde{G}(x,y) = \left(-\Delta_x + \frac{R(x)}{4} + m^2\right)\widetilde{G}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} - P(x,y).} \tag{3.101b}$$

These Green functions are orthogonal to the projector:

$$\boxed{\int \mathrm{d}^2z\sqrt{g}\, P(x,z)\widetilde{S}(z,y) = 0, \qquad \int \mathrm{d}^2z\sqrt{g}\, P(x,z)\widetilde{G}(z,y) = 0.} \tag{3.102}$$

When there are no zero-modes and on the torus (in Majorana basis), this reduces to:

$$\boxed{N_0 = 0 \text{ or } g = 1: \qquad \int \mathrm{d}^2z\sqrt{g}\,\widetilde{S}(z,y) = 0, \qquad \int \mathrm{d}^2z\sqrt{g}\,\widetilde{G}(z,y) = 0,} \tag{3.103}$$

since $P(x,y) = 0$ in these cases (Section 3.3.3).

The tilde functions can be easily related to the un-tilde Green functions using the expression (3.56) in terms of modes and the projector (3.82):

$$\boxed{\widetilde{S}(x,y) = S(x,y) - \frac{\gamma_* P(x,y)}{m},} \tag{3.104a}$$

$$\boxed{\widetilde{G}(x,y) = G(x,y) - \frac{P(x,y)}{m^2}.} \tag{3.104b}$$

Note that (3.97) still holds when using $\widetilde{G}$ everywhere thanks to (3.102). The γ_* matrix in the RHS of (3.104a) is necessary to ensure that (3.93a) and (3.101a) are compatible:

$$\begin{aligned}(\mathrm{i}\nabla\!\!\!/ + m\gamma_*)S(x,y) &= (\mathrm{i}\nabla\!\!\!/ + m\gamma_*)\left(\widetilde{S}(x,y) + \frac{\gamma_* P(x,y)}{m}\right) \\ &= \frac{\delta(x-y)}{\sqrt{g}} - \cancel{P(x,y)} + \cancel{P(x,y)}.\end{aligned}$$

The relation (3.96) becomes:

$$\boxed{\widetilde{G}(x,y) = \int \mathrm{d}^2 z \sqrt{g}\, \widetilde{S}(x,z)\widetilde{S}(z,y)} \tag{3.105}$$

which can be found by solving the equation $\widetilde{S} = D\widetilde{G}$ and using that S is orthogonal with P. We can also use (3.104):

$$\begin{aligned}
\widetilde{G}_{xy} = G_{xy} - \frac{P_{xy}}{m^2} &= \int \mathrm{d}^2 z \sqrt{g}\, S_{xz} S_{zy} - \frac{P_{xy}}{m^2} \\
&= \int \mathrm{d}^2 z \sqrt{g} \left(\widetilde{S}_{xz} + \frac{\gamma_* P_{xz}}{m} \right) \left(\widetilde{S}_{zy} + \frac{\gamma_* P_{zy}}{m} \right) - \frac{P_{xy}}{m^2} \\
&= \int \mathrm{d}^2 z \sqrt{g}\, \widetilde{S}_{xz} \widetilde{S}_{zy} + \cancel{\frac{1}{m} \int \mathrm{d}^2 z \sqrt{g}\, (\widetilde{S}_{xz} \gamma_* P_{zy} + \gamma_* P_{xz} \widetilde{S}_{zy})} \\
&\quad + \cancel{\frac{1}{m^2} \int \mathrm{d}^2 z \sqrt{g}\, P_{xz} P_{zy}} - \cancel{\frac{P_{xy}}{m^2}}.
\end{aligned}$$

where we used that P and $\widetilde{S}$ are orthogonal and that $\gamma_* P = P \gamma_*$ from (3.87).

A solution to the equation

$$D^2 \Psi(x) = \eta(x) \tag{3.106}$$

reads

$$\Psi(x) = \int \mathrm{d}^2 y \sqrt{g}\, G(x,y) \eta(y) + \int \mathrm{d}^2 y \sqrt{g}\, P(x,y) \Psi(y). \tag{3.107}$$

We need to add the components of Ψ proportional to the zero-modes. Similarly, the solution to

$$D^2 \Psi(x) = \eta(x) - \int \mathrm{d}^2 y \sqrt{g}\, P(x,y) \eta(y) \tag{3.108}$$

(the source is orthogonal to the zero-modes) is:

$$\Psi(x) = \int \mathrm{d}^2 y \sqrt{g}\, \widetilde{G}(x,y) \eta(y) + \int \mathrm{d}^2 y \sqrt{g}\, P(x,y) \Psi(y). \tag{3.109}$$

A first consistency condition is obtained by integrating (3.101b) over a surface and using the fact that the Laplacian is a total derivative

$$\int \mathrm{d}^2 x \sqrt{g} \left(\frac{R}{4} + m^2 \right) \widetilde{G}(x,y) = 1 - \int \mathrm{d}^2 x \sqrt{g}\, P(x,y). \tag{3.110}$$

The $m = 0$ Green functions $\widetilde{S}^{(0)}$ and $\widetilde{G}^{(0)}$ are then:

$$\boxed{D_x \widetilde{S}^{(0)}(x,y) = \mathrm{i} \nabla\!\!\!/_x \widetilde{S}^{(0)}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} - P(x,y),} \tag{3.111a}$$

$$\boxed{D_x^2 \widetilde{G}^{(0)}(x,y) = \left(-\Delta_x + \frac{R(x)}{4} \right) \widetilde{G}^{(0)}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} - P(x,y).} \tag{3.111b}$$

Note that the projection is identical for all operators.

Note that

$$\widetilde{G}(x,y) = \sum_{s \geq 0} (-1)^s m^{2s} \sum_{n \neq 0} \frac{\Psi_n(x) \Psi_n(y)^\dagger}{\Lambda_n^{(0)s+1}} = \widetilde{G}^{(0)}(x,y) + O(m^2), \tag{3.112}$$

which follows by using (3.62):

$$\widetilde{G}(x,y) = \sum_{n\neq 0} \frac{\Psi_n(x)\Psi_n(y)^\dagger}{\Lambda_n} = \sum_{n\neq 0} \frac{\Psi_n(x)\Psi_n(y)^\dagger}{\Lambda_n^{(0)} + m^2}$$
$$= \sum_{n\neq 0} \frac{\Psi_n(x)\Psi_n(y)^\dagger}{\Lambda_n^{(0)}} \sum_{s\geq 0} (-1)^s \left(\frac{m^2}{\Lambda_n}\right)^s.$$

As a consequence of (3.57), the mode expansion of the Green function $\widetilde{G}$ can also be written as:

$$\boxed{\widetilde{G}(x,y) = \sum_{n\neq 0} \frac{(\nabla\!\!\!/\,\Psi_n(x))(\nabla\!\!\!/\,\Psi_n(y))^\dagger}{\Lambda_n(\Lambda_n - m^2)}.} \tag{3.113}$$

The integrated Green function is denoted by:

$$\boxed{\Psi_G[g] := \int \mathrm{d}^2x \sqrt{g}\, \mathrm{tr}_D\, \widetilde{G}(x).} \tag{3.114}$$

4 Conformal variations

In this section, we study the Weyl transformation of the different quantities used to compute the effective action. A Weyl transformation of the metric reads

$$g_{\mu\nu} = \mathrm{e}^{2\phi} \hat{g}_{\mu\nu}. \tag{4.1}$$

For an infinitesimal parameter $\delta\phi$, we have:

$$\delta g_{\mu\nu} = 2\delta\phi\, g_{\mu\nu}. \tag{4.2}$$

This implies that

$$\delta g^{-1/2} = -2\delta\phi\, g^{-1/2}. \tag{4.3}$$

4.1 Zero-modes

The zero-modes transform as:

$$\boxed{\psi_{i,0} = \mathrm{e}^{-\frac{\phi}{2}} \hat{\psi}_{i,0}.} \tag{4.4}$$

Indeed, from the transformation (A.50e), it is obvious that $\hat{\psi}_{i,0} = \mathrm{e}^{\frac{\phi}{2}} \psi_{i,0}$ is a zero-mode of $\hat{D}$ if $\psi_{i,0}$ is a zero-mode of D:

$$D\psi_{i,0} = 0 \quad \Longrightarrow \quad \hat{D}\hat{\psi}_{i,0} = 0. \tag{4.5}$$

This directly implies that the number of zero-modes is a conformal invariant [37, prop. 1.3]:

$$\delta N_0 = 0. \tag{4.6}$$

This is consistent with the transformation (2.34).

The variation of the other modes is discussed in Section 4.4. It cannot be as simple as the transformation (2.34) for the field, in particular, because of the mass and of the change in the normalization condition.

The inner-product matrix (3.81) transforms as:

$$\boxed{\kappa_{ij} = \int \mathrm{d}^2x \sqrt{\hat{g}}\, \mathrm{e}^{\phi}\, \hat{\psi}_{0,i}(x)^\dagger \hat{\psi}_{0,j}(x)} \tag{4.7}$$

such that

$$\boxed{\delta\kappa_{ij} = \int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x)\, \psi_{0,i}(x)^\dagger \psi_{0,j}(x).} \tag{4.8}$$

The fact that κ_{ij} is a functional of the metric shows that orthonormality of the zero-modes cannot generically be preserved under a transformation [11]. We can also write the variation of the inverse metric:

$$\boxed{\delta\kappa^{ij} = -\int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x)\, \kappa^{ik} \psi_{0,k}(x)^\dagger \psi_{0,\ell}(x)\, \kappa^{\ell j}} \tag{4.9}$$

from

$$\delta\kappa^{-1} = -\kappa^{-1} \delta\kappa \kappa^{-1}. \tag{4.10}$$

We have the useful formula:

$$\boxed{\delta \ln \det \kappa_{ij} = \operatorname{tr} \int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D P(x).} \tag{4.11}$$

This follows from:

$$\begin{aligned}
\delta \ln \det \kappa_{ij} = \delta \operatorname{tr} \ln \kappa_{ij} &= \operatorname{tr} \kappa^{ij} \delta\kappa_{jk} = \kappa^{ij} \delta\kappa_{ij} \\
&= \int \mathrm{d}^2x \sqrt{g}\, \psi_{0,i}(x)^\dagger \kappa^{ij} \psi_{0,j}(x)\, \delta\phi(x) \\
&= \int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D P(x).
\end{aligned}$$

The variation of the projector (3.82) reads:

$$\boxed{\delta P_{xy} = -\frac{1}{2}(\delta\phi_x + \delta\phi_y) P_{xy} - \int \mathrm{d}^2z \sqrt{g}\, \delta\phi_z\, P_{xz} P_{zy}.} \tag{4.12}$$

following from (sum over i and j is implicit):

$$\begin{aligned}
\delta P(x,y) &= \delta\psi_{0,i}(x) \kappa^{ij} \psi_{0,j}(y)^\dagger + \psi_{0,i}(x) \kappa^{ij} \delta\psi_{0,j}(y)^\dagger + \psi_{0,i}(x) \delta\kappa^{ij} \psi_{0,j}(y)^\dagger \\
&= -\frac{1}{2}\, \psi_{0,i}(x) \kappa^{ij} \psi_{0,j}(y)^\dagger \big(\delta\phi(x) + \delta\phi(y)\big) \\
&\quad - \int \mathrm{d}^2z \sqrt{g}\, \delta\phi(z)\, \psi_{0,i}(x) \kappa^{ik} \psi_{0,k}(z)^\dagger \psi_{0,\ell}(z)\, \kappa^{\ell j} \psi_{0,j}(y)^\dagger \\
&= -\frac{1}{2}\big(\delta\phi(x) + \delta\phi(y)\big) P(x,y) - \int \mathrm{d}^2z \sqrt{g}\, \delta\phi(z)\, P(x,z) P(z,y).
\end{aligned}$$

Note the similarity with the variation of the Green function (4.21). We can check that it is compatible with the projector condition (3.84):

$$\begin{aligned}
\delta \int \mathrm{d}^2 w \sqrt{g}\, P_{xw} P_{wy} &= 2 \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_w\, P_{xw} P_{wy} + \int \mathrm{d}^2 w \sqrt{g}\, \delta P_{xw} P_{wy} + \int \mathrm{d}^2 w \sqrt{g}\, P_{xw} \delta P_{wy} \\
&= 2 \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_w\, P_{xw} P_{wy} \\
&\quad - \int \mathrm{d}^2 w \sqrt{g} \left(-\frac{1}{2} (\delta\phi_x + \delta\phi_w) P_{xw} - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, P_{xz} P_{zw} \right) P_{wy} \\
&\quad + \int \mathrm{d}^2 w \sqrt{g}\, P_{xw} \left(-\frac{1}{2} (\delta\phi_w + \delta\phi_y) P_{wy} - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, P_{wz} P_{zy} \right) \\
&= 2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, P_{xz} P_{zy} - \frac{1}{2} (\delta\phi_x + \delta\phi_y) P_{xy} \\
&\quad - \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_z P_{xz} P_{zy} - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \int \mathrm{d}^2 w \sqrt{g}\, P_{xz} P_{zw} P_{wy} \\
&\quad - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \int \mathrm{d}^2 w \sqrt{g}\, P_{xw} P_{wz} P_{zy} \\
&= - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, P_{xz} P_{zy} - \frac{1}{2} (\delta\phi_x + \delta\phi_y) P_{xy},
\end{aligned}$$

where we used (3.84) multiple times.

[HE: Find the finite transformation of the projector (see Appendix C for some attemps).] $\Leftarrow 4$

Example: scalar field The case of a massive scalar field X [35, 40] provides a simple example. Since the scalar Laplacian is Weyl invariant, see (A.51a), the single zero-mode is also invariant:

$$\hat{\psi}_0 = \psi_0. \tag{4.13}$$

However, if ψ_0 is normalized in the metric $g_{\mu\nu}$:

$$\kappa = \int \mathrm{d}^2 x \sqrt{g}\, \psi_0^2 = 1 \quad \Longrightarrow \quad \psi_0 = \frac{1}{\sqrt{A}}, \tag{4.14}$$

where A is the area of the surface with the metric $g_{\mu\nu}$, then it is not normalized in the metric $\hat{g}_{\mu\nu}$:

$$\hat{\kappa} = \int \mathrm{d}^2 x \sqrt{\hat{g}}\, \psi_0^2 = \frac{\hat{A}}{A}. \tag{4.15}$$

Another possibility is to use the zero-mode $\psi_0 = \hat{\psi}_0 = 1$ such that $\kappa = A$ and $\hat{\kappa} = \hat{A}$. This shows that it is necessary to keep track of the normalization of the zero-modes.[7] When there is a single zero-mode, one can trade expressions in terms of the zero-mode for expressions in terms of the area which has a simple variation, as was done in [35, 40]. However, this is not possible in the presence of multiple zero-modes.

4.2 Green functions

The infinitesimal variations of Green functions can be found by varying the Green equations (3.93) and solving for the variation. For $m = 0$, we could directly find the finite transformation since D transforms homogeneously under a Weyl transformation.

[7] A familiar context where the inner-product matrices of zero-modes appear is in the definition of the worldsheet functional integral [34].

4.2.1 D Green function

The variation of (3.93a) gives:

$$\boxed{D_x \delta S(x,y) = -(\delta\phi(x) + \delta\phi(y))\,\frac{\delta(x-y)}{\sqrt{g}} - (\delta D_x) S(x,y),} \tag{4.16}$$

such that convolution with the Green function yields

$$\boxed{\delta S(x,y) = -(\delta\phi(x) + \delta\phi(y)) S(x,y) - \int \mathrm{d}^2 z \sqrt{g}\, S(x,z)(\delta D_z) S(z,y).} \tag{4.17}$$

Note that we need to symmetrize $\delta\phi(x)$ in the first expression to get a symmetric expression in the second line. However, inserting the explicit expression of δD would in any case give a symmetric result at the end even without symmetrizing now.

Using the formula (A.50), one finds

$$\delta D = -\delta\phi\,(D - m\gamma_*) + \frac{\mathrm{i}}{2}\,(\partial\!\!\!/ \delta\phi)\,\psi, \tag{4.18}$$

following from:

$$\delta D = \mathrm{i}\,\delta \nabla\!\!\!\!/ = -\mathrm{i}\delta\phi\,\nabla\!\!\!\!/ + \frac{\mathrm{i}}{2}\,(\partial\!\!\!/ \delta\phi)\,\psi. \tag{4.19}$$

Upon acting on the Green functions, the variation can be simplified further by using the Green function definitions: (3.93):

$$(\delta D_x) S(x,y) = -\frac{1}{2}\,(\delta\phi(x) + \delta\phi(y))\,\frac{\delta(x-y)}{\sqrt{g}} + \left(m\gamma_*\,\delta\phi(x) + \frac{\mathrm{i}}{2}\,\partial\!\!\!/ \delta\phi(x)\right) S(x,y). \tag{4.20}$$

This can be plugged in the expressions of the variations (for simplicity the functional arguments are written as indices):

$$\begin{aligned}
\delta S_{xy} &= -(\delta\phi_x + \delta\phi_y) S_{xy} - \int \mathrm{d}^2 z \sqrt{g}\, S_{xz} (\delta D_z) S_{zy} \\
&= -(\delta\phi_x + \delta\phi_y) S_{xy} - \int \mathrm{d}^2 z \sqrt{g}\, S_{xz} \left[-\frac{1}{2}\,(\delta\phi_z + \delta\phi_y)\,\frac{\delta_{zy}}{\sqrt{g}} + \left(m\gamma_*\,\delta\phi_z + \frac{\mathrm{i}}{2}\,\partial\!\!\!/ \delta\phi_z\right) S_{zy} \right] \\
&= -\frac{1}{2}(\delta\phi_x + \delta\phi_y) S_{xy} - \int \mathrm{d}^2 z \sqrt{g}\, S_{xz} \left(m\gamma_* \delta\phi_z + \frac{\mathrm{i}}{2}\,(\partial\!\!\!/_z \delta\phi_z) \right) S_{zy} \\
&= -\frac{1}{2}(\delta\phi_x + \delta\phi_y) S_{xy} + \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \left(-m\, S_{xz}\gamma_* S_{zy} + \frac{\mathrm{i}}{2}\,(S_{xz} \overleftarrow{\nabla\!\!\!\!/}_z S_{zy} + S_{xz} \nabla\!\!\!\!/_z S_{zy}) \right) \\
&= -\frac{1}{2}(\delta\phi_x + \delta\phi_y) S_{xy} - m \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, S_{xz}\gamma_* S_{zy} \\
&\quad + \frac{1}{2} \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \left(-S_{xz}(-\mathrm{i}\overleftarrow{\nabla\!\!\!\!/}_z + m\gamma_*) S_{zy} + S_{xz}(\mathrm{i}\nabla\!\!\!\!/_z + m\gamma_*) S_{zy} \right) \\
&= -\frac{1}{2}(\delta\phi_x + \delta\phi_y) S_{xy} - m \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, S_{xz}\gamma_* S_{zy} \\
&\quad + \frac{1}{2} \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \cancel{\left(-\frac{\delta_{xz}}{\sqrt{g}} S_{zy} + S_{xz} \frac{\delta_{zy}}{\sqrt{g}} \right)},
\end{aligned}$$

where we used (3.93a) and (3.98) to get the last equality. Finally, we obtain

$$\boxed{\delta S_{xy} = -\frac{1}{2}(\delta\phi_x + \delta\phi_y) S_{xy} - m \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, S_{xz}\gamma_* S_{zy}.} \tag{4.21}$$

Let's look for the variation of the Green function without zero-modes:

$$
\begin{aligned}
\delta\widetilde{S}_{xy} &= \delta S_{xy} - \frac{\gamma_*}{m}\,\delta P_{xy} \\
&= \frac{1}{2m}(\delta\phi_x + \delta\phi_y)\gamma_* P_{xy} + \frac{1}{m}\int \mathrm{d}^2z\sqrt{g}\,\delta\phi(z)\,\gamma_* P_{xz}P_{zy} \\
&\qquad - \frac{1}{2}(\delta\phi_x + \delta\phi_y)S_{xy} - m\int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z\,S_{xz}\gamma_* S_{zy} \\
&= -\frac{1}{2}(\delta\phi_x + \delta\phi_y)\widetilde{S}_{xy} + \frac{1}{m}\int \mathrm{d}^2z\sqrt{g}\,\delta\phi(z)\,\gamma_* P_{xz}P_{zy} \\
&\qquad - m\int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z\left(\widetilde{S}_{xz} + \frac{\gamma_* P_{xz}}{m}\right)\gamma_*\left(\widetilde{S}_{zy} + \frac{\gamma_* P_{zy}}{m}\right) \\
&= -\frac{1}{2}(\delta\phi_x + \delta\phi_y)\widetilde{S}_{xy} - m\int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z\,\widetilde{S}_{xz}\gamma_*\widetilde{S}_{zy} \\
&\qquad - \int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z\,(\widetilde{S}_{xz}P_{zy} + \gamma_* P_{xz}\gamma_*\widetilde{S}_{zy})
\end{aligned}
$$

where we used (3.104a), then (4.12). We then get:

$$
\boxed{\begin{aligned}
\delta\widetilde{S}_{xy} = &-\frac{1}{2}(\delta\phi_x + \delta\phi_y)\widetilde{S}_{xy} - m\int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z\,\widetilde{S}_{xz}\gamma_*\widetilde{S}_{zy} \\
&- \int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z\,(\widetilde{S}_{xz}P_{zy} + P_{xz}\widetilde{S}_{zy}).
\end{aligned}} \tag{4.22}
$$

Let's look at the variation of the constraint (3.102) [16, app. B]:

$$
\begin{aligned}
0 &= \delta\int \mathrm{d}^2z\sqrt{g}\,P_{xz}\widetilde{S}_{zy} \\
&= 2\int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z P_{xz}\widetilde{S}_{zy} + \int \mathrm{d}^2z\sqrt{g}\,\delta P_{xz}\widetilde{S}_{zy} + \int \mathrm{d}^2z\sqrt{g}\,P_{xz}\delta\widetilde{S}_{zy} \\
&= 2\int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z P_{xz}\widetilde{S}_{zy} \\
&\quad + \int \mathrm{d}^2z\sqrt{g}\left[-\frac{1}{2}(\delta\phi_x + \delta\phi_z)P_{xz} - \int \mathrm{d}^2w\sqrt{g}\,\delta\phi_w\,P_{xw}P_{wz}\right]\widetilde{S}_{zy} \\
&\quad + \int \mathrm{d}^2z\sqrt{g}\,P_{xz}\Big[-\frac{1}{2}(\delta\phi_y + \delta\phi_z)\widetilde{S}_{zy} - m\int \mathrm{d}^2w\sqrt{g}\,\delta\phi_w\,\widetilde{S}_{zw}\gamma_*\widetilde{S}_{wy} \\
&\qquad\qquad - \int \mathrm{d}^2w\sqrt{g}\,\delta\phi_w\,(\widetilde{S}_{zw}P_{wy} + P_{zw}\widetilde{S}_{wy})\Big] \\
&= 2\int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z P_{xz}\widetilde{S}_{zy} - \frac{1}{2}(\delta\phi_x + \delta\phi_y)\cancel{\int \mathrm{d}^2z\sqrt{g}\,P_{xz}\widetilde{S}_{zy}} \\
&\quad - \int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z\,P_{xz}\widetilde{S}_{zy} - \int \mathrm{d}^2w\sqrt{g}\,\delta\phi_w\,P_{xw}\cancel{\int \mathrm{d}^2z\sqrt{g}\,P_{wz}\widetilde{S}_{zy}} \\
&\quad - \int \mathrm{d}^2w\sqrt{g}\,\delta\phi_w\cancel{\left[\int \mathrm{d}^2z\sqrt{g}\,P_{xz}\widetilde{S}_{zw}\right]}\left(m\,\gamma_*\widetilde{S}_{wy} + P_{wy}\right) \\
&\quad - \int \mathrm{d}^2w\sqrt{g}\,\delta\phi_w\left(\int \mathrm{d}^2z\sqrt{g}\,P_{xz}P_{zw}\right)\widetilde{S}_{wy} \\
&= 2\int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z P_{xz}\widetilde{S}_{zy} - \int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z\,P_{xz}\widetilde{S}_{zy} - \int \mathrm{d}^2w\sqrt{g}\,\delta\phi_w\,P_{xw}\widetilde{S}_{wy},
\end{aligned}
$$

where we have used (4.12) and (4.22), and then (3.102) to simplify. The result is trivially zero, however, it can be helpful to derive the variation of $G\psi(x)$ (convolution of G and ψ), where ψ is an arbitrary spinor [16, app. B]. This provides a good test that the variations are correct.

The finite transformation of $\widetilde{S}^{(0)}(x,y)$ reads [16, eq. (3.32)]:

$$\boxed{\begin{aligned}\widetilde{S}^{(0)}_{g,xy} &= \mathrm{e}^{-\frac{\phi_x}{2}}\widetilde{S}^{(0)}_{\hat{g},xy}\mathrm{e}^{-\frac{\phi_y}{2}} + \int \mathrm{d}^2w\sqrt{g}\int \mathrm{d}^2z\sqrt{g}\,P_{g,xw}\mathrm{e}^{-\frac{\phi_w}{2}}\widetilde{S}^{(0)}_{\hat{g},wz}\mathrm{e}^{-\frac{\phi_z}{2}}P_{g,zy}\\ &\quad - \int \mathrm{d}^2z\sqrt{g}\,P_{g,xz}\mathrm{e}^{-\frac{\phi_z}{2}}\widetilde{S}^{(0)}_{\hat{g},zy}\mathrm{e}^{-\frac{\phi_y}{2}} - \int \mathrm{d}^2z\sqrt{g}\,\mathrm{e}^{-\frac{\phi_x}{2}}\widetilde{S}^{(0)}_{\hat{g},xz}\mathrm{e}^{-\frac{\phi_z}{2}}P_{g,zy}.\end{aligned}} \tag{4.23}$$

This generalizes the formula found in [41–45] for the massless scalar. This is because the Dirac operator transforms covariantly under Weyl transformations. It can be rewritten as:

$$\boxed{\begin{aligned}\widetilde{S}^{(0)}_{g,xy} &= \mathrm{e}^{-\frac{1}{2}(\phi_x+\phi_y)}\widetilde{S}^{(0)}_{\hat{g},xy} + \int \mathrm{d}^2w\sqrt{\hat{g}}\int \mathrm{d}^2z\sqrt{\hat{g}}\,\mathrm{e}^{\frac{3}{2}(\phi_w+\phi_z)}P_{g,xw}\widetilde{S}^{(0)}_{\hat{g},wz}P_{g,zy}\\ &\quad - \mathrm{e}^{-\frac{\phi_y}{2}}\int \mathrm{d}^2z\sqrt{\hat{g}}\,\mathrm{e}^{\frac{3}{2}\phi_z}P_{g,xz}\widetilde{S}^{(0)}_{\hat{g},zy} - \mathrm{e}^{-\frac{\phi_x}{2}}\int \mathrm{d}^2z\sqrt{\hat{g}}\,\mathrm{e}^{\frac{3}{2}\phi_z}\widetilde{S}^{(0)}_{\hat{g},xz}P_{g,zy}.\end{aligned}} \tag{4.24}$$

Let's check that the infinitesimal form of (4.23) reproduces (4.22) with $m = 0$. The first term reads:

$$\mathrm{e}^{-\frac{\delta\phi_x}{2}}\widetilde{S}^{(0)}_{\hat{g},xy}\mathrm{e}^{-\frac{\delta\phi_y}{2}} \sim \widetilde{S}^{(0)}_{\hat{g},xy} - \frac{1}{2}(\delta\phi_x + \delta\phi_y)\widetilde{S}^{(0)}_{\hat{g},xy}.$$

This reproduces the first term. Then, the second term reads:

$$\begin{aligned}&\int \mathrm{d}^2w\sqrt{g}\int \mathrm{d}^2z\sqrt{g}\,P_{xw}\mathrm{e}^{-\frac{\delta\phi_w}{2}}\widetilde{S}^{(0)}_{\hat{g},wz}\mathrm{e}^{-\frac{\delta\phi_z}{2}}P_{zy}\\ &\qquad = \frac{3}{2}\int \mathrm{d}^2w\sqrt{\hat{g}}\int \mathrm{d}^2z\sqrt{\hat{g}}\,(\delta\phi_w + \delta\phi_z)\hat{P}_{xw}\widetilde{S}^{(0)}_{\hat{g},wz}\hat{P}_{zy}\\ &\qquad\quad - \frac{1}{2}\int \mathrm{d}^2w\sqrt{\hat{g}}\int \mathrm{d}^2z\sqrt{\hat{g}}\,(\delta\phi_x + \delta\phi_y + \delta\phi_w + \delta\phi_z)\hat{P}_{xw}\widetilde{S}^{(0)}_{\hat{g},wz}\hat{P}_{zy}\\ &\qquad\quad - \int \mathrm{d}^2w\sqrt{\hat{g}}\int \mathrm{d}^2z\sqrt{\hat{g}}\int \mathrm{d}^2z'\sqrt{\hat{g}}\,\delta\phi_{z'}\hat{P}_{xz'}P_{z'w}\widetilde{S}^{(0)}_{\hat{g},wz}\hat{P}_{zy}\\ &\qquad\quad - \int \mathrm{d}^2w\sqrt{\hat{g}}\int \mathrm{d}^2z\sqrt{\hat{g}}\int \mathrm{d}^2z'\sqrt{\hat{g}}\,\delta\phi_{z'}\hat{P}_{xw}\widetilde{S}^{(0)}_{\hat{g},wz}\hat{P}_{zy}\hat{P}_{zz'}P_{z'y}.\end{aligned}$$

where we used (4.12) to get the first equality. All terms vanish because they each contain an integral of $\widetilde{G}^{(0)}$ and P without $\delta\phi$ which vanish due to the orthogonality condition (3.102). We will see that this term is necessary for solving the Green function equation. The third term reads:

$$\begin{aligned}&\int \mathrm{d}^2z\sqrt{g}\,P_{xz}\mathrm{e}^{-\frac{\delta\phi_z}{2}}\widetilde{S}^{(0)}_{\hat{g},zy}\mathrm{e}^{-\frac{\delta\phi_y}{2}}\\ &\qquad = -\frac{1}{2}\int \mathrm{d}^2z\sqrt{\hat{g}}\,(\cancel{\delta\phi_x} - 3\delta\phi_z)\hat{P}_{xz}\widetilde{S}^{(0)}_{\hat{g},zy} - \frac{1}{2}\int \mathrm{d}^2z\sqrt{\hat{g}}\,(\cancel{\delta\phi_x} + \delta\phi_z)\hat{P}_{xz}\widetilde{S}^{(0)}_{\hat{g},zy}\\ &\qquad\quad - \cancel{\int \mathrm{d}^2z\sqrt{\hat{g}}\int \mathrm{d}^2w\sqrt{\hat{g}}\,\hat{P}_{xw}\hat{P}_{wz}\widetilde{S}^{(0)}_{\hat{g},zy}}\\ &\qquad = \int \mathrm{d}^2z\sqrt{\hat{g}}\,\delta\phi_z\hat{P}_{xz}\widetilde{S}^{(0)}_{\hat{g},zy}.\end{aligned}$$

Performing the same manipulations with the fourth term and summing all contributions together correctly reproduces (4.22).

Next, we want to check that $\widetilde{S}_g^{(0)}$ as given by (4.23) satisfies the massless Green equation (3.59) if $\widetilde{S}_{\hat{g}}^{(0)}$ solves the massless Green equation:

$$\begin{aligned} D_{g,x}^{(0)}\widetilde{S}_{g,xy}^{(0)} &= \mathrm{e}^{-\frac{3}{2}\phi_x}\, D_{\hat{g},x}^{(0)}\mathrm{e}^{\frac{\phi_x}{2}}\Big[\mathrm{e}^{-\frac{\phi_x}{2}}\,\widetilde{S}_{\hat{g},xy}^{(0)}\mathrm{e}^{-\frac{\phi_y}{2}} + \int \mathrm{d}^2w\sqrt{g}\int \mathrm{d}^2z\sqrt{g}\,P_{g,xw}\mathrm{e}^{-\frac{\phi_w}{2}}\,\widetilde{S}_{\hat{g},wz}^{(0)}\mathrm{e}^{-\frac{\phi_z}{2}}\,P_{g,zy} \\ &\qquad - \int \mathrm{d}^2z\sqrt{g}\,P_{g,xz}\mathrm{e}^{-\frac{\phi_z}{2}}\,\widetilde{S}_{\hat{g},zy}^{(0)}\mathrm{e}^{-\frac{\phi_y}{2}} - \int \mathrm{d}^2z\sqrt{g}\,\mathrm{e}^{-\frac{\phi_x}{2}}\,\widetilde{S}_{\hat{g},xz}^{(0)}\mathrm{e}^{-\frac{\phi_z}{2}}\,P_{g,zy}\Big] \\ &= \mathrm{e}^{-\frac{3}{2}\phi_x-\frac{\phi_y}{2}}\left(\frac{\delta_{xy}}{\sqrt{\hat{g}}} - P_{\hat{g},xy}\right) - \mathrm{e}^{-\frac{3}{2}\phi_x}\int \mathrm{d}^2z\sqrt{\hat{g}}\,\mathrm{e}^{2\phi_z}\left(\frac{\delta_{xz}}{\sqrt{\hat{g}}} - P_{\hat{g},xz}\right)\mathrm{e}^{-\frac{\phi_z}{2}}\,P_{g,zy} \\ &= \frac{\delta_{xy}}{\sqrt{g}} - P_{g,xy} - \mathrm{e}^{-\frac{3}{2}\phi_x-\frac{\phi_y}{2}}P_{\hat{g},xy} + \mathrm{e}^{-\frac{3}{2}\phi_x}\int \mathrm{d}^2z\sqrt{\hat{g}}\,\mathrm{e}^{\frac{3}{2}\phi_z}P_{\hat{g},xz}P_{g,zy}, \end{aligned}$$

where we used (A.50e). We have assumed that $P_{g,x\cdot} = \mathrm{e}^{-\frac{\phi_x}{2}}(\cdots)$, such that $D_{\hat{g},x}^{(0)}$ kills the second and third terms by acting only on $P_{\hat{g},x\cdot}$. We need the last two terms to cancel with each other:

$$P_{\hat{g},xy}\mathrm{e}^{-\frac{\phi_y}{2}} \stackrel{?}{=} \int \mathrm{d}^2z\sqrt{\hat{g}}\,\mathrm{e}^{\frac{3}{2}\phi_z}P_{\hat{g},xz}P_{g,zy}.$$

[HE: Prove this relation and show that the Green equation is satisfied.] ⇐ 5

Since we don't know the finite transformation of P, we can at least check if the relation above is compatible with its infinitesimal variation. Let's check if this relation is compatible with (4.12):

$$\begin{aligned} -\frac{\delta\phi_y}{2}\,P_{xy} &= \frac{3}{2}\int \mathrm{d}^2z\sqrt{g}\,\delta\phi_z P_{xz}P_{zy} \\ &\qquad - \int \mathrm{d}^2z\sqrt{g}\,P_{xz}\left[\frac{1}{2}(\delta\phi_z + \delta\phi_y)P_{zy} + \int \mathrm{d}^2w\sqrt{g}\,\delta\phi_w\,P_{zw}P_{wy}\right]. \end{aligned}$$

We see that the RHS correctly equals the LHS after using (3.84).

Finally, it remains to check that $\widetilde{S}_g^{(0)}$ as given by (4.23) is orthogonal to P if $\widetilde{S}_{\hat{g}}^{(0)}$

is orthogonal to $\hat{P}$:

$$
\begin{aligned}
&\int \mathrm{d}^2 w \sqrt{g}\, P_{g,xw}\, \widetilde{S}^{(0)}_{g,wy} \\
&\quad = \int \mathrm{d}^2 w \sqrt{g}\, P_{g,xw} \bigg[\mathrm{e}^{-\frac{1}{2}(\phi_w + \phi_y)} \widetilde{S}^{(0)}_{\hat{g},wy} + \int \mathrm{d}^2 z \sqrt{g} \int \mathrm{d}^2 z' \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}(\phi_z + \phi_{z'})} P_{g,wz} \widetilde{S}^{(0)}_{\hat{g},zz'} P_{g,z'y} \\
&\qquad\qquad - \mathrm{e}^{-\frac{\phi_y}{2}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{g,wz} \widetilde{S}^{(0)}_{\hat{g},zy} - \mathrm{e}^{-\frac{\phi_w}{2}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} \widetilde{S}^{(0)}_{\hat{g},wz} P_{g,zy} \bigg] \\
&\quad = \mathrm{e}^{-\frac{\phi_y}{2}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{g,xw} \widetilde{S}^{(0)}_{\hat{g},wy} \\
&\qquad + \int \mathrm{d}^2 w \sqrt{g} \int \mathrm{d}^2 z \sqrt{g}\, \mathrm{e}^{\frac{3}{2}\phi_z} \int \mathrm{d}^2 z' \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_{z'}} P_{g,xw} P_{g,wz} \widetilde{S}^{(0)}_{\hat{g},zz'} P_{g,z'y} \\
&\qquad - \mathrm{e}^{-\frac{\phi_y}{2}} \int \mathrm{d}^2 w \sqrt{g} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{g,xw} P_{g,wz} \widetilde{S}^{(0)}_{\hat{g},zy} \\
&\qquad - \int \mathrm{d}^2 w \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_w} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{g,xw} \widetilde{S}^{(0)}_{\hat{g},wz} P_{g,zy} \\
&\quad = \cancel{\mathrm{e}^{-\frac{\phi_y}{2}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{g,xw} \widetilde{S}^{(0)}_{\hat{g},wy}} + \int \mathrm{d}^2 z \sqrt{g}\, \mathrm{e}^{\frac{3}{2}\phi_z} \int \mathrm{d}^2 z' \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_{z'}} P_{g,xz} \widetilde{S}^{(0)}_{\hat{g},zz'} P_{g,z'y} \\
&\qquad \cancel{- \mathrm{e}^{-\frac{\phi_y}{2}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{g,xz} \widetilde{S}^{(0)}_{\hat{g},zy}} - \int \mathrm{d}^2 w \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_w} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{g,xw} \widetilde{S}^{(0)}_{\hat{g},wz} P_{g,zy} \\
&\quad = \int \mathrm{d}^2 z \sqrt{g}\, \mathrm{e}^{\frac{3}{2}\phi_z} \int \mathrm{d}^2 z' \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_{z'}} P_{g,xz} \widetilde{S}^{(0)}_{\hat{g},zz'} P_{g,z'y} \\
&\qquad - \int \mathrm{d}^2 w \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_w} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{g,xw} \widetilde{S}^{(0)}_{\hat{g},wz} P_{g,zy} \\
&\quad = 0,
\end{aligned}
$$

where we used (3.84) for the metric g to get the third equality. [HE: I am surprised that it was not necessary to use (4.23) for the metric $\hat{g}$. I think that it's because it was not necessary to write P_g in terms of $P_{\hat{g}}$.] ⇐ 6

Particular case: no zero-mode In the absence of zero-modes, the transformation (4.23) becomes directly:

$$
\boxed{\widetilde{S}^{(0)}_{g,xy} = \mathrm{e}^{-\frac{1}{2}(\phi_x + \phi_y)} \widetilde{S}^{(0)}_{\hat{g},xy}.} \tag{4.25}
$$

Particular case: torus, odd spin structure We have seen in Section 3.3.3 that there is a single zero-mode for the torus with odd spin structure. In this case, the projector reads:

$$
P(x, y) = \frac{1}{A}\, 1_2. \tag{4.26}
$$

[HE: There is something strange here because varying the previous equation does not seem to give (4.12).] This allows to simplify the finite transformation (4.23): ⇐ 7

$$
\boxed{
\begin{aligned}
\widetilde{S}^{(0)}_{g,xy} = \mathrm{e}^{-\frac{1}{2}(\phi_x + \phi_y)} \widetilde{S}^{(0)}_{\hat{g},xy} &+ \frac{1}{A^2} \int \mathrm{d}^2 w \sqrt{\hat{g}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}(\phi_w + \phi_z)} \widetilde{S}^{(0)}_{\hat{g},wz} \\
&- \frac{1}{A} \mathrm{e}^{-\frac{\phi_y}{2}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} \widetilde{S}^{(0)}_{\hat{g},zy} - \frac{1}{A} \mathrm{e}^{-\frac{\phi_x}{2}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} \widetilde{S}^{(0)}_{\hat{g},xz}.
\end{aligned}
} \tag{4.27}
$$

4.2.2 D^2 Green function

We follow the same method: the variation of (3.93b) is

$$\boxed{D_x^2 \delta G(x,y) = -(\delta\phi(x) + \delta\phi(y))\, \frac{\delta(x-y)}{\sqrt{g}} - (\delta D_x^2) G(x,y),} \tag{4.28}$$

and its solution reads:

$$\boxed{\delta G(x,y) = -(\delta\phi(x) + \delta\phi(y)) G(x,y) - \int \mathrm{d}^2 z \sqrt{g}\, G(x,z)(\delta D_z^2) G(z,y).} \tag{4.29}$$

We have

$$\boxed{\delta D^2 = -2\delta\phi\, (D^2 - m^2) - (\partial^\mu \delta\phi)\, \nabla_\mu + (\not\partial \delta\phi) \not\nabla - \frac{1}{2}\, \Delta\, \delta\phi,} \tag{4.30}$$

which follows from using (A.51):

$$\delta D^2 = -\delta \not\nabla^2 = 2\delta\phi\, \not\nabla^2 - (\partial^\mu \delta\phi)\, \nabla_\mu + (\not\partial \delta\phi) \not\nabla - \frac{1}{2}\, \Delta\, \delta\phi.$$

Action on the Green function, we get, after using (3.93b):

$$\begin{aligned}(\delta D_x^2) G(x,y) = &- (\delta\phi(x) + \delta\phi(y))\, \frac{\delta(x-y)}{\sqrt{g}} \\ &+ \left(2m^2\, \delta\phi - (\partial^\mu \delta\phi)\, \nabla_\mu + (\not\partial \delta\phi) \not\nabla - \frac{1}{2}\, \Delta\, \delta\phi \right) G(x,y).\end{aligned} \tag{4.31}$$

We can now compute the variation of the Green function:

$$\begin{aligned}
\delta G_{xy} &= -(\delta\phi_x + \delta\phi_y) G_{xy} - \int \mathrm{d}^2 z \sqrt{g}\, G_{xz} (\delta D_z^2) G_{zy} \\
&= - \int \mathrm{d}^2 z \sqrt{g}\, G_{xz} \left(2m^2\, \delta\phi_z - (\partial_z^\mu \delta\phi_z)\, \nabla_{z\mu} + (\not\partial_z \delta\phi_z) \not\nabla_z - \frac{1}{2}\, \Delta_z\, \delta\phi_z \right) G_{zy} \\
&= - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \Big[2m^2\, G_{xz} G_{zy} + \cancel{G_{xz} \overleftarrow{\nabla}_{z\mu} \nabla_{z\mu} G_{zy}} + G_{xz}\, \Delta_z\, G_{zy} \\
&\qquad - G_{xz} \overleftarrow{\not\nabla}_z \not\nabla_z G_{zy} - G_{xz} \not\nabla_z^2 G_{zy} - \frac{1}{2}\, G_{xz} \overleftarrow{\Delta}_z G_{zy} \\
&\qquad - \cancel{G_{xz} \overleftarrow{\nabla}_{z\mu} \nabla_z^\mu G_{zy}} - \frac{1}{2}\, G_{xz}\, \Delta_z\, G_{zy} \Big] \\
&= - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \Big[2m^2\, G_{xz} G_{zy} + \frac{1}{2}\, G_{xz}\, \Delta_z\, G_{zy} - \frac{1}{2}\, G_{xz} \overleftarrow{\Delta}_z G_{zy} \\
&\qquad - G_{xz} \overleftarrow{\not\nabla}_z \not\nabla_z G_{zy} - G_{xz} \not\nabla_z^2 G_{zy} \Big] \\
&= - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \Big[2m^2\, G_{xz} G_{zy} + \frac{1}{2}\, G_{xz} \not\nabla_z^2 G_{zy} - \frac{1}{2}\, G_{xz} \overleftarrow{\not\nabla}{}_z^2 G_{zy} \\
&\qquad - G_{xz} \overleftarrow{\not\nabla}_z \not\nabla_z G_{zy} - G_{xz} \not\nabla_z^2 G_{zy} \Big] \\
&= - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \Big[m^2\, G_{xz} G_{zy} + \frac{1}{2}\, G_{xz} D_z^2 G_{zy} + \frac{1}{2}\, G_{xz} \overleftarrow{D}{}_z^2 G_{zy} - G_{xz} \overleftarrow{\not\nabla}_z \not\nabla_z G_{zy} \Big] \\
&= - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \Big[m^2\, G_{xz} G_{zy} + \frac{1}{2}\, G_{xz} \frac{\delta_{zy}}{\sqrt{g}} + \frac{1}{2}\, \frac{\delta_{xz}}{\sqrt{g}} G_{zy} - G_{xz} \overleftarrow{\not\nabla}_z \not\nabla_z G_{zy} \Big],
\end{aligned}$$

where we used $\nabla\!\!\!/^2 = \Delta - R/4$ and $D^2 = -\nabla\!\!\!/^2 + m^2$, and for the last equality (3.93b). Finally, we find:

$$\boxed{\delta G_{xy} = -\frac{1}{2}(\delta\phi_x + \delta\phi_y) G_{xy} - m^2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, G_{xz} G_{zy} + \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, G_{xz} \overleftarrow{\nabla\!\!\!/}_z \nabla\!\!\!/_z G_{zy}.} \tag{4.32}$$

Unfortunately, it is not possible to simplify the third term. When we introduce the ζ-function regularization in Section 5, we will see that the trace over Dirac indices and spacetime positions has a simple expression, see (5.61).

It may be useful to introduce the operator $D = \mathrm{i}\nabla\!\!\!/ + m\gamma_*$ in the last term since they act in a simple way on the modes:[8]

$$\boxed{\begin{aligned}\delta G_{xy} = &-\frac{1}{2}(\delta\phi_x + \delta\phi_y) G_{xy} - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, G_{xz} \overleftarrow{D}_z D_z G_{zy} \\ &- m \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, G_{xz} (\overleftarrow{D}_z \gamma_* + \gamma_* D_z) G_{zy}\end{aligned}} \tag{4.33}$$

following from

$$\begin{aligned}\delta G_{xy} &= -\frac{1}{2}(\delta\phi_x + \delta\phi_y) G_{xy} - m^2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, G_{xz} G_{zy} \\ &\quad + \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, G_{xz} (-\mathrm{i}\overleftarrow{\nabla\!\!\!/}_z)(\mathrm{i}\nabla\!\!\!/_z) G_{zy}. \\ &= -\frac{1}{2}(\delta\phi_x + \delta\phi_y) G_{xy} - m^2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, G_{xz} G_{zy} \\ &\quad + \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, G_{xz} (\overleftarrow{D}_z - m\gamma_*)(D_z - m\gamma_*) G_{zy}\end{aligned}$$

Using the relation (3.95), one can write:

$$\boxed{\begin{aligned}\delta G_{xy} = &-\frac{1}{2}(\delta\phi_x + \delta\phi_y) G_{xy} + \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, S_{xz} S_{zy} \\ &- m \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, (S_{xz}\gamma_* G_{zy} + G_{xz}\gamma_* S_{zy})\end{aligned}} \tag{4.34}$$

We can also find this expression by computing the variation of (3.96).

We also need to compute the variation of the Green function without zero-modes (3.104b) in terms of itself. Writing (3.104b) and using (4.12) and (4.32), we have:

$$\begin{aligned}\delta \widetilde{G}_{xy} &= \delta G_{xy} - \frac{\delta P_{xy}}{m^2} \\ &= \frac{1}{m^2}\left(\frac{1}{2}(\delta\phi_x + \delta\phi_y) P_{xy} + \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, P_{xz} P_{zy}\right) \\ &\quad - \frac{1}{2}(\delta\phi_x + \delta\phi_y)\left(\widetilde{G}_{xy} + \frac{P_{xy}}{m^2}\right) - m^2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \left(\widetilde{G}_{xz} + \frac{P_{xz}}{m^2}\right)\left(\widetilde{G}_{zy} + \frac{P_{zy}}{m^2}\right) \\ &\quad + \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z \left(\widetilde{G}_{xz} + \frac{P_{xz}}{m^2}\right) \overleftarrow{\nabla\!\!\!/}_z \nabla\!\!\!/_z \left(\widetilde{G}_{zy} + \frac{P_{zy}}{m^2}\right),\end{aligned}$$

[8] However, we will see that it is simpler to manipulate (4.32) because $\{\nabla\!\!\!/ \Psi_n\}$ forms a complete basis of eigenmodes.

Using that $\not{\nabla}_x P(x,y) = 0$ since P is built from zero-modes and simplifying, we get:

$$\boxed{\begin{aligned}\delta\widetilde{G}_{xy} = &-\frac{1}{2}(\delta\phi_x + \delta\phi_y)\widetilde{G}_{xy} - m^2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, \widetilde{G}_{xz}\widetilde{G}_{zy} \\ &- \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, \left(\widetilde{G}_{xz} P_{zy} + P_{xz}\widetilde{G}_{zy}\right) + \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, \widetilde{G}_{xz} \overleftarrow{\not{\nabla}}_z \not{\nabla}_z \widetilde{G}_{zy}.\end{aligned}} \tag{4.35}$$

In the case of the scalar field, it was found useful to replace $\delta\phi$ by δK in the third term [35, 40]. Inserting (A.57), we have:

$$\begin{aligned}\int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, \widetilde{G}_{xz} P_{zy} &= \int \mathrm{d}^2 z \sqrt{g}\, \left(\frac{\delta A}{2A} + \frac{A}{4}\,\Delta\,\delta K\right) \widetilde{G}_{xz} P_{zy} \\ &= \frac{\delta A}{2A} \cancel{\int \mathrm{d}^2 z \sqrt{g}\, \widetilde{G}_{xz} P_{zy}} + \frac{A}{4} \int \mathrm{d}^2 z \sqrt{g}\, \delta K_z (\Delta_z \widetilde{G}_{xz}) P_{zy} \\ &= \frac{A}{4} \int \mathrm{d}^2 z \sqrt{g}\, \delta K_z \left(-D_z^2 + \frac{R_z}{4} + m^2\right) \widetilde{G}_{xz} P_{zy} \\ &= -\frac{A}{4}\, \delta K_x\, P_{xy} + \frac{A}{4} \int \mathrm{d}^2 z \sqrt{g}\, \delta K_z\, P_{xz} P_{zy} \\ &\qquad + \frac{A}{16} \int \mathrm{d}^2 z \sqrt{g}\, \delta K_z\, R_z\, \widetilde{G}_{xz}\, P_{zy} + \frac{m^2 A}{4} \int \mathrm{d}^2 z \sqrt{g}\, \delta K_z\, \widetilde{G}_{xz} P_{zy},\end{aligned}$$

since P and $\widetilde{G}$ are orthogonal (3.102), using Green's second identity (A.16). In total, we have:

$$\boxed{\begin{aligned}\delta\widetilde{G}_{xy} = &-\frac{1}{2}(\delta\phi_x + \delta\phi_y)\widetilde{G}_{xy} + \frac{A}{4}\,(\delta K_x + \delta K_y)\, P_{xy} - \frac{A}{2} \int \mathrm{d}^2 z \sqrt{g}\, \delta K_z\, P_{xz} P_{zy} \\ &- \frac{A}{16} \int \mathrm{d}^2 z \sqrt{g}\, \delta K_z\, R_z (\widetilde{G}_{xz}\, P_{zy} + P_{xz}\, \widetilde{G}_{zy}) \\ &- m^2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, \widetilde{G}_{xz}\widetilde{G}_{zy} - \frac{m^2 A}{4} \int \mathrm{d}^2 z \sqrt{g}\, \delta K_z (\widetilde{G}_{xz} P_{zy} + P_{xz}\widetilde{G}_{zy}) \\ &+ \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, \widetilde{G}_{xz} \overleftarrow{\not{\nabla}}_z \not{\nabla}_z \widetilde{G}_{zy}.\end{aligned}} \tag{4.36}$$

[HE: We cannot proceed like for the scalar case because the projector on zero-modes is not constant.] $\Leftarrow$ 8

Let's look at the variation of the constraint (3.102) [16, app. B]:

$$
\begin{aligned}
0 &= \delta \int \mathrm{d}^2 z \sqrt{g}\, P_{xz} \widetilde{G}_{zy} \\
&= 2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z P_{xz} \widetilde{G}_{zy} + \int \mathrm{d}^2 z \sqrt{g}\, \delta P_{xz} \widetilde{G}_{zy} + \int \mathrm{d}^2 z \sqrt{g}\, P_{xz} \delta \widetilde{G}_{zy} \\
&= 2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z P_{xz} \widetilde{G}_{zy} \\
&\quad + \int \mathrm{d}^2 z \sqrt{g} \left[-\frac{1}{2} (\cancel{\delta\phi_x} + \delta\phi_z) P_{xz} - \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_w\, P_{xw} P_{wz} \right] \widetilde{G}_{zy} \\
&\quad + \int \mathrm{d}^2 z \sqrt{g}\, P_{xz} \bigg[-\frac{1}{2} (\delta\phi_z + \cancel{\delta\phi_y}) \widetilde{G}_{zy} - m^2 \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_w\, \widetilde{G}_{zw} \widetilde{G}_{wy} \\
&\qquad\qquad - \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_w \left(\widetilde{G}_{zw} P_{wy} + P_{zw} \widetilde{G}_{wy} \right) \\
&\qquad\qquad + \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_w\, \widetilde{G}_{zw} \overleftarrow{\not{\nabla}}_w \not{\nabla}_w \widetilde{G}_{wy} \bigg] \\
&= 2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z P_{xz} \widetilde{G}_{zy} - \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, P_{xz} \widetilde{G}_{zy} \\
&\quad - \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_w\, P_{xw} \left(\cancel{\int \mathrm{d}^2 z \sqrt{g}\, P_{wz} \widetilde{G}_{zy}} \right) \\
&\quad + \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_w \left(\cancel{\int \mathrm{d}^2 z \sqrt{g}\, P_{xz} \widetilde{G}_{zw}} \right) \left(- m^2 \widetilde{G}_{wy} - P_{wy} + \overleftarrow{\not{\nabla}}_w \not{\nabla}_w \widetilde{G}_{wy} \right) \\
&\quad - \int \mathrm{d}^2 w \sqrt{g}\, \delta\phi_w \left(\int \mathrm{d}^2 z \sqrt{g}\, P_{xz} P_{zw} \right) \widetilde{G}_{wy}
\end{aligned}
$$

where we have used (4.12) and (4.22), and then (3.102) to simplify. The RHS vanishes identically.

Finally, we can get the finite transformation when $m = 0$ from (3.105) and (4.23):

$$
\begin{aligned}
\widetilde{G}^{(0)}_{g,xy} &= \int \mathrm{d}^2 \zeta \sqrt{g}\, \widetilde{S}^{(0)}_{g,x\zeta} \widetilde{S}^{(0)}_{g,\zeta y} \\
&= \int \mathrm{d}^2 \zeta \sqrt{g}\, \widetilde{S}^{(0)}_{g,x\zeta} \bigg[\mathrm{e}^{-\frac{1}{2}(\phi_\zeta + \phi_y)} \widetilde{S}^{(0)}_{\hat{g},\zeta y} + \cancel{\int \mathrm{d}^2 w \sqrt{g} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}(\phi_w + \phi_z)} P_{g,\zeta w} \widetilde{S}^{(0)}_{\hat{g},wz}} P_{g,zy} \\
&\qquad\qquad - \mathrm{e}^{-\frac{\phi_y}{2}} \cancel{\int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{g,\zeta z} \widetilde{S}^{(0)}_{\hat{g},zy}} - \mathrm{e}^{-\frac{\phi_\zeta}{2}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} \widetilde{S}^{(0)}_{\hat{g},\zeta z} P_{g,zy} \bigg] \\
&= \int \mathrm{d}^2 \zeta \sqrt{g}\, \widetilde{S}^{(0)}_{g,x\zeta} \left[\mathrm{e}^{-\frac{1}{2}(\phi_\zeta + \phi_y)} \widetilde{S}^{(0)}_{\hat{g},\zeta y} - \mathrm{e}^{-\frac{\phi_\zeta}{2}} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\frac{3}{2}\phi_z} \widetilde{S}^{(0)}_{\hat{g},\zeta z} P_{g,zy} \right].
\end{aligned}
$$

To get the second equality, we have replace only one Green function with (4.23), and used the orthogonality condition of $\widetilde{S}$ and P in the metric g. [HE: Simplify after finding the finite transformation of the projector.] ⇐ 9

Particular case: no zero-mode If $N_0 = 0$, we have $P(x, y) = 0$ such that:

$$
\boxed{\delta \widetilde{G}_{xy} = -\frac{1}{2} (\delta\phi_x + \delta\phi_y) \widetilde{G}_{xy} - m^2 \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, \widetilde{G}_{xz} \widetilde{G}_{zy} + \int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, \widetilde{G}_{xz} \overleftarrow{\not{\nabla}}_z \not{\nabla}_z \widetilde{G}_{zy}.} \tag{4.37}
$$

We can also obtain the finite transformation from (4.25):

$$
\boxed{\widetilde{G}^{(0)}_{g,xy} = \mathrm{e}^{-\frac{1}{2}(\phi_x + \phi_y)} \int \mathrm{d}^2 z \sqrt{\hat{g}}\, \mathrm{e}^{\phi_z} \widetilde{S}^{(0)}_{\hat{g},xz} \widetilde{S}^{(0)}_{\hat{g},zy}.} \tag{4.38}
$$

Particular case: torus, odd spin structure For the torus with one zero-mode, using $P(x,y) = 1_2/A$, (4.36) is:

$$\boxed{\begin{aligned}\delta\widetilde{G}_{xy} = &-\frac{1}{2}(\delta\phi_x + \delta\phi_y)\widetilde{G}_{xy} + \frac{1}{4}(\delta K_x + \delta K_y) - \frac{1}{2A}\int \mathrm{d}^2 z\sqrt{g}\,\delta K_z \\ &- \frac{1}{16}\int \mathrm{d}^2 z\sqrt{g}\,\delta K_z\, R_z(\widetilde{G}_{xz} + \widetilde{G}_{zy}) \\ &- m^2\int \mathrm{d}^2 z\sqrt{g}\,\delta\phi_z\,\widetilde{G}_{xz}\widetilde{G}_{zy} - \frac{m^2}{4A}\int \mathrm{d}^2 z\sqrt{g}\,\delta K_z(\widetilde{G}_{xz} + \widetilde{G}_{zy}) \\ &+ \int \mathrm{d}^2 z\sqrt{g}\,\delta\phi_z\,\widetilde{G}_{xz}\overleftarrow{\nabla\!\!\!\!/}_z\nabla\!\!\!\!/_z\widetilde{G}_{zy}.\end{aligned}} \tag{4.39}$$

The form of $\widetilde{G}^{(0)}_{g,xy}$ can be simplified in this case:

$$\begin{aligned}\widetilde{G}^{(0)}_{g,xy} = \mathrm{e}^{-\frac{1}{2}(\phi_x+\phi_y)}\int \mathrm{d}^2 z\sqrt{g}\,\mathrm{e}^{-\phi_z}\,\widetilde{S}^{(0)}_{\hat{g},xz}\widetilde{S}^{(0)}_{\hat{g},zy} \\ - \frac{1}{A}\int \mathrm{d}^2\zeta\sqrt{\hat{g}}\int \mathrm{d}^2 z\sqrt{\hat{g}}\,\mathrm{e}^{\frac{3}{2}(\phi_z+\phi_\zeta)}\,\widetilde{S}^{(0)}_{g,x\zeta}\widetilde{S}^{(0)}_{\hat{g},\zeta z}.\end{aligned} \tag{4.40}$$

4.3 Geodesic length

The infinitesimal variation of the geodesic length reads [40]:

$$\delta\ell(x,y)^2 = \ell(x,y)^2\,\big(\delta\phi(x) + \delta\phi(y)\big) + O(\ell^4). \tag{4.41}$$

This can be rewritten as:

$$\delta\ln\big(\mu^2\ell(x,y)^2\big) = \delta\phi(x) + \delta\phi(y) + O(\ell^2). \tag{4.42}$$

[HE: prove these relations] ⇐ 10

4.4 Eigenvalues and eigenmodes

The variation of the eigenvalues and eigenmodes under the perturbation (4.2) can be found by applying perturbation theory [35, sec. 3.2.2, 40, sec. 2.3] (see [46, sec. 5.1] for a review). In the rest of this subsection, we assume that all eigenvalues are independent to avoid the difficulties arising in degenerate perturbation theory: the validity of the formulas in the general case follows by continuity [35, p. 15].

First, we note that the normalization condition (3.52) is not preserved and gives the relation:

$$\boxed{\langle\delta\Psi_m|\Psi_n\rangle + \langle\Psi_m|\delta\Psi_n\rangle = -2\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\,.} \tag{4.43}$$

This gives immediately the contribution to the variation $\delta\Psi_n$ which is proportional to Ψ_n:

$$\langle\Psi_n|\delta\Psi_n\rangle = -\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\,. \tag{4.44}$$

Note that, in usual perturbation theory in quantum mechanics, the RHS is zero. Multiplying with $\langle\Psi_n|$ on the right and summing over n gives (after renaming $n \leftrightarrow m$):

$$\langle\delta\Psi_n| = -2\langle\Psi_n|\,\delta\phi - \sum_{n\in\mathbb{Z}}\langle\Psi_n|\delta\Psi_m\rangle\langle\Psi_m|\,. \tag{4.45}$$

Note that it includes the zero-modes.

The variation of the eigenvalue equation (3.51b) is:

$$\delta D^2 \left|\Psi_n\right\rangle + D^2 \delta \left|\Psi_n\right\rangle = \delta\Lambda_n \left|\Psi_n\right\rangle + \Lambda_n\, \delta \left|\Psi_n\right\rangle . \tag{4.46}$$

Applying $\left\langle\Psi_n\right|$ yields the equation:

$$\boxed{\delta\Lambda_n = \left\langle\Psi_n\right| \delta D^2 \left|\Psi_n\right\rangle .} \tag{4.47}$$

Next, multiplying on the left by $\left\langle\Psi_m\right|$:

$$\begin{aligned}\left\langle\Psi_m\right| D^2 \left|\delta\Psi_n\right\rangle - \Lambda_n \left\langle\Psi_m|\delta\Psi_n\right\rangle &= \delta\Lambda_n \left\langle\Psi_m|\Psi_n\right\rangle - \left\langle\Psi_m\right| \delta D^2 \left|\Psi_n\right\rangle \\ (\Lambda_m - \Lambda_n) \left\langle\Psi_m|\delta\Psi_n\right\rangle &= \delta\Lambda_n \delta_{mn} - \left\langle\Psi_m\right| \delta D^2 \left|\Psi_n\right\rangle .\end{aligned}$$

Note that this is trivial if $m = n$. The next step is to multiply with $\left|\Psi_m\right\rangle$, sum over $m \neq n$, add $\left|\Psi_n\right\rangle \left\langle\Psi_n|\delta\Psi_n\right\rangle$ on both sides and use (4.44) in the RHS:

$$\boxed{\delta \left|\Psi_n\right\rangle = -\left\langle\Psi_n\right| \delta\phi \left|\Psi_n\right\rangle \left|\Psi_n\right\rangle + \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{1}{\Lambda_n - \Lambda_m} \left\langle\Psi_m\right| \delta D^2 \left|\Psi_n\right\rangle \left|\Psi_m\right\rangle .} \tag{4.48}$$

Using the expression (4.30) for δD^2 gives:

$$\begin{aligned}\delta\Lambda_n =& -2(\Lambda_n - m^2)\left\langle\Psi_n\right| \delta\phi \left|\Psi_n\right\rangle + \left\langle\Psi_n\right| (\partial\!\!\!/ \delta\phi) \nabla\!\!\!\!/ \left|\Psi_n\right\rangle \\ & - \left\langle\Psi_n\right| (\partial^\mu \delta\phi)\, \nabla_\mu \left|\Psi_n\right\rangle - \frac{1}{2}\left\langle\Psi_n\right| \Delta\, \delta\phi \left|\Psi_n\right\rangle ,\end{aligned} \tag{4.49a}$$

$$\begin{aligned}\delta \left|\Psi_n\right\rangle =& -\left\langle\Psi_n\right| \delta\phi \left|\Psi_n\right\rangle \left|\Psi_n\right\rangle - 2 \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{\Lambda_n - m^2}{\Lambda_n - \Lambda_m} \left\langle\Psi_m\right| \delta\phi \left|\Psi_n\right\rangle \left|\Psi_m\right\rangle \\ & - \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{1}{\Lambda_n - \Lambda_m} \Bigg[-\left\langle\Psi_m\right| (\partial\!\!\!/ \delta\phi) \nabla\!\!\!\!/ \left|\Psi_n\right\rangle + \left\langle\Psi_m\right| (\partial^\mu \delta\phi)\, \nabla_\mu \left|\Psi_n\right\rangle \\ & + \frac{1}{2} \left\langle\Psi_m\right| \Delta\, \delta\phi \left|\Psi_n\right\rangle \Bigg] \left|\Psi_m\right\rangle .\end{aligned} \tag{4.49b}$$

We can compute each bracket independently. The first term in the parenthesis in the second line is:

$$\begin{aligned}\left\langle\Psi_m\right| (\partial\!\!\!/ \delta\phi) \nabla\!\!\!\!/ \left|\Psi_n\right\rangle &= \int \mathrm{d}^2x \sqrt{g}\, \Psi_m(x)^\dagger (\partial\!\!\!/ \delta\phi) \nabla\!\!\!\!/ \Psi_n(x) \\ &= -\int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x) \Big[\Psi_m(x)^\dagger \nabla\!\!\!\!/^2 \Psi_n(x) + \Psi_m(x)^\dagger \overleftarrow{\nabla\!\!\!\!/} \nabla\!\!\!\!/ \Psi_n(x) \Big] \\ &= -\int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x) \Psi_m(x)^\dagger \left(-D^2 + m^2 \right) \Psi_n(x) \\ &\quad - \int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x) \Psi_m(x)^\dagger (\overleftarrow{D} - m\gamma_*)(D - m\gamma_*) \Psi_n(x) \\ &= (\Lambda_n - m^2)\left\langle\Psi_m\right| \delta\phi \left|\Psi_n\right\rangle - (\lambda_m\lambda_n + m^2)\left\langle\Psi_m\right| \delta\phi \left|\Psi_n\right\rangle \\ &\quad + m(\lambda_m + \lambda_n)\left\langle\Psi_m\right| \delta\phi\, \gamma_* \left|\Psi_n\right\rangle .\end{aligned}$$

After simplification, we get:

$$\begin{aligned}\left\langle\Psi_m\right| (\partial\!\!\!/ \delta\phi) \nabla\!\!\!\!/ \left|\Psi_n\right\rangle &= (\Lambda_n - m^2)\left\langle\Psi_m\right| \delta\phi \left|\Psi_n\right\rangle - \left\langle\Psi_m\right| \overleftarrow{\nabla\!\!\!\!/}\, \delta\phi\, \nabla\!\!\!\!/ \left|\Psi_n\right\rangle \\ &= \left[(\Lambda_n - m^2) + (\lambda_m\lambda_n + m^2) \right] \left\langle\Psi_m\right| \delta\phi \left|\Psi_n\right\rangle \\ &\quad + m(\lambda_m + \lambda_n)\left\langle\Psi_m\right| \delta\phi\, \gamma_* \left|\Psi_n\right\rangle .\end{aligned} \tag{4.50}$$

The second term is:

$$\begin{aligned}
\langle\Psi_m|\,(\partial^\mu\delta\phi)\,\nabla_\mu\,|\Psi_n\rangle &= \int \mathrm{d}^2x\sqrt{g}\,\Psi_m(x)^\dagger(\partial^\mu\delta\phi)\,\nabla_\mu\,\Psi_n(x)\\
&= -\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\Big[\Psi_m(x)^\dagger\Delta\Psi_n(x) + \Psi_m(x)^\dagger\overleftarrow{\nabla}^\mu\,\nabla_\mu\,\Psi_n(x)\Big]\\
&= -\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\Psi_m(x)^\dagger\left(-D^2+m^2+\frac{R}{4}\right)\Psi_n(x)\\
&\quad -\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\Psi_m(x)^\dagger\overleftarrow{\nabla}^\mu\,\nabla_\mu\,\Psi_n(x).
\end{aligned}$$

After simplification, we get:

$$\langle\Psi_m|\,(\partial^\mu\delta\phi)\,\nabla_\mu\,|\Psi_n\rangle = \left(\Lambda_n - m^2 - \frac{R}{4}\right)\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle - \langle\Psi_m|\,\overleftarrow{\nabla}^\mu\delta\phi\;\nabla_\mu\,|\Psi_n\rangle\,. \tag{4.51}$$

Finally, the third term is:

$$\begin{aligned}
\langle\Psi_m|\,\Delta\,\delta\phi\,|\Psi_n\rangle &= \int \mathrm{d}^2x\sqrt{g}\,\Psi_m(x)^\dagger(\Delta\,\delta\phi)\not{\nabla}\Psi_n(x)\\
&= \int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\Big[\Psi_m(x)^\dagger\Delta\Psi_n(x) + \Psi_m(x)^\dagger\overleftarrow{\Delta}\,\Psi_n(x)\\
&\qquad + 2\,\Psi_m(x)^\dagger\overleftarrow{\nabla}^\mu\,\nabla_\mu\,\Psi_n(x)\Big]\\
&= \int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\Big[\Psi_m(x)^\dagger\left(-D^2+m^2+\frac{R}{4}\right)\Psi_n(x)\\
&\qquad + \Psi_m(x)^\dagger\left(-\overleftarrow{D}^2+m^2+\frac{R}{4}\right)\Psi_n(x)\Big]\\
&\quad + 2\,\langle\Psi_m|\,\overleftarrow{\nabla}^\mu\delta\phi\;\nabla_\mu\,|\Psi_n\rangle\,.
\end{aligned}$$

After simplification, we get:

$$\langle\Psi_m|\,\Delta\,\delta\phi\,|\Psi_n\rangle = \left(-\Lambda_m - \Lambda_n + 2m^2 + \frac{R}{2}\right)\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle + 2\langle\Psi_m|\,\overleftarrow{\nabla}^\mu\delta\phi\;\nabla_\mu\,|\Psi_n\rangle\,. \tag{4.52}$$

Combining all three terms together, we get:

$$\begin{aligned}
&\langle\Psi_m|\,(\not{\partial}\delta\phi)\not{\nabla}\,|\Psi_n\rangle - \langle\Psi_m|\,(\partial^\mu\delta\phi)\,\nabla_\mu\,|\Psi_n\rangle - \frac{1}{2}\,\langle\Psi_m|\,\Delta\,\delta\phi\,|\Psi_n\rangle\\
&= \cancel{(\Lambda_n - m^2)\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle} - \langle\Psi_m|\,\overleftarrow{\not{\nabla}}\,\delta\phi\not{\nabla}\,|\Psi_n\rangle - \left(\cancel{\Lambda_n - m^2} - \frac{\cancel{R}}{\cancel{4}}\right)\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\\
&\quad + \cancel{\langle\Psi_m|\,\overleftarrow{\nabla}^\mu\delta\phi\;\nabla_\mu\,|\Psi_n\rangle} - \frac{1}{2}\left(-\Lambda_m - \Lambda_n + 2m^2 + \frac{\cancel{R}}{\cancel{2}}\right)\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\\
&\quad - \cancel{\langle\Psi_m|\,\overleftarrow{\nabla}^\mu\delta\phi\;\nabla_\mu\,|\Psi_n\rangle},
\end{aligned}$$

which yields finally:

$$\begin{aligned}
&\langle\Psi_m|\,(\not{\partial}\delta\phi)\not{\nabla}\,|\Psi_n\rangle - \langle\Psi_m|\,(\partial^\mu\delta\phi)\,\nabla_\mu\,|\Psi_n\rangle - \frac{1}{2}\,\langle\Psi_m|\,\Delta\,\delta\phi\,|\Psi_n\rangle\\
&\qquad = \left(\frac{\Lambda_m+\Lambda_n}{2} - m^2\right)\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle - \langle\Psi_m|\,\overleftarrow{\not{\nabla}}\,\delta\phi\not{\nabla}\,|\Psi_n\rangle\,.
\end{aligned} \tag{4.53}$$

This allows to simplify the variation of the eigenvalues and eigenmodes:

$$\boxed{\delta\Lambda_n = -(\Lambda_n - m^2)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle - \langle\Psi_n|\,\overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\,,} \tag{4.54a}$$

$$\boxed{\begin{aligned}\delta\,|\Psi_n\rangle = &-\frac{1}{2}\,\delta\phi\,|\Psi_n\rangle - \frac{1}{2}\,\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\;|\Psi_n\rangle - \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}}\frac{\Lambda_n - m^2}{\Lambda_n - \Lambda_m}\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\;|\Psi_m\rangle\\ &- \sum_{m\neq 0,n}\frac{1}{\Lambda_n - \Lambda_m}\langle\Psi_m|\,\overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\;|\Psi_m\rangle\,.\end{aligned}} \tag{4.54b}$$

following from:

$$\begin{aligned}\delta\,|\Psi_n\rangle &= -\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\;|\Psi_n\rangle - 2\sum_{\substack{m\in\mathbb{Z}\\ m\neq n}}\frac{\Lambda_n - m^2}{\Lambda_n - \Lambda_m}\,\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\;|\Psi_m\rangle\\ &\quad + \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}}\frac{1}{\Lambda_n - \Lambda_m}\left[\left(\frac{\Lambda_m + \Lambda_n}{2} - m^2\right)\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle - \langle\Psi_m|\,\overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\right]|\Psi_m\rangle\\ &= -\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\;|\Psi_n\rangle + \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}}\frac{1}{\Lambda_n - \Lambda_m}\left(\frac{\Lambda_m - \Lambda_n}{2} - \Lambda_n + m^2\right)\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\;|\Psi_m\rangle\\ &\quad - \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}}\frac{1}{\Lambda_n - \Lambda_m}\langle\Psi_m|\,\overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\;|\Psi_m\rangle\\ &= -\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\;|\Psi_n\rangle - \frac{1}{2}\sum_{\substack{m\in\mathbb{Z}\\ m\neq n}}\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\;|\Psi_m\rangle\\ &\quad - \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}}\frac{\Lambda_n - m^2}{\Lambda_n - \Lambda_m}\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\;|\Psi_m\rangle - \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}}\frac{1}{\Lambda_n - \Lambda_m}\langle\Psi_m|\,\overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\;|\Psi_m\rangle\,.\end{aligned}$$

We obtain the result above by using the resolution of the identity. Note that the term $m = 0$ in the last sum does not contribute since $\nabla\!\!\!/\,\Psi_0 = 0$.

Note that extending the formula (4.54a) to the zero-modes correctly gives $\delta\Lambda_0 = 0$ since $\Lambda_0 = m^2$ and $\nabla\!\!\!/\,\Psi_0 = 0$. [HE: Check what happens for $\delta\Psi_0$ and if we need to include $m = 0$] ⇐ 11

5 Gravitational action

The gravitational action (3.6) is defined as the WZW action for the metrics $(g_{\mu\nu}, \hat{g}_{\mu\nu})$:

$$S_{\text{grav}}[g, \hat{g}] := S_{\text{eff}}[g] - S_{\text{eff}}[\hat{g}] \tag{5.1}$$

where the effective action is (3.4):

$$S_{\text{eff}} = -\frac{1}{4}\ln\det\frac{D^2}{\mu^2}, \tag{5.2}$$

where the scale μ has also been introduced here to make the argument of the logarithm dimensionless [47, p. 188]. It can be properly derived by relating the measure for the field with the measures for the modes [40, p. 4]. The simplest method to compute is to consider two metrics related by an infinitesimal Weyl factor:

$$\delta g_{\mu\nu} = 2 g_{\mu\nu} \delta\phi. \tag{5.3}$$

However, we first need to regularize the expression of the effective action which is divergent as it stands.

We have found in (3.56b) the expression for the Green function G in terms of modes:

$$G(x, y) = \sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n} \Psi_n(x)\Psi_n(y)^\dagger. \tag{5.4}$$

It is well known that this the Green function diverges at coincident points and behaves as

$$G(x, y) \sim_{x\sim y} -\frac{1}{4\pi} \ln \ell(x, y)^2, \tag{5.5}$$

where $\ell(x, y)$ is the geodesic length between points x and y. [HE: prove it] For this reason, it is necessary to introduce a regularization. A simple regularization is to just remove this singularity and to define the regularized Green function at coincident points as: ⇐ 12

$$G_R(x) := \lim_{y\to x} \left(G(x, y) + \frac{1}{4\pi} \ln \left(\mu^2 \ell(x, y)^2\right), \right) \tag{5.6}$$

where μ is a scale inserted for dimensionality. However, We will consider ζ-regularization in the rest of this paper. In general, we will omit the second position for all bi-local functions when $x = y$, i.e. $G(x) := G(x, x)$.

5.1 Spectral regularization

First, we define the bi-local ζ-function for the operator D^2:

$$\boxed{\zeta(s, x, y) := \sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n^s} \Psi_n(x)\Psi_n(y)^\dagger.} \tag{5.7}$$

and the corresponding integrated version:

$$\boxed{\zeta(s) := \int \mathrm{d}^2x \; \mathrm{tr}_D\, \zeta(s, x) = \sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n^s},} \tag{5.8}$$

where tr_D is the trace over Dirac indices.

The zeta function can be obtained as the Laplace transform of the heat kernel $K(t, x, y)$:

$$\zeta(s, x, y) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1}\, K(t, x, y), \qquad \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1}\, K(t). \tag{5.9}$$

The heat kernel can be expressed in terms of modes as:

$$\boxed{K(t, x, y) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\Lambda_n t} \Psi_n(x)\Psi_n(y)^\dagger, \qquad K(t) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\Lambda_n t},} \tag{5.10}$$

which is a solution of the diffusion equation:

$$\boxed{\left(\frac{\mathrm{d}}{\mathrm{d}t} + D^2\right) K(t, x, y) = 0, \qquad K(t, x, y) \sim_{t\sim 0} \frac{\delta(x-y)}{\sqrt{g}}\, 1_2.} \tag{5.11}$$

It also dies at $t \to \infty$ when $m^2 > 0$, otherwise it reduces to the projector on zero-modes:

$$\lim_{t\to\infty} K(t,x,y) = \begin{cases} 0 & m^2 > 0, \\ P(x,y) & m^2 = 0. \end{cases} \tag{5.12}$$

The Green function can be recovered either as the $s \to 1$ limit of the ζ-function, or as the integral over t of the heat kernel:

$$G(x,y) = \lim_{s\to 1} \zeta(s,x,y) = \int_0^\infty \mathrm{d}t\, K(t,x,y). \tag{5.13}$$

The boundary condition above directly demonstrates the short distance singularity of the Green function.

Finally, it can be helpful to introduce the generalized heat kernel

$$\boxed{K(s,t,x,y) := \sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{-t\Lambda_n}}{\Lambda_n^s}\, \Psi_n(x)\Psi_n(y)^\dagger,} \tag{5.14}$$

which allows to recover both the ζ-function and heat kernel for $t = 0$ and $s = 0$ respectively.

The behaviour of K for small t is related to the asymptotics of the eigenvalues Λ_n and eigenfunctions Ψ_n for large n, which in turn is related to the short-distance properties of the Riemann surface. It is well-known that the small-t asymptotics is given in terms of local expressions of the curvature and its derivatives and that on a compact manifold without boundaries one has:

$$K(t,x,y) \sim_{t\sim 0} \frac{1}{4\pi t}\, \mathrm{e}^{-\ell(x,y)^2/4t} \sum_{k\geq 0} a_k(x,y)\, t^k. \tag{5.15}$$

The Seeley-DeWitt expansion $a_k(x,y)$ coefficients can be computed recursively using normal coordinates around x. This simplies for $x = y$ since $\ell(x,x) = 0$:

$$K(t,x) \sim_{t\sim 0} \frac{1}{4\pi t} \sum_{k\geq 0} a_k(x)\, t^k. \tag{5.16}$$

and this is the only expansion we will need. The first two coefficients at coincident points [47, p. 194]:

$$\boxed{a_0(x) = 1_2, \qquad a_1(x) = -\left(\frac{R}{12} + m^2\right) 1_2,} \tag{5.17}$$

where 1_2 is the 2-dimensional identity matrix for Dirac indices. This gives the expression:

$$K(t) \sim_{t\sim 0} \frac{A}{2\pi t} - \frac{\chi}{6} - \frac{m^2 A}{2\pi} + O(t) \tag{5.18}$$

The ζ-function has poles at:

$$s \in 1 - k, \qquad k \in \mathbb{N}. \tag{5.19}$$

For $s = -k \in -\mathbb{N}$, the ζ-function admits an analytic continuation:

$$\boxed{\zeta(-k,x) = \frac{(-1)^k\, k!}{4\pi}\, a_{k+1}(x).} \tag{5.20}$$

Moreover, the residue at $s = 1$ is:

$$\boxed{\mathrm{Res}_{s=1}\,\zeta(s,x) = \lim_{s\to 1}(s-1)\,\zeta(s,x) = \frac{1}{4\pi}\,a_0(x) = \frac{1}{4\pi}\,1_2.} \tag{5.21}$$

Using (5.27), the heat kernel K is easily related to the massless heat kernel as:

$$\boxed{K(t,x,y) = \mathrm{e}^{-m^2 t} K^{(0)}(t,x,y).} \tag{5.22}$$

Indeed, the derivative of $\mathrm{e}^{-m^2 t}$ is equivalent to shifting $D^{(0)2}$ by m^2, which corresponds to D^2.

Next, we need to convert the expressions for the functions without zero-modes. We have:

$$\boxed{\widetilde{\zeta}(s,x,y) = \zeta(s,x,y) - \frac{P(x,y)}{m^{2s}}, \qquad \widetilde{\zeta}(s) = \zeta(s) - \frac{N_0}{m^{2s}},} \tag{5.23}$$

where N_0 is the number of zero-modes (3.86). As a special case, we have:

$$\widetilde{\zeta}(0,x,y) = \zeta(0,x,y) - P(x,y), \qquad \widetilde{\zeta}(0) = \zeta(0) - N_0. \tag{5.24}$$

Note that we also have the simple equality between the massless ζ-function at $s = 0$ and the ζ-function without zero-modes:

$$\widetilde{\zeta}^{(0)}(0,x,y) = \widetilde{\zeta}(0,x,y) \tag{5.25}$$

since they do not depend on the eigenvalues for $s = 0$, and this is the only place where the mass appears.

The heat kernel without zero-modes has the following limits as $t \to \infty$:

$$\boxed{\lim_{t\to\infty} \widetilde{K}(t,x,y) = 0, \qquad \lim_{t\to\infty} \mathrm{e}^{m^2 t}\widetilde{K}(t,x,y) = 0,} \tag{5.26}$$

generalizing (5.12). The second is stronger and follows from the fact that Λ_1 is strictly greater than m^2. A useful relation is:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right)\widetilde{K}(t,x,y) = \mathrm{e}^{-m^2 t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{m^2 t}\widetilde{K}(t,x,y)\right). \tag{5.27}$$

The regularized ζ-function is defined by removing this pole:

$$\boxed{\zeta_R(s,x) := \zeta(s,x) - \frac{\mathrm{Res}_{s=1}\,\zeta(s,x)}{s-1} = \zeta(s,x) - \frac{1}{4\pi(s-1)}\,1_2,} \tag{5.28}$$

where μ is the same scale as the one appearing in (5.2). We will see below why it is needed. Then, the ζ-regularized Green function at coincident points is defined has:

$$\boxed{G_\zeta(x) := \lim_{s\to 1}\left(\mu^{2s-2}\,\zeta(s,x) - \frac{1}{4\pi(s-1)}\,1_2\right) = \zeta_R(1,x) + \frac{1}{4\pi}\,1_2\,\ln\mu^2.} \tag{5.29}$$

The same holds for the Green function without zero-modes, and one finds that (3.104b) generalizes:

$$\widetilde{G}_\zeta(x) := \lim_{s\to 1}\mu^{2s-2}\,\widetilde{\zeta}_R(s,x) = G_\zeta(x) - \frac{1}{m^2}\,P(x). \tag{5.30}$$

It is possible to show that G_ζ and G_R defined in (5.6) differ only by a constant $(\gamma/2\pi)$ [48, 40, sec. 2.2].

The spectral regularization of the Green function also provides a regularization of the determinant appearing in the effective action (3.4) [47, sec. 5.3.3]:

$$S_{\text{eff}} = -\frac{1}{4} \sum_{n \neq 0} \ln \frac{\Lambda_n}{\mu^2}. \tag{5.31}$$

The sum can be regularized with the ζ-function as

$$\sum_{n \neq 0} \ln \frac{\Lambda_n}{\mu^2} = \lim_{s \to 0} \sum_{n \neq 0} \frac{1}{\Lambda_n^s} \ln \frac{\Lambda_n}{\mu^2}, \tag{5.32}$$

giving:

$$\boxed{S_{\text{eff}} = \frac{1}{4} \big(\widetilde{\zeta}'(0) + \widetilde{\zeta}(0) \ln \mu^2 \big).} \tag{5.33}$$

The first term follows by noting that:

$$\frac{\mathrm{d}}{\mathrm{d}s} \frac{1}{\Lambda_n^s} = - \frac{\ln \Lambda_n}{\Lambda_n^s}. \tag{5.34}$$

The ζ-function at coincident points $\zeta(s, x)$ is a scalar for $s \neq 1$. In particular, we have

$$\Delta_0 \operatorname{tr}_D \zeta(s, x) = \operatorname{tr}_D \Delta_{1/2 \otimes 1/2} \zeta(s, x), \tag{5.35}$$

where Δ_s is the spin-s Laplacian (this relation is obviously true only at coincident points and not for $x \neq y$).

We have seen that $\{\Psi_n\}$ is a complete basis for D^2. This implies that the ζ-function (5.7) can be written as:

$$\boxed{\widetilde{\zeta}(s, x, y) = \sum_{n \neq 0} \frac{\nabla\!\!\!/ \Psi_n(x) \big(\nabla\!\!\!/ \Psi_n(y)\big)^\dagger}{\Lambda_n^s (\Lambda_n - m^2)},} \tag{5.36}$$

where the additional factor in the denominator indicates that the basis is not normalized. This allows to obtain the relation:

$$\boxed{\sum_{n \neq 0} \frac{\nabla\!\!\!/ \Psi_n(x) \big(\nabla\!\!\!/ \Psi_n(y)\big)^\dagger}{\Lambda_n^{s+1}} = \widetilde{\zeta}(s, x, y) - m^2 \, \widetilde{\zeta}(s+1, x, y),} \tag{5.37}$$

which follows by inserting $(\Lambda_n - m^2)$ in the numerator and denominator and using (5.36):

$$\sum_{n \neq 0} \frac{\nabla\!\!\!/ \Psi_n(x) \big(\nabla\!\!\!/ \Psi_n(y)\big)^\dagger}{\Lambda_n^{s+1}} = \sum_{n \neq 0} \frac{(\Lambda_n - m^2)}{\Lambda_n^{s+1} (\Lambda_n - m^2)} \nabla\!\!\!/ \Psi_n(x) \big(\nabla\!\!\!/ \Psi_n(y)\big)^\dagger.$$

Using the ζ-function, we can rewrite the mass expansion (3.112) of the Green function as:

$$\widetilde{G}(x, y) = \widetilde{G}^{(0)}(x, y) + \sum_{s \geq 1} (-1)^s m^{2s} \widetilde{\zeta}(s+1, x, y). \tag{5.38}$$

[HE: Compute $\Delta \operatorname{tr} \zeta(s, x)$ and use this to find which equation solves $\Delta \operatorname{tr} G(x, y)$] $\Leftarrow 13$

5.2 Computation of the gravitational action

The goal of this section is to compute the ζ-regularized gravitational action. We consider the infintesimal variation of (5.33) under the Weyl transformation (4.2):

$$\boxed{\delta S_{\text{eff}} = \frac{1}{4}\big(\delta\zeta'(0) + \delta\zeta(0)\ln\mu^2\big).} \tag{5.39}$$

Hence, we need to find the variation of the ζ-function. This can be written in terms of the variation of the eigenvalues:

$$\boxed{\delta\zeta(s) = -s\sum_{n\in\mathbb{Z}}\frac{\delta\Lambda_n}{\Lambda_n^{s+1}}.} \tag{5.40}$$

Since $\delta\Lambda_0 = 0$, both ζ-function with and without zero-modes have the same variations:

$$\delta\widetilde{\zeta}(s) = \delta\zeta(s). \tag{5.41}$$

This is convenient because it allows to treat both the massless and massive cases at the same time.

5.2.1 Variations of spectral functions

First, we can then use (5.37) to write:

$$\boxed{\sum_{n\neq 0}\frac{1}{\Lambda_n^{s+1}}\langle\Psi_n|\overleftarrow{\nabla}\delta\phi\nabla|\Psi_n\rangle = \sum_{n\neq 0}\frac{1}{\Lambda_n^{s}}\langle\Psi_n|\delta\phi|\Psi_n\rangle - m^2\sum_{n\neq 0}\frac{1}{\Lambda_n^{s+1}}\langle\Psi_n|\delta\phi|\Psi_n\rangle\,.} \tag{5.42}$$

ζ-function We first derive the variation $\delta\zeta(s,x,y)$. To simplify the computation, we assume that there is a single zero-mode $N_0 = 1$: the same results could be obtained by keeping N_0 arbitrary but using $P(x,y)$ and κ. Using (4.54), we have:

$$
\begin{aligned}
\delta\zeta(s,x,y) &= \delta \sum_{n\in\mathbb{Z}} \frac{\Psi_n(x)\Psi_n(y)^\dagger}{\Lambda_n^s} \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n^s}\, \delta\Psi_n(x)\Psi_n(y)^\dagger + \sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n^s}\, \Psi_n(x)\delta\Psi_n(y)^\dagger - s\sum_{n\in\mathbb{Z}} \frac{\delta\Lambda_n}{\Lambda_n^{s+1}}\, \Psi_n(x)\Psi_n(y)^\dagger \\
&= -\sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n^s}\bigg[\frac{1}{2}\,\delta\phi(x)\Psi_n(x) + \frac{1}{2}\,\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\,\Psi_n(x) \\
&\qquad\qquad + \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{\Lambda_n - m^2}{\Lambda_n - \Lambda_m} \langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\,\Psi_m(x) \\
&\qquad\qquad + \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{1}{\Lambda_n - \Lambda_m} \langle\Psi_m|\,\overleftarrow{\nabla}\delta\phi\,\overrightarrow{\nabla}\,|\Psi_n\rangle\,\Psi_m(x)\bigg]\Psi_n(y)^\dagger \\
&\quad - \sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n^s}\,\Psi_n(x)\bigg[\frac{1}{2}\,\delta\phi(y)\Psi_n(y)^\dagger + \frac{1}{2}\,\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\,\Psi_n(y)^\dagger \\
&\qquad\qquad\qquad + \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{\Lambda_n - m^2}{\Lambda_n - \Lambda_m} \langle\Psi_n|\,\delta\phi\,|\Psi_m\rangle\,\Psi_m(y)^\dagger \\
&\qquad\qquad\qquad + \sum_{\substack{m\in\mathbb{Z}\\ m\neq n}} \frac{1}{\Lambda_n - \Lambda_m} \langle\Psi_n|\,\overleftarrow{\nabla}\delta\phi\,\overrightarrow{\nabla}\,|\Psi_m\rangle\,\Psi_m(y)^\dagger\bigg] \\
&\quad + \sum_{n\neq 0} \frac{s}{\Lambda_n^{s+1}}\left[(\Lambda_n - m^2)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle + \langle\Psi_n|\,\overleftarrow{\nabla}\delta\phi\,\overrightarrow{\nabla}\,|\Psi_n\rangle\right]\Psi_n(x)\Psi_n(y)^\dagger \\
&= -\frac{1}{2}(\delta\phi(x) + \delta\phi(y))\widetilde{\zeta}(s,x,y) + \sum_{n\in\mathbb{Z}} \frac{1}{\Lambda_n^s}\left(s - 1 - \frac{m^2 s}{\Lambda_n}\right)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\,\Psi_n(x)\Psi_n(y)^\dagger \\
&\quad - \sum_{\substack{m,n\in\mathbb{Z}\\ m\neq n}} \left[\frac{\Lambda_m^{s-1} - \Lambda_n^{s-1}}{\Lambda_n^{s-1}\Lambda_m^{s-1}(\Lambda_n - \Lambda_m)} + \frac{m^2(\Lambda_n^s - \Lambda_m^s)}{\Lambda_n^s\Lambda_m^s(\Lambda_n - \Lambda_m)}\right]\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\,\Psi_m(x)\Psi_n(y)^\dagger \\
&\quad + \sum_{\substack{m,n\neq 0\\ m\neq n}} \frac{\Lambda_n^s - \Lambda_m^s}{\Lambda_n^s\Lambda_m^s(\Lambda_n - \Lambda_m)} \langle\Psi_m|\,\overleftarrow{\nabla}\delta\phi\,\overrightarrow{\nabla}\,|\Psi_n\rangle\,\Psi_m(x)\Psi_n(y)^\dagger \\
&\quad + \sum_{n\neq 0} \frac{s}{\Lambda_n^{s+1}} \langle\Psi_n|\,\overleftarrow{\nabla}\delta\phi\,\overrightarrow{\nabla}\,|\Psi_n\rangle\,\Psi_n(x)\Psi_n(y)^\dagger.
\end{aligned}
$$

To get the fourth equality, we have exchanged $m \leftrightarrow n$ in the fourth to sixth lines, and used the relations:

$$
\begin{aligned}
\frac{1}{\Lambda_n - \Lambda_m}\left[\frac{\Lambda_n - m^2}{\Lambda_n^s} - \frac{\Lambda_m - m^2}{\Lambda_m^s}\right] &= \frac{1}{\Lambda_n^s\Lambda_m^s(\Lambda_n - \Lambda_m)}\left[\Lambda_m^s(\Lambda_n - m^2) - \Lambda_n^s(\Lambda_m - m^2)\right] \\
&= \frac{1}{\Lambda_n^s\Lambda_m^s(\Lambda_n - \Lambda_m)}\left[\Lambda_m\Lambda_n(\Lambda_m^{s-1} - \Lambda_n^{s-1}) + m^2(\Lambda_n^s - \Lambda_m^s)\right], \\
\frac{1}{\Lambda_n - \Lambda_m}\left[\frac{1}{\Lambda_n^s} - \frac{1}{\Lambda_m^s}\right] &= \frac{\Lambda_m^s - \Lambda_n^s}{\Lambda_n^s\Lambda_m^s(\Lambda_n - \Lambda_m)}.
\end{aligned}
$$

Hence, we have:

$$\boxed{\begin{aligned}\delta\widetilde{\zeta}(s,x,y) =& -\frac{1}{2}(\delta\phi(x)+\delta\phi(y))\zeta(s,x,y)\\ &+\sum_{n\in\mathbb{Z}}\frac{1}{\Lambda_n^s}\left(s-1-\frac{m^2 s}{\Lambda_n}\right)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\,\Psi_n(x)\Psi_n(y)^\dagger\\ &+\sum_{n\neq 0}\frac{s}{\Lambda_n^{s+1}}\langle\Psi_n|\,\overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\,\Psi_n(x)\Psi_n(y)^\dagger\\ &+\sum_{\substack{m,n\in\mathbb{Z}\\ m\neq n}}\left[\frac{\Lambda_n^{s-1}-\Lambda_m^{s-1}}{\Lambda_n^{s-1}\Lambda_m^{s-1}(\Lambda_n-\Lambda_m)}-\frac{m^2(\Lambda_n^s-\Lambda_m^s)}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)}\right]\\ &\qquad\times\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\,\Psi_m(x)\Psi_n(y)^\dagger\\ &+\sum_{\substack{m,n\neq 0\\ m\neq n}}\frac{\Lambda_n^s-\Lambda_m^s}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)}\langle\Psi_m|\,\overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\,\Psi_m(x)\Psi_n(y)^\dagger.\end{aligned}} \tag{5.43}$$

Using the following limit:

$$\begin{aligned}\lim_{y\to x}\frac{x^s-y^s}{x-y} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(x^s-(x+\epsilon)^s\right) = -x^s\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(1-(1+\epsilon/x)^s\right)\\ &= -x^s\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(-\epsilon s/x\right) = s\,x^{s-1},\end{aligned}$$

we obtain the limits with eigenvalues ratios:

$$\lim_{m\to n}\frac{\Lambda_n^s-\Lambda_m^s}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)} = \frac{s}{\Lambda_n^{s+1}}, \tag{5.44a}$$

$$\lim_{m\to n}\left[\frac{\Lambda_n^{s-1}-\Lambda_m^{s-1}}{\Lambda_n^{s-1}\Lambda_m^{s-1}(\Lambda_n-\Lambda_m)}-\frac{m^2(\Lambda_n^s-\Lambda_m^s)}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)}\right] = \frac{s-1}{\Lambda_n^s}-\frac{sm^2}{\Lambda_n^{s+1}}. \tag{5.44b}$$

This shows that we can rewrite (5.43) as:

$$\boxed{\begin{aligned}\delta\zeta(s,x,y) =& -\frac{1}{2}(\delta\phi(x)+\delta\phi(y))\zeta(s,x,y)\\ &+\sum_{m,n\in\mathbb{Z}}\left[\frac{\Lambda_n^{s-1}-\Lambda_m^{s-1}}{\Lambda_n^{s-1}\Lambda_m^{s-1}(\Lambda_n-\Lambda_m)}-\frac{m^2(\Lambda_n^s-\Lambda_m^s)}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)}\right]\\ &\qquad\times\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\,\Psi_m(x)\Psi_n(y)^\dagger\\ &+\sum_{m,n\neq 0}\frac{\Lambda_n^s-\Lambda_m^s}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)}\langle\Psi_m|\,\overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\,\Psi_m(x)\Psi_n(y)^\dagger.\end{aligned}} \tag{5.45}$$

by evaluating properly the case $m=n$.

Next, we want to find the variation for the ζ-function without zero-modes:

$$
\begin{aligned}
\delta\widetilde{\zeta}(s,x,y) &= \delta\zeta(s,x,y) - \frac{\delta P(x,y)}{m^{2s}} \\
&= -\frac{1}{2}(\delta\phi(x)+\delta\phi(y))\zeta(s,x,y) \\
&\quad + \sum_{m,n\in\mathbb{Z}} \left[\frac{\Lambda_n^{s-1}-\Lambda_m^{s-1}}{\Lambda_n^{s-1}\Lambda_m^{s-1}(\Lambda_n-\Lambda_m)} - \frac{m^2(\Lambda_n^s-\Lambda_m^s)}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)}\right] \\
&\qquad\quad \times \langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\, \Psi_m(x)\Psi_n(y)^\dagger \\
&\quad + \sum_{m,n\neq 0} \frac{\Lambda_n^s-\Lambda_m^s}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)} \langle\Psi_m|\,\overleftarrow{\not{\nabla}}\,\delta\phi\,\not{\nabla}\,|\Psi_n\rangle\, \Psi_m(x)\Psi_n(y)^\dagger \\
&\quad + \frac{1}{2}(\delta\phi(x)+\delta\phi(y))\frac{P(x,y)}{m^{2s}} + \frac{1}{m^{2s}}\int \mathrm{d}^2z\sqrt{g}\,\delta\phi(z)\,P(x,z)P(z,y) \\
&= -\frac{1}{2}(\delta\phi(x)+\delta\phi(y))\widetilde{\zeta}(s,x,y) \\
&\quad + \sum_{m,n\neq 0} \left[\frac{\Lambda_n^{s-1}-\Lambda_m^{s-1}}{\Lambda_n^{s-1}\Lambda_m^{s-1}(\Lambda_n-\Lambda_m)} - \frac{m^2(\Lambda_n^s-\Lambda_m^s)}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)}\right] \\
&\qquad\quad \times \langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\, \Psi_m(x)\Psi_n(y)^\dagger \\
&\quad + \sum_{n\neq 0} \left[\frac{\Lambda_n^{s-1}-m^{2(s-1)}}{\Lambda_n^{s-1}m^{2(s-1)}(\Lambda_n-m^2)} - \frac{m^2(\Lambda_n^s-m^{2s})}{\Lambda_n^s m^{2s}(\Lambda_n-m^2)}\right] \\
&\qquad\quad \times \langle\psi_{0,i}|\,\delta\phi\,|\Psi_n\rangle\, \kappa^{ij}\psi_{0,j}(x)\Psi_n(y)^\dagger \\
&\quad + \sum_{m\neq 0} \left[\frac{m^{2(s-1)}-\Lambda_m^{s-1}}{m^{2(s-1)}\Lambda_m^{s-1}(m^2-\Lambda_m)} - \frac{m^2(m^{2s}-\Lambda_m^s)}{m^{2s}\Lambda_m^s(m^2-\Lambda_m)}\right] \\
&\qquad\quad \times \langle\Psi_m|\,\delta\phi\,|\psi_{0,i}\rangle\, \Psi_m(x)\kappa^{ij}\psi_{0,j}(y)^\dagger \\
&\quad + \left(\frac{\cancel{s}-1}{m^{2s}} - \cancel{\frac{sm^2}{m^{2(s+1)}}}\right)\cancel{\langle\psi_{0,i}|\,\delta\phi\,|\psi_{0,k}\rangle\,\kappa^{ij}\psi_{0,j}(x)\kappa^{k\ell}\psi_{0,\ell}(y)^\dagger} \\
&\quad + \sum_{m,n\neq 0} \frac{\Lambda_n^s-\Lambda_m^s}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)} \langle\Psi_m|\,\overleftarrow{\not{\nabla}}\,\delta\phi\,\not{\nabla}\,|\Psi_n\rangle\, \Psi_m(x)\Psi_n(y)^\dagger \\
&\quad + \cancel{\frac{1}{m^{2s}}\int \mathrm{d}^2z\sqrt{g}\,\delta\phi(z)\,P(x,z)P(z,y)}.
\end{aligned}
$$

using (4.12) and (5.44). After simplification, we find:

$$
\begin{aligned}
\delta\widetilde{\zeta}(s,x,y) = &-\frac{1}{2}(\delta\phi(x)+\delta\phi(y))\widetilde{\zeta}(s,x,y) \\
&+\sum_{m,n\neq 0}\left[\frac{\Lambda_n^{s-1}-\Lambda_m^{s-1}}{\Lambda_n^{s-1}\Lambda_m^{s-1}(\Lambda_n-\Lambda_m)}-\frac{m^2(\Lambda_n^s-\Lambda_m^s)}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)}\right] \\
&\qquad\times\langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\,\Psi_m(x)\Psi_n(y)^\dagger \\
&+\sum_{n\neq 0}\left[\frac{\Lambda_n^{s-1}-m^{2(s-1)}}{\Lambda_n^{s-1}m^{2(s-1)}(\Lambda_n-m^2)}-\frac{m^2(\Lambda_n^s-m^{2s})}{\Lambda_n^s m^{2s}(\Lambda_n-m^2)}\right] \\
&\qquad\times\langle\psi_{0,i}|\,\delta\phi\,|\Psi_n\rangle\,\kappa^{ij}\psi_{0,j}(x)\Psi_n(y)^\dagger \\
&+\sum_{m\neq 0}\left[\frac{m^{2(s-1)}-\Lambda_m^{s-1}}{m^{2(s-1)}\Lambda_m^{s-1}(m^2-\Lambda_m)}-\frac{m^2(m^{2s}-\Lambda_m^s)}{m^{2s}\Lambda_m^s(m^2-\Lambda_m)}\right] \\
&\qquad\times\langle\Psi_m|\,\delta\phi\,|\psi_{0,i}\rangle\,\Psi_m(x)\kappa^{ij}\psi_{0,j}(y)^\dagger \\
&+\sum_{m,n\neq 0}\frac{\Lambda_n^s-\Lambda_m^s}{\Lambda_n^s\Lambda_m^s(\Lambda_n-\Lambda_m)}\langle\Psi_m|\,\overleftarrow{\not\nabla}\,\delta\phi\,\not\nabla\,|\Psi_n\rangle\,\Psi_m(x)\Psi_n(y)^\dagger .
\end{aligned}
\tag{5.46}
$$

Next, we are interested in obtaining $\delta\zeta(s)$. First, we integrate (5.45) over $x=y$ and take the Dirac trace, using

$$
\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\Psi_m(x)\Psi_n(x)^\dagger = \delta_{mn}. \tag{5.47}
$$

This implies that only terms with $m=n$ in the sum contributes, such that double sums reduce to a single sum. We find:

$$
\begin{aligned}
\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta\zeta(s,x) = (s-2)&\sum_{n\in\mathbb{Z}}\frac{1}{\Lambda_n^s}\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle - m^2 s\sum_{n\in\mathbb{Z}}\frac{1}{\Lambda_n^{s+1}}\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle \\
&+\sum_{n\neq 0}\left[\frac{\Lambda_n^{s-1}-m^{2(s-1)}}{\Lambda_n^{s-1}m^{2(s-1)}(\Lambda_n-m^2)}-\frac{m^2(\Lambda_n^s-m^{2s})}{\Lambda_n^s m^{2s}(\Lambda_n-m^2)}\right] \\
&\qquad\times\langle\psi_{0,i}|\,\delta\phi\,|\Psi_n\rangle\,\kappa^{ij}\psi_{0,j}(x)\Psi_n(y)^\dagger \\
&+\sum_{m\neq 0}\left[\frac{m^{2(s-1)}-\Lambda_m^{s-1}}{m^{2(s-1)}\Lambda_m^{s-1}(m^2-\Lambda_m)}-\frac{m^2(m^{2s}-\Lambda_m^s)}{m^{2s}\Lambda_m^s(m^2-\Lambda_m)}\right] \\
&\qquad\times\langle\Psi_m|\,\delta\phi\,|\psi_{0,i}\rangle\,\Psi_m(x)\kappa^{ij}\psi_{0,j}(y)^\dagger \\
&+s\sum_{n\in\mathbb{Z}}\frac{1}{\Lambda_n^{s+1}}\langle\Psi_n|\,\overleftarrow{\not\nabla}\,\delta\phi\,\not\nabla\,|\Psi_n\rangle .
\end{aligned}
\tag{5.48}
$$

Using (5.42), this finally gives:

$$
\begin{aligned}
\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta\zeta(s,x) &= 2\sum_{n\in\mathbb{Z}}\frac{1}{\Lambda_n^s}\left(s-1-\frac{m^2 s}{\Lambda_n}\right)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle \\
&= 2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\Big((s-1)\zeta(s,x)-m^2 s\,\zeta(s+1,x)\Big).
\end{aligned}
\tag{5.49}
$$

Next, we want to obtain the trace of $\delta\widetilde{\zeta}(s,x,y)$ from (5.46). We see immediately that the additional cross-terms with zero-modes and non-zero modes in (5.46) compared to (5.45) do not contribute since these modes are orthogonal. Hence, the only difference with respect to (5.49) is to use tilde functions:

$$\boxed{\begin{aligned}\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta\widetilde{\zeta}(s,x) &= 2\sum_{n\neq 0}\frac{1}{\Lambda_n^s}\left(s-1-\frac{m^2 s}{\Lambda_n}\right)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle \\ &= 2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left((s-1)\widetilde{\zeta}(s,x)-m^2 s\,\widetilde{\zeta}(s+1,x)\right).\end{aligned}} \tag{5.50}$$

Let's check if (5.49) and (5.50) are consistent. Using (5.23), we can write:

$$\begin{aligned}(s-1)\zeta(s,x)-m^2 s\,\zeta(s+1,x) &= (s-1)\left(\widetilde{\zeta}(s,x)+\frac{P(x)}{m^{2s}}\right)-m^2 s\left(\widetilde{\zeta}(s+1,x)+\frac{P(x)}{m^{2(s+1)}}\right) \\ &= (s-1)\widetilde{\zeta}(s,x)-m^2 s\,\widetilde{\zeta}(s+1,x)-\frac{P(x)}{m^{2s}}\end{aligned}$$

such that

$$\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta\zeta(s,x) = \int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta\widetilde{\zeta}(s,x)-\frac{2}{m^{2s}}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\,P(x). \tag{5.51}$$

We can also use (5.23) and (4.12):

$$\begin{aligned}\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta\zeta(s,x) &= \int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\left(\delta\widetilde{\zeta}(s,x)+\frac{\delta P(x)}{m^{2s}}\right) \\ &= \int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta\widetilde{\zeta}(s,x)-\frac{1}{m^{2s}}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\,P(x) \\ &\quad -\frac{1}{m^{2s}}\int \mathrm{d}^2x\sqrt{g}\int \mathrm{d}^2y\sqrt{g}\,\delta\phi(y)\,\mathrm{tr}_D\,P(x,y)P(y,x).\end{aligned}$$

Using (3.84) to simplify the last term correctly reproduces the formula above.

To find $\delta\widetilde{\zeta}(s)$, we need to add the variation of the conformal factor:

$$\delta\widetilde{\zeta}(s) = 2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\,\delta\widetilde{\zeta}(s,x)+\int \mathrm{d}^2x\sqrt{g}\,\mathrm{tr}_D\,\delta\widetilde{\zeta}(s,x). \tag{5.52}$$

Adding this with (5.49) gives:

$$\boxed{\begin{aligned}\delta\zeta(s)=\delta\widetilde{\zeta}(s) &= 2s\sum_{n\neq 0}\frac{1}{\Lambda_n^s}\left(1-\frac{m^2}{\Lambda_n}\right)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle \\ &= 2s\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left(\widetilde{\zeta}(s,x)-m^2\widetilde{\zeta}(s+1,x)\right).\end{aligned}} \tag{5.53}$$

As an exercise, let's check that evaluating (5.40) gives the same formula:

$$\delta\widetilde{\zeta}(s) = -s\sum_{n\neq 0}\frac{\delta\Lambda_n}{\Lambda_n^{s+1}} = s\sum_{n\neq 0}\frac{1}{\Lambda_n^{s+1}}\left((\Lambda_n-m^2)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle+\langle\Psi_n|\,\overleftarrow{\nabla}\,\delta\phi\,\nabla\,|\Psi_n\rangle\right).$$

Using (5.42), we recover (5.53).

We also need to compute the derivative:

$$\boxed{\begin{aligned}\delta\widetilde{\zeta}'(s) &= 2\left(1+s\,\frac{\mathrm{d}}{\mathrm{d}s}\right)\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left(\widetilde{\zeta}(s,x)-m^2\widetilde{\zeta}(s+1,x)\right)\\ &= 2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left(\widetilde{\zeta}(s,x)-m^2\widetilde{\zeta}(s+1,x)\right)\\ &\qquad + 2s\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left(\widetilde{\zeta}'(s,x)-m^2\widetilde{\zeta}'(s+1,x)\right).\end{aligned}} \tag{5.54}$$

We can now evaluate the variation and its derivative at $s=0$. Fist, we have:

$$\boxed{\delta\widetilde{\zeta}(0) = -\frac{m^2}{\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)} \tag{5.55}$$

using that $\widetilde{\zeta}(s,x)$ is regular at $s=0$ and

$$\lim_{s\to 0} s\,\widetilde{\zeta}(s+1,x) = \lim_{s\to 0} s\,\zeta(s+1,x) = \mathrm{Res}_{s=1}\,\zeta(s,x) = \frac{1}{4\pi}\,1_2.$$

The first equality follows by using (5.24) and that the projector term is independent of s. The last equality follows from (5.21).

Similarly, we find for the derivative:

$$\boxed{\begin{aligned}\delta\widetilde{\zeta}'(0) = &-\frac{1}{12\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)R(x) - 2m^2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\left(\mathrm{tr}_D\,\widetilde{G}_\zeta(x)+\frac{1}{4\pi}\right)\\ &- 2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\,P(x) + \frac{m^2}{\pi}\,\ln\mu^2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x).\end{aligned}} \tag{5.56}$$

This follows by noting first that

$$\lim_{s\to 0} s\,\widetilde{\zeta}'(s,x) = 0 \tag{5.57}$$

and that

$$\lim_{s\to 0}\left(1+s\,\frac{\mathrm{d}}{\mathrm{d}s}\right)\widetilde{\zeta}(s+1,x) = \widetilde{\zeta}_R(1,x) = \widetilde{G}_\zeta(x) - \frac{1}{4\pi}\,1_2\,\ln\mu^2, \tag{5.58}$$

where the regularizd ζ-function and Green function were defined in (5.28) and (5.29). We have:

$$\begin{aligned}\delta\widetilde{\zeta}'(0) &= 2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left(\frac{a_1(x)}{4\pi}-P(x)\right)\\ &\qquad - 2m^2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left(\widetilde{G}_\zeta(x)+\frac{1}{4\pi}\,1_2\,\ln\mu^2\right)\\ &= -2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left(\frac{1}{4\pi}\left(\frac{R}{12}+m^2\right)1_2+P(x)\right)\\ &\qquad - 2m^2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\,\widetilde{G}_\zeta(x) + \frac{m^2}{\pi}\,\ln\mu^2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x).\end{aligned}$$

Green function For $s = 1$, we correctly reproduce the variation of the Green function (4.32) from (5.43):

$$\boxed{\begin{aligned}\delta G(x,y) &= \delta\zeta(1,x,y) \\ &= -\frac{1}{2}(\delta\phi(x)+\delta\phi(y))\zeta(1,x,y) - \sum_{m,n\in\mathbb{Z}} \frac{m^2}{\Lambda_n\Lambda_m} \langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\, \Psi_m(x)\Psi_n(y)^\dagger \\ &\quad + \sum_{m,n\in\mathbb{Z}} \frac{1}{\Lambda_n\Lambda_m} \langle\Psi_m|\, \overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\, \Psi_m(x)\Psi_n(y)^\dagger.\end{aligned}} \tag{5.59}$$

Similarly, we can check that we reproduce correctly the variation (4.35) of the Green function without zero-modes:

$$\boxed{\begin{aligned}\delta\widetilde{G}(x,y) &= \delta\widetilde{\zeta}(1,x,y) \\ &= -\frac{1}{2}(\delta\phi(x)+\delta\phi(y))\widetilde{\zeta}(1,x,y) - \sum_{m,n\neq 0} \frac{m^2}{\Lambda_n\Lambda_m} \langle\Psi_m|\,\delta\phi\,|\Psi_n\rangle\, \Psi_m(x)\Psi_n(y)^\dagger \\ &\quad - \sum_{n\neq 0} \frac{1}{\Lambda_n} \langle\psi_{0,i}|\,\delta\phi\,|\Psi_n\rangle\, \kappa^{ij}\psi_{0,j}(x)\Psi_n(y)^\dagger \\ &\quad - \sum_{m\neq 0} \frac{1}{\Lambda_n} \langle\Psi_m|\,\delta\phi\,|\psi_{0,i}\rangle\, \Psi_m(x)\kappa^{ij}\psi_{0,j}(y)^\dagger \\ &\quad + \sum_{m,n\neq 0} \frac{1}{\Lambda_n\Lambda_m} \langle\Psi_m|\, \overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\, \Psi_m(x)\Psi_n(y)^\dagger.\end{aligned}} \tag{5.60}$$

We have seen that it is not possible to simplify the variation (4.32) or (5.59) of the Green function further without taking the trace over Dirac indices and spacetime, which is what we will need to compute the action. The integrated trace of the Green function is given by (5.49) at $s = 1$:

$$\boxed{\int d^2x\sqrt{g}\; \mathrm{tr}_D\, \delta\widetilde{G}(x) = \int d^2x\sqrt{g}\, \delta\widetilde{\zeta}(1,x) = -2m^2 \int d^2x\sqrt{g}\, \delta\phi(x)\, \mathrm{tr}_D\, \widetilde{\zeta}(2,x).} \tag{5.61}$$

Finally, we want to compute the variation of the regularized Green function (5.29). The simplest way is to use the fact that $\delta G_\zeta = \delta G_R$ since G_R defined in (5.6) differs from G_ζ by a constant.

$$\delta\widetilde{G}_\zeta(x) := \lim_{y\to x} \left(\delta\widetilde{G}(x,y) + \frac{1}{4\pi}\,\delta\ln\left(\mu^2\ell(x,y)^2\right)\right) \tag{5.62}$$

[HE: How to get this from (5.29)?] The variation of the geodesic distance is given in (4.42) ⇐ 14

$$\delta\ln\left(\mu^2\ell(x,y)^2\right) = \delta\phi(x) + \delta\phi(y), \tag{5.63}$$

and the one of the Green function in (4.32) or in (5.59). Again, we cannot proceed further without integrating the variation. Using (5.61), we find:

$$\boxed{\int d^2x\sqrt{g}\; \mathrm{tr}_D\, \delta\widetilde{G}_\zeta(x) = -2m^2 \int d^2x\sqrt{g}\, \delta\phi(x)\, \mathrm{tr}_D\, \widetilde{\zeta}(2,x) + \frac{1}{\pi}\int d^2x\sqrt{g}\, \delta\phi(x).} \tag{5.64}$$

Finally, we find the variation of the integrated Green function (3.114):

$$\boxed{\delta\Psi_G[g] = 2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\;\mathrm{tr}_D\left(\widetilde{G}(x) - m^2\,\widetilde{\zeta}(2,x)\right) + \frac{1}{\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x),} \tag{5.65}$$

following from:

$$\begin{aligned}
\delta\Psi_G[g] &= \delta\int \mathrm{d}^2x\sqrt{g}\;\mathrm{tr}_D\,\widetilde{G}(x) \\
&= 2\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\,\widetilde{G}(x) + \int \mathrm{d}^2x\sqrt{g}\;\mathrm{tr}_D\,\delta\widetilde{G}(x).
\end{aligned}$$

Heat kernel The variation of the integrated heat kernel (5.10) is:

$$\boxed{\delta\widetilde{K}(t) = -2t\left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right)\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\,\widetilde{K}(t,x),} \tag{5.66}$$

This follows from:

$$\begin{aligned}
\delta\widetilde{K}(t) &= -t\sum_{n\neq 0}\mathrm{e}^{-\Lambda_n t}\,\delta\Lambda_n \\
&= t\sum_{n\neq 0}\mathrm{e}^{-\Lambda_n t}\left((\Lambda_n - m^2)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle + \langle\Psi_n|\,\overleftarrow{\nabla\!\!\!/}\,\delta\phi\,\nabla\!\!\!/\,|\Psi_n\rangle\right) \\
&= 2t\sum_{n\neq 0}\mathrm{e}^{-\Lambda_n t}\left((\Lambda_n - m^2)\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\right) \\
&= -2t\left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right)\sum_{n\neq 0}\mathrm{e}^{-\Lambda_n t}\,\langle\Psi_n|\,\delta\phi\,|\Psi_n\rangle\,,
\end{aligned}$$

using (4.54a) and (5.42) with $s = -1$. Using (5.22), the variation can be simplified further as:

$$\boxed{\delta\widetilde{K}(t) = -2t\,\mathrm{e}^{-m^2 t}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\,\frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{K}^{(0)}(t,x),} \tag{5.67}$$

where $\widetilde{K}^{(0)}$ is the massless heat kernel.

5.2.2 Variations of effective action

Summing (5.55) and (5.56) together, we obtain the variation of the effective action (5.33):

$$\boxed{\begin{aligned}\delta S_{\text{eff}} = &-\frac{1}{48\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)R(x) - \frac{m^2}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left(\widetilde{G}_\zeta(x) + \frac{1}{4\pi}\right) \\ &-\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\,P(x).\end{aligned}} \tag{5.68}$$

We see that the terms in $\ln\mu^2$ cancel. Moreover, we can use the relation (3.104b) to combine the projector and the Green function without zero-modes:

$$\boxed{\delta S_{\text{eff}} = -\frac{1}{48\pi}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)R(x) - \frac{m^2}{2}\int \mathrm{d}^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}_D\left(G_\zeta(x) + \frac{1}{4\pi}\right).} \tag{5.69}$$

In order to integrate this equation for a finite transformation, we need to express each term in terms of a total variation. The first term is easily recognized as the variation of the Liouville action

$$\delta S_{\mathrm{L}} = \frac{1}{4\pi} \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x) R(x). \tag{5.70}$$

The third term is the variation of the log of the determinant of the zero-mode inner-product matrix (4.11):

$$\delta \ln \det \kappa = \operatorname{tr} \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D P(x). \tag{5.71}$$

At this intermediate stage, we have:

$$\boxed{\delta S_{\mathrm{eff}} = -\frac{1}{12}\, \delta S_{\mathrm{L}} - \frac{1}{2}\, \delta \ln \det \kappa - \frac{m^2}{2} \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D \left(\widetilde{G}_\zeta(x) + \frac{1}{4\pi} \right).} \tag{5.72}$$

The last term can be rewritten as:

$$2 \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D \widetilde{G}_\zeta(x) = \delta \int \mathrm{d}^2 x \sqrt{g}\, \operatorname{tr}_D \widetilde{G}_\zeta(x) - \int \mathrm{d}^2 x \sqrt{g}\, \operatorname{tr}_D \delta \widetilde{G}_\zeta(x). \tag{5.73}$$

The second term in this last expression can be expressed as (5.64):

$$\int \mathrm{d}^2 x \sqrt{g}\, \operatorname{tr}_D \delta \widetilde{G}(x) = -2m^2 \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D \widetilde{\zeta}(2, x) + \frac{1}{\pi} \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x).$$

In total, we have:

$$\begin{aligned}
&\int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D \left(\widetilde{G}_\zeta(x) + \frac{1}{4\pi} \right) \\
&\quad = \frac{1}{2\pi} \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x) + \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D \widetilde{G}_\zeta(x) \\
&\quad = \frac{1}{2\pi} \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x) + \frac{1}{2}\, \delta \int \mathrm{d}^2 x \sqrt{g}\, \operatorname{tr}_D \widetilde{G}_\zeta(x) - \frac{1}{2} \int \mathrm{d}^2 x \sqrt{g}\, \operatorname{tr}_D \delta \widetilde{G}_\zeta(x) \\
&\quad = \cancel{\frac{1}{2\pi} \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)} + \frac{1}{2}\, \delta \int \mathrm{d}^2 x \sqrt{g}\, \operatorname{tr}_D \widetilde{G}_\zeta(x) \\
&\qquad\quad + m^2 \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D \widetilde{\zeta}(2, x) - \cancel{\frac{1}{2\pi} \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)}.
\end{aligned}$$

To summarize the current state of the action, we have:

$$\boxed{\begin{aligned}
\delta S_{\mathrm{eff}} = &-\frac{1}{12}\, \delta S_{\mathrm{L}} - \frac{1}{2}\, \delta \ln \det \kappa - \frac{m^2}{4}\, \delta \int \mathrm{d}^2 x \sqrt{g}\, \operatorname{tr}_D \widetilde{G}_\zeta(x) \\
&- \frac{m^4}{2} \int \mathrm{d}^2 x \sqrt{g}\, \delta\phi(x)\, \operatorname{tr}_D \widetilde{\zeta}(2, x).
\end{aligned}} \tag{5.74}$$

We need to express the term with $\widetilde{\zeta}(2, x)$ as a total variation.

First, we rewrite $\widetilde{\zeta}(2, x)$ in terms of the heat kernel using (5.9):

$$\widetilde{\zeta}(2, x, y) = \frac{1}{\cancel{\Gamma(2)}} \int_0^\infty \mathrm{d}t\, t\, \widetilde{K}(t, x, y). \tag{5.75}$$

Let's recall (5.27) which we will need several times:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right) \widetilde{K}(t,x,y) = \mathrm{e}^{-m^2 t} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{m^2 t} \widetilde{K}(t,x,y)\right). \tag{5.76}$$

This allows to prove two useful relations. The first is:

$$\int_0^\infty \mathrm{d}t\,(m^2 t + 1) \left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right) \widetilde{K}(t,x,y) = -\widetilde{K}(0,x,y) + m^4 \int_0^\infty \mathrm{d}t\, t\, \widetilde{K}(t,x,y), \tag{5.77}$$

which allows to obtain $t\widetilde{K}(t)$. It follows from:

$$\begin{aligned}
&\int_0^\infty \mathrm{d}t\,(m^2 t + 1) \left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right) \widetilde{K}(t,x,y) \\
&\quad = \int_0^\infty \mathrm{d}t\,(m^2 t + 1)\, \mathrm{e}^{-m^2 t} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{m^2 t} \widetilde{K}(t,x,y)\right) \\
&\quad = \int_0^\infty \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} \left((m^2 t + 1) \widetilde{K}(t,x,y)\right) - \int_0^\infty \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} \left((m^2 t + 1) \mathrm{e}^{-m^2 t}\right) \mathrm{e}^{m^2 t}\, \widetilde{K}(t,x,y) \\
&\quad = \cancel{\lim_{t\to\infty} (m^2 t + 1) \widetilde{K}(t,x,y)} - \widetilde{K}(0,x,y) - \int_0^\infty \mathrm{d}t\, \left(\cancel{m^2} - m^2(m^2 t + \cancel{1})\right) \widetilde{K}(t,x,y).
\end{aligned}$$

We have to find another identity involving the heat kernel at $t = 0$:

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{m^2 t} \left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right) \widetilde{K}(t,x,y) = -\widetilde{K}(0,x,y). \tag{5.78}$$

This follows from:

$$\begin{aligned}
\int_0^\infty \mathrm{d}t\, \mathrm{e}^{m^2 t} \left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right) \widetilde{K}(t,x,y) &= \int_0^\infty \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{m^2 t} \widetilde{K}(t,x,y)\right) \\
&= \cancel{\lim_{t\to\infty} \mathrm{e}^{m^2 t} \widetilde{K}(t,x,y)} - \widetilde{K}(0,x,y).
\end{aligned}$$

The first term cancels because $\Lambda_1 > m^2$, see (5.26). Thus, we find that:

$$m^4 \int_0^\infty \mathrm{d}t\, t\, \widetilde{K}(t,x,y) = -\int_0^\infty \mathrm{d}t (\mathrm{e}^{m^2 t} - m^2 t - 1) \left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right) \widetilde{K}(t,x,y). \tag{5.79}$$

Finally, we can use (5.66)

$$\delta\widetilde{K}(t) = -2t \left(\frac{\mathrm{d}}{\mathrm{d}t} + m^2\right) \int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x)\, \mathrm{tr}_D\, \widetilde{K}(t,x), \tag{5.80}$$

to obtain:

$$\begin{aligned}
m^4 \int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x)\, \mathrm{tr}_D\, \widetilde{\zeta}(2,x) &= m^4 \int \mathrm{d}^2x \sqrt{g}\, \delta\phi(x) \int_0^\infty \mathrm{d}t\, t\, \widetilde{K}(t,x) \\
&= \frac{1}{2} \int_0^\infty \frac{\mathrm{d}t}{t} (\mathrm{e}^{m^2 t} - m^2 t - 1) \delta\widetilde{K}(t).
\end{aligned} \tag{5.81}$$

The action becomes:

$$\boxed{\begin{aligned}
\delta S_{\text{eff}} = &-\frac{1}{12}\, \delta S_{\text{L}} - \frac{1}{2}\, \delta \ln\det\kappa - \frac{m^2}{2}\, \delta \int \mathrm{d}^2x \sqrt{g}\, \mathrm{tr}_D\, \widetilde{G}_\zeta(x) \\
&- \frac{1}{4} \int_0^\infty \frac{\mathrm{d}t}{t} (\mathrm{e}^{m^2 t} - m^2 t - 1) \delta\widetilde{K}(t).
\end{aligned}} \tag{5.82}$$

We can now integrate (5.82) to find the effective action:

$$\boxed{\begin{aligned} S_{\text{grav}} = &- \frac{1}{12} S_{\text{L}}[g, \hat{g}] - \frac{1}{2} \ln \frac{\det \kappa[g]}{\det \kappa[\hat{g}]} - \frac{m^2}{2} \left(\Psi_G[g] - \Psi_G[\hat{g}] \right) \\ &- \frac{1}{4} \int_0^\infty \frac{\mathrm{d}t}{t} \left(\mathrm{e}^{m^2 t} - m^2 t - 1 \right) \left(\widetilde{K}_g(t) - \widetilde{K}_{\hat{g}}(t) \right). \end{aligned}} \tag{5.83}$$

where we have defined the integrated regularized Green function $\Psi_{G,\zeta}[g]$ generalizes (3.114):

$$\boxed{\Psi_{G,\zeta}[g] := \int \mathrm{d}^2x \sqrt{g} \, \mathrm{tr}_D \, \widetilde{G}_{g,\zeta}(x).} \tag{5.84}$$

Note that all functionals have well-defined $m^2 \to 0$ limit, such that one correctly gets the Liouville action.

5.3 Small mass expansion

In this section, we want to study the small expansion of (5.83). We will focus on the first correction to the Liouville action. First, we note that the last term does not contribute:

$$\frac{1}{4} \int_0^\infty \frac{\mathrm{d}t}{t} \left(\mathrm{e}^{m^2 t} - m^2 t - 1 \right) \left(\widetilde{K}_g(t) - \widetilde{K}_{\hat{g}}(t) \right) = O(m^4) \tag{5.85}$$

as can be seen by expanding the exponential in powers of m^2. Moreover, we can use (3.112)

$$\widetilde{G}(x,y) = \widetilde{G}^{(0)}(x,y) + O(m^2) \tag{5.86}$$

to use the massless Green function instead. As a consequence, the action (5.83) up to $O(m^2)$ reads:

$$\boxed{S_{\text{grav}} = -\frac{1}{12} S_{\text{L}}[g, \hat{g}] - \frac{1}{2} \ln \frac{\det \kappa[g]}{\det \kappa[\hat{g}]} - \frac{m^2}{2} \left(\Psi_G^{(0)}[g] - \Psi_G^{(0)}[\hat{g}] \right) + O(m^4).} \tag{5.87}$$

At this order, the variaton of the effective action reads:

$$\boxed{\begin{aligned} \delta S_{\text{eff}} &= -\frac{1}{12} \delta S_{\text{L}} - \frac{1}{2} \delta \ln \det \kappa - \frac{m^2}{2} \int \mathrm{d}^2x \sqrt{g} \, \delta\phi(x) \, \mathrm{tr}_D \left(\widetilde{G}_\zeta^{(0)}(x) + \frac{1}{4\pi} \right) + O(m^4) \\ &= -\frac{1}{12} \delta S_{\text{L}} - \frac{1}{2} \delta \ln \det \kappa - \frac{m^2}{4} \delta \int \mathrm{d}^2x \sqrt{g} \, \mathrm{tr}_D \, \widetilde{G}_\zeta^{(0)}(x) + O(m^4). \end{aligned}} \tag{5.88}$$

In order to compute the third term, we take the $m^2 \to 0$ limit of (5.65) and add the regularization term:

$$\delta \Psi_{G,\zeta}[g] = 2 \int \mathrm{d}^2x \sqrt{g} \, \delta\phi(x) \, \mathrm{tr}_D \left(\widetilde{G}_\zeta^{(0)}(x) + \frac{1}{2\pi} \right) + O(m^2). \tag{5.89}$$

Next, inserting the relation (A.57), we get:

$$\begin{aligned} \int \mathrm{d}^2x \sqrt{g} \, \delta\phi(x) \, \mathrm{tr}_D \, \widetilde{G}_\zeta^{(0)}(x) &= \int \mathrm{d}^2x \sqrt{g} \left(\frac{\delta A}{2A} + \frac{A}{4} \, \Delta \, \delta K(x) \right) \mathrm{tr}_D \, \widetilde{G}_\zeta^{(0)}(x) \\ &= \frac{\delta A}{2A} \int \mathrm{d}^2x \sqrt{g} \, \mathrm{tr}_D \, \widetilde{G}_\zeta^{(0)}(x) + \frac{A}{4} \int \mathrm{d}^2x \sqrt{g} \, \delta K(x) \, \mathrm{tr}_D \, \Delta \, \widetilde{G}_\zeta^{(0)}(x) \\ &= \frac{\delta A}{2A} \int \mathrm{d}^2x \sqrt{g} \, \mathrm{tr}_D \, \widetilde{G}_\zeta^{(0)}(x) + \frac{A}{4} \int \mathrm{d}^2x \sqrt{g} \, \delta K(x) \, \mathrm{tr}_D \, \Delta \, \widetilde{G}_\zeta^{(0)}(x) \end{aligned}$$

where we have used Green's second identity (A.16). [HE: Can we evaluate $\mathrm{tr}_D\, \Delta\, \widetilde{G}^{(0)}_\zeta(x)$ by generalizing [35, sec. 3.2.5, app. A]?] ⇐ 15

A Formulas

A.1 Conventions

Greek indices refer to the curved frame and Latin indices (a, b, etc.) to the local flat frame. Explicit indices are denoted by letters in the first case, $\mu = (t, x)$, and numbers in the second case, $a = 0, 1$. We follow the conventions of [19, 24] (they have minor divergences).

A.1.1 Lorentzian coordinates

The coordinates are denoted as $x^\mu = (t, x)$ in Lorentzian signature. The metric in the local flat frame is

$$\eta_{ab} = \mathrm{diag}(-1, 1) \tag{A.1}$$

such that

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 = -\mathrm{d}x^+ \mathrm{d}x^-, \tag{A.2}$$

where the light-cone coordinates read [19, sec. 2.3]:

$$x^\pm := t \pm x. \tag{A.3}$$

The metric $g_{\mu\nu}$ is related to the local flat metric through the zweibeins:

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \tag{A.4}$$

The Levi–Civita antisymmetric tensor is defined in the local frame as [19, p. 512]:

$$\varepsilon_{01} = 1, \qquad \varepsilon^{01} = -1. \tag{A.5}$$

We have $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = -2$. In curved space, it corresponds to a density tensor rather than to a tensor:

$$\varepsilon_{\mu\nu} := e_\mu^a e_\nu^b \varepsilon_{ab}, \qquad \varepsilon^{\mu\nu} := g^{\mu\rho} g^{\nu\sigma} \varepsilon_{\rho\sigma} \tag{A.6}$$

such that

$$\varepsilon_{tx} = \sqrt{\det g}, \qquad \varepsilon^{tx} = -\frac{1}{\sqrt{\det g}}. \tag{A.7}$$

Note that it is possible to define a tensor (called ϵ symbol as $\epsilon_{\mu\nu} := \sqrt{\det g}\, \varepsilon_{\mu\nu}$).

A.1.2 Euclidean coordinates

The Wick rotation from Lorentzian time t to Euclidean time τ reads:

$$t = -\mathrm{i}\tau \tag{A.8}$$

such that

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + \mathrm{d}x^2 = \mathrm{d}z \mathrm{d}\bar{z}, \tag{A.9}$$

where the complex coordinates are defined as

$$z = \tau + \mathrm{i}x = \mathrm{i}x^+, \qquad \bar{z} = \tau - \mathrm{i}x = \mathrm{i}x^-. \tag{A.10}$$

As a consequence, we have

$$\partial = -\mathrm{i}\partial_+, \qquad \bar{\partial} = -\mathrm{i}\partial_-. \tag{A.11}$$

We have:

$$\begin{gathered}
\partial = \frac{1}{2}(\partial_t - \mathrm{i}\partial_x), \qquad \bar{\partial} = \frac{1}{2}(\partial_t + \mathrm{i}\partial_x), \\
V^z = V^t + \mathrm{i}V^x \qquad V^{\bar{z}} = V^t - \mathrm{i}V^x, \\
\mathrm{d}^2x := \mathrm{d}t\mathrm{d}x = \frac{1}{2}\,\mathrm{d}^2z, \qquad \mathrm{d}^2z := \mathrm{d}z\mathrm{d}\bar{z}, \\
\delta(z) = \frac{1}{2}\,\delta(x).
\end{gathered} \tag{A.12}$$

The Levi–Civita antisymmetric tensor is defined in the local frame as:

$$\varepsilon_{01} = 1, \qquad \varepsilon^{01} = 1. \tag{A.13}$$

We have $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = 2$. In complex coordinates, it reads:

$$\varepsilon_{z\bar{z}} = \frac{\mathrm{i}}{2}, \qquad \varepsilon^{z\bar{z}} = -2\mathrm{i}. \tag{A.14}$$

Note that γ_* takes the same value in both the local and curved frames. Indeed, in the curved frame, it is given by $\varepsilon_{\mu\nu}\gamma^\mu\gamma^\nu = \gamma_*$ since $\varepsilon_{\mu\nu}$ is a density and contains $\sqrt{g}$. As a consequence, it is Weyl invariant.

A.2 Geometry, covariant derivatives and Green functions

All formulas in this subsection and in the following ones are in d-dimensional Lorentzian signature.

Green's second identity states that:

$$\int \mathrm{d}^d x\sqrt{g}\,\Big(f\,\nabla_\mu(h\,\nabla^\mu\,g) - g\,\nabla_\mu(h\,\nabla^\mu\,f)\Big) = 0 \tag{A.15}$$

for any three functions f, g and h. This implies in particular:

$$\int \mathrm{d}^2 d\sqrt{g}\,\Big(f\,\Delta\,g - g\,\Delta\,f\Big) = 0. \tag{A.16}$$

Curvature The Christoffel connection $\Gamma^\rho_{\mu\nu}$ reads:

$$\Gamma^\rho_{\mu\nu} := \frac{1}{2}\,g^{\rho\sigma}\big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\big). \tag{A.17}$$

We have the relations:

$$\Gamma^\nu_{\mu\nu} = \frac{1}{2}\,g^{\nu\rho}\partial_\mu g_{\nu\rho} = \frac{1}{2}\,\partial_\mu \ln g, \tag{A.18a}$$

$$g^{\mu\sigma}\Gamma^\nu_{\rho\sigma} + g^{\nu\sigma}\Gamma^\mu_{\rho\sigma} = \partial_\rho g^{\mu\nu}, \tag{A.18b}$$

$$g_{\nu\sigma}\Gamma^\sigma_{\mu\rho} + g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} = \partial_\rho g_{\mu\nu}, \tag{A.18c}$$

$$g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - g_{\nu\sigma}\Gamma^\sigma_{\mu\rho} = \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho}. \tag{A.18d}$$

This follows from:

$$\begin{aligned} g_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \pm g_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} &= \frac{1}{2}\,\delta^{\tau}_{\nu}(\partial_\mu g_{\rho\tau} + \partial_\rho g_{\mu\tau} - \partial_\tau g_{\mu\rho}) \pm \frac{1}{2}\,\delta^{\tau}_{\mu}(\partial_\nu g_{\rho\tau} + \partial_\rho g_{\nu\tau} - \partial_\tau g_{\nu\rho}) \\ &= \frac{1}{2}\,(\partial_\mu g_{\rho\nu} + \partial_\rho g_{\mu\nu} - \partial_\nu g_{\mu\rho} \pm \partial_\nu g_{\rho\mu} \pm \partial_\rho g_{\nu\mu} \mp \partial_\mu g_{\nu\rho}) \\ &= \frac{1}{2}\,((1\mp 1)\partial_\mu g_{\nu\rho} + (1\pm 1)\partial_\rho g_{\mu\nu} - (1\mp 1)\partial_\nu g_{\mu\rho}). \end{aligned}$$

The spin connection is denoted by $\omega_{\mu ab}$ and is related to $\Gamma^{\rho}_{\mu\nu}$ as:

$$\Gamma^{\rho}_{\mu\nu} = e^{\rho}_{a}(\partial_\mu e^{a}_{\nu} + \omega_{\mu}{}^{a}{}_{b}e^{b}_{\nu}). \tag{A.19}$$

This can be rewritten as:

$$\omega_{\mu}{}^{a}{}_{b} = e^{\nu}_{b}(e^{a}_{\rho}\Gamma^{\rho}_{\mu\nu} - \partial_\mu e^{a}_{\nu}). \tag{A.20}$$

The Riemann tensor and its contractions are defined as [24, sec. 7.10]:

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} := \partial_\mu\Gamma^{\rho}_{\nu\sigma} - \partial_\nu\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\tau}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\rho}_{\nu\tau}\Gamma^{\tau}_{\mu\sigma}, \tag{A.21a}$$

$$R_{\mu\nu} := R_{\rho\mu}{}^{\rho}{}_{\nu}. \tag{A.21b}$$

Note that $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ and $R_{\mu\nu} = R_{\nu\mu}$ only when there is no torsion. In that case, one can also write:

$$R^{\mu}{}_{\nu\rho\sigma} := \partial_\rho\Gamma^{\mu}_{\nu\sigma} - \partial_\sigma\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\rho\tau}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\tau}\Gamma^{\tau}_{\nu\rho}, \tag{A.22a}$$

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}. \tag{A.22b}$$

The Einstein tensor is:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}\,R g_{\mu\nu}. \tag{A.23}$$

Covariant derivatives The covariant derivative ∇_μ always contain all the connections appropriate for the object it acts on. It acts on a scalar f, spinor ψ and vectors A^a and A^μ, and forms A_μ and A_a as:

$$\nabla_\mu f = \partial_\mu f, \tag{A.24a}$$

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4}\,\omega_{\mu ab}\gamma^{ab}\psi, \tag{A.24b}$$

$$\nabla_\mu A^a = \partial_\mu A^\nu + \omega_{\mu}{}^{a}{}_{b}A^b, \tag{A.24c}$$

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^{\nu}_{\mu\rho}A^\rho, \tag{A.24d}$$

$$\nabla_\mu A_a = \partial_\mu A_\nu + \omega_{\mu a}{}^{b}A_b, \tag{A.24e}$$

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^{\rho}_{\mu\nu}A_\rho. \tag{A.24f}$$

The Laplacian is defined as:

$$\Delta := g^{\mu\nu}\nabla_\mu\nabla_\nu. \tag{A.25}$$

It satisfies:

$$\Delta(ST) = S\Delta T + T\Delta S + 2\nabla S\cdot\nabla T, \tag{A.26a}$$

$$\Delta(f(T)) = f''(T)(\nabla T)^2 + f'(T)\Delta T, \tag{A.26b}$$

$$\Delta(\mathrm{e}^{\alpha\phi}T) = \mathrm{e}^{\alpha\phi}\Big(\Delta T + \alpha\,T(\alpha\,(\nabla\phi)^2 + \Delta\phi) + 2\alpha\,\nabla\phi\cdot\nabla T\Big), \tag{A.26c}$$

where f is a function (with the prile denoting the derivative with respect to the argument), T and S are tensor fields, and ϕ a scalar field. This second relation follows from:

$$\Delta(f(T)) = g^{\mu\nu}\nabla_\mu(\nabla_\nu f(T)) = g^{\mu\nu}\nabla_\mu(f'(T)\nabla_\nu T) = f''(T)(\nabla T)^2 + f'(T)\Delta T.$$

Combining the first and second equations give:

$$\begin{aligned}\Delta(\mathrm{e}^{\alpha\phi}T) &= \mathrm{e}^{\alpha\phi}\Delta T + T\Delta\mathrm{e}^{\alpha\phi} + 2\nabla\mathrm{e}^{\alpha\phi}\cdot\nabla T \\ &= \mathrm{e}^{\alpha\phi}\Delta T + \alpha\,\mathrm{e}^{\alpha\phi}\,T(\alpha\,(\nabla\phi)^2 + \Delta\phi) + 2\alpha\,\mathrm{e}^{\alpha\phi}\nabla\phi\cdot\nabla T\end{aligned}$$

We have the anti-commutators:

$$[\nabla_\mu, \nabla_\nu]f = 0, \tag{A.27a}$$

$$[\nabla_\mu, \nabla_\nu]A^\rho = R_{\mu\nu}{}^\rho{}_\sigma A^\sigma, \tag{A.27b}$$

$$[\nabla_\mu, \nabla_\nu]\psi = \frac{1}{4}\,R_{\mu\nu ab}\gamma^{ab}\psi. \tag{A.27c}$$

When ∇_μ acts on an object with a vector index, we introduce a new covariant derivative D_μ from which the Levi–Civita connection Γ_μ for this vector has been extracted:

$$\mathrm{D}_\mu := \nabla_\mu - \Gamma_\mu. \tag{A.28}$$

In that case, the Laplacian can be written as:

$$\Delta = \frac{1}{\sqrt{g}}\,\mathrm{D}_\mu(\sqrt{g}g^{\mu\nu}\nabla_\nu). \tag{A.29}$$

Similarly, the divergence of a vector density reads:

$$\nabla_\mu(\sqrt{g}v^\mu) = \mathrm{D}_\mu(\sqrt{g}v^\mu). \tag{A.30}$$

Given an operator D_x, the Green function $G(x, y)$ is defined as

$$D_x G(x, y) = \frac{\delta(x - y)}{\sqrt{g}} - \text{zero-modes} \tag{A.31}$$

A.3 Scalar field

Useful references are [49, 50, app. A, 51].

The scalar conformal Laplacian reads:

$$\Delta_c := \Delta - \frac{d - 2}{4(d - 1)}\,R. \tag{A.32}$$

The geodesic distance can be used to extract the singularity of the scalar Green function [35, p. 15]:

$$G(x, y) = G_{\text{reg}}(x, y) - \frac{1}{4\pi}\,\ln \ell(x, y)^2, \qquad \Delta_x G_{\text{reg}}(x, y) = 0. \tag{A.33}$$

This relation is still true if one removes a finite number of modes from the Green functions, and hence the singularity is related to the large number behavior of modes and eigenvalues.

A.4 Fermion field

We have the relation:

$$\gamma^{\mu\nu} R_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} = -R. \tag{A.34}$$

The covariant derivative reads:

$$\boxed{\nabla_\mu \psi := \left(\partial_\mu + \frac{1}{4}\, \omega_{\mu ab} \gamma^{ab}\right) \psi.} \tag{A.35}$$

The covariant derivative of gamma matrices vanishes:

$$\boxed{0 = \nabla_\mu \gamma^\nu = \partial_\mu \gamma^\nu + \Gamma^\nu_{\mu\rho} \gamma^\rho + \frac{1}{4}\, \omega_{\mu ab} [\gamma^{ab}, \gamma^\nu].} \tag{A.36}$$

The square of the curved Dirac operator $\not\nabla$ is

$$\boxed{\not\nabla^2 = \Delta - \frac{R}{4}} \tag{A.37}$$

since

$$\begin{aligned}\not\nabla^2 \psi &= \gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu \psi = (g^{\mu\nu} + \gamma^{\mu\nu}) \nabla_\mu \nabla_\nu \psi \\ &= \Delta \psi + \gamma^{\mu\nu} [\nabla_\mu, \nabla_\nu] \psi = \Delta \psi + \frac{1}{4}\, \gamma^{\mu\nu} R_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} \psi,\end{aligned}$$

and using (A.34).

Note that one has:

$$\begin{aligned}\not\partial^2 &= \gamma^\mu \partial_\mu (\gamma^\nu \partial_\nu) = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + \gamma^\mu (\partial_\mu \gamma^\nu) \partial_\nu \\ &= \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - \gamma^\mu \gamma^\rho \Gamma^\nu_{\mu\rho} \partial_\nu - \frac{1}{2}\, \gamma^\mu \omega_\mu \gamma_* \gamma^\nu \partial_\nu \\ &= g^{\mu\nu} (\partial_\mu + \Gamma_\mu) \partial_\nu - \frac{1}{2}\, \gamma^\mu \omega_\mu \gamma_* \gamma^\nu \partial_\nu.\end{aligned}$$

The third equality follows from (A.36), the fourth from the index symmetry of the partial derivatives and Christoffel connection which leads to the symmetrization of the gamma matrices.

We want to express Δ more explicitly:

$$\begin{aligned}\Delta &= g^{\mu\nu} \partial_\mu \partial_\nu + a^\mu \partial_\mu + b, \\ a^\mu &:= -g^{\nu\rho} \Gamma^\mu{}_{\nu\rho} + \frac{1}{2}\, \omega^\mu \gamma_*, \\ b &:= \frac{1}{16}\, g^{\mu\nu} \omega_\mu \omega_\nu + \frac{1}{4}\, \partial^\mu \omega_\mu \gamma_* - \frac{1}{4}\, g^{\nu\rho} \Gamma^\mu{}_{\nu\rho} \omega_\mu \gamma_*,\end{aligned} \tag{A.38}$$

which follows from:

$$\begin{aligned}\Delta \psi &= g^{\mu\nu} \nabla_\mu \left(\partial_\nu + \frac{1}{4}\, \omega_\nu \gamma_*\right) \psi \\ &= g^{\mu\nu} \left(\partial_\mu + \frac{1}{4}\, \omega_\mu \gamma_*\right) \left(\partial_\nu + \frac{1}{4}\, \omega_\nu \gamma_*\right) \psi - g^{\mu\nu} \Gamma^\rho{}_{\mu\nu} \left(\partial_\rho + \frac{1}{4}\, \omega_\rho \gamma_*\right) \psi.\end{aligned}$$

Another way to write the Laplacian is to write it as a combination of the identity and γ_*:

$$\Delta = \tilde{\Delta}_0\, 1_2 + \Delta_*\, \gamma_*, \qquad \tilde{\Delta}_0 := \Delta_0 + \frac{\omega^2}{16}, \qquad \Delta_* := \frac{1}{2}\,\omega\cdot\partial + \frac{1}{4}\,(\partial\cdot\omega) - \frac{1}{4}\, g^{\nu\rho}\Gamma^{\mu}{}_{\nu\rho}\omega_\mu, \tag{A.39}$$

where $\Delta_0 = g^{\mu\nu}\,\nabla_\mu\,\partial_\nu$ is the Laplacian scalar.

A quadratic functional integral (in Euclidean signature) for a Dirac fermion ψ leads to:

$$\int \mathrm{d}\psi\mathrm{d}\bar{\psi}\, \mathrm{e}^{-\langle\bar{\psi}, M\psi\rangle} = \det M, \tag{A.40}$$

while for a Majorana fermion it leads to

$$\int \mathrm{d}\psi\, \mathrm{e}^{-\frac{1}{2}\langle\bar{\psi}, M\psi\rangle} = \sqrt{\det M}. \tag{A.41}$$

The effective actions are defined by

$$W_D = -\det\ln M, \qquad W_M = -\frac{1}{2}\,\det\ln M. \tag{A.42}$$

Two dimensions All formulas in this subsection are in Lorentzian space. To obtain the formulas in Euclidean space, it is sufficient to rescale $\gamma_* \to -\mathrm{i}\gamma_*$.

The covariant derivative (A.35) becomes:

$$\boxed{\nabla_\mu\,\psi = \left(\partial_\mu + \frac{1}{4}\,\omega_\mu\gamma_*\right)\psi, \qquad \omega_\mu := \omega_{\mu ab}\epsilon^{ab},} \tag{A.43}$$

using the relation (A.44).

We have the following relation:

$$\gamma^{\mu\nu} = \epsilon^{\mu\nu}\gamma_*. \tag{A.44}$$

Equation (A.36) is equivalent to:

$$\boxed{0 = \nabla_\mu\,\gamma^\nu = \partial_\mu\gamma^\nu + \Gamma^\nu_{\mu\rho}\gamma^\rho + \frac{1}{4}\,\omega_\mu\gamma_*\gamma^\nu.} \tag{A.45}$$

We have:

$$\boxed{\partial_\mu(\sqrt{g}\gamma^\mu) = \frac{\sqrt{g}}{4}\,\not{\omega}\gamma_*} \tag{A.46}$$

since

$$\partial_\mu(\sqrt{g}\gamma^\mu) = (\partial_\mu + \Gamma_\mu)(\sqrt{g}\gamma^\mu) = -\frac{\sqrt{g}}{4}\,\omega_\mu\gamma_*\gamma^\mu = \frac{\sqrt{g}}{4}\,\not{\omega}\gamma_*$$

where where the first equality follows from (A.30) and the second from (A.36).

This implies:

$$\not{\nabla} = \not{\partial} + \frac{1}{4\sqrt{g}}\,\partial_\mu(\sqrt{g}\gamma^\mu) \tag{A.47}$$

following from

A.5 Conformal variations

We provide formulas for a Weyl transformation of the metric:

$$g_{\mu\nu} = \mathrm{e}^{2\phi}\hat{g}_{\mu\nu}, \qquad e^a_\mu = \mathrm{e}^{\phi}\hat{e}^a_\mu. \tag{A.48}$$

For references, see for instance [52, app. D, 53, app. G, 16, app. A, 11, sec. 4, 48, sec. 5.3, app. A].

Gamma matrices and geometry tensors transform as [52, app. D, 11, sec. 4, 16, app. A] (see also [53, app. G]):

$$\sqrt{g} = \mathrm{e}^{D\phi}\sqrt{\hat{g}}, \tag{A.49a}$$

$$\gamma^\mu = \mathrm{e}^{-\phi}\hat{\gamma}^\mu, \tag{A.49b}$$

$$\Gamma_{\mu\nu}{}^{\rho} = \hat{\Gamma}_{\mu\nu}{}^{\rho} + \delta^\rho_\mu\,\partial_\nu\phi + \delta^\rho_\nu\,\partial_\mu\phi - \hat{g}_{\mu\nu}\,\partial^\rho\phi, \tag{A.49c}$$

$$\omega_{\mu ab} = \hat{\omega}_{\mu ab} + \hat{e}_{\mu a}\hat{e}^\nu_b\partial_\nu\phi - \hat{e}_{\mu b}\hat{e}^\nu_a\partial_\nu\phi, \tag{A.49d}$$

$$\begin{aligned} R_{\mu\nu}{}^{\rho}{}_{\sigma} = \hat{R}_{\mu\nu}{}^{\rho}{}_{\sigma} &+ \delta^\rho_\nu\,\hat{\nabla}_\mu\partial_\sigma\phi - \hat{g}_{\nu\sigma}\,\hat{\nabla}_\mu\partial^\rho\phi - \delta^\rho_\mu\,\hat{\nabla}_\nu\partial_\sigma\phi + \hat{g}_{\mu\sigma}\,\hat{\nabla}_\nu\partial^\rho\phi \\ &+ \delta^\rho_\mu\,\partial_\nu\phi\partial_\sigma\phi - \delta^\rho_\nu\,\partial_\mu\phi\partial_\sigma\phi + \hat{g}_{\nu\sigma}\,\partial_\mu\phi\partial^\rho\phi - \hat{g}_{\mu\sigma}\,\partial_\nu\phi\partial^\rho\phi \\ &+ \delta^\rho_\nu\hat{g}_{\mu\sigma}(\partial\phi)^2 - \delta^\rho_\mu\hat{g}_{\nu\sigma}(\partial\phi)^2. \end{aligned} \tag{A.49e}$$

$$R_{\mu\nu} = \hat{R}_{\mu\nu} + (d-2)\Big(\partial_\mu\phi\partial_\nu\phi - \hat{g}_{\mu\nu}(\partial\phi)^2 - \hat{\nabla}_\mu\partial_\nu\phi\Big) - \hat{g}_{\mu\nu}\hat{\Delta}\phi, \tag{A.49f}$$

$$R = \mathrm{e}^{-2\phi}\Big(\hat{R} - (d-1)(d-2)(\partial\phi)^2 - 2(d-1)\hat{\Delta}\phi\Big), \tag{A.49g}$$

$$G_{\mu\nu} = \hat{G}_{\mu\nu} + (d-2)\left[\partial_\mu\phi\partial_\nu\phi + \frac{d-3}{2}\,\hat{g}_{\mu\nu}(\partial\phi)^2 - \hat{\nabla}_\mu\partial_\nu\phi + \hat{g}_{\mu\nu}\hat{\Delta}\phi\right]. \tag{A.49h}$$

The gamma matrix transformation follows from:

$$\gamma^\mu = e^\mu_a\gamma^a = \mathrm{e}^{-\phi}\,\hat{e}^\mu_a\gamma^a = \mathrm{e}^{-\phi}\hat{\gamma}^\mu.$$

For the affine connection, we get:

$$\begin{aligned} \Gamma^\rho_{\mu\nu} &= \frac{1}{2}\,g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\ &= \hat{\Gamma}^\rho_{\mu\nu} + \frac{1}{2}\,\mathrm{e}^{-2\phi}\hat{g}^{\rho\sigma}\left(\hat{g}_{\nu\sigma}\partial_\mu\mathrm{e}^{2\phi} + \hat{g}_{\mu\sigma}\partial_\nu\mathrm{e}^{2\phi} - \hat{g}_{\mu\nu}\partial_\sigma\mathrm{e}^{2\phi}\right) \\ &= \hat{\Gamma}^\rho_{\mu\nu} + \delta^\rho_\nu\,\partial_\mu\phi + \delta^\rho_\mu\,\partial_\nu\phi - \hat{g}_{\mu\nu}\,\partial^\rho\phi. \end{aligned}$$

Then, (A.19) can be used to find the transformation of the spin connection:

$$\begin{aligned} \omega_\mu{}^a{}_b &= e^\nu_b\left(e^a_\rho\Gamma^\rho_{\mu\nu} - \partial_\mu e^a_\nu\right) \\ &= \mathrm{e}^{-\phi}\hat{e}^\nu_b\Big(\mathrm{e}^{\phi}\hat{e}^a_\rho(\hat{\Gamma}^\rho_{\mu\nu} + \delta^\rho_\nu\,\partial_\mu\phi + \delta^\rho_\mu\,\partial_\nu\phi - \hat{g}_{\mu\nu}\,\partial^\rho\phi) - \partial_\mu(\mathrm{e}^{\phi}\hat{e}^a_\nu)\Big) \\ &= \hat{\omega}_\mu{}^a{}_b + \hat{e}^\nu_b\left(\cancel{\hat{e}^a_\nu\,\partial_\mu\phi} + \hat{e}^a_\mu\,\partial_\nu\phi - \hat{e}^a_\rho\hat{g}_{\mu\nu}\,\partial^\rho\phi - \cancel{\hat{e}^a_\nu\,\partial_\mu\phi}\right) \\ &= \hat{\omega}_\mu{}^a{}_b + \hat{e}^a_\mu\hat{e}^\nu_b\,\partial_\nu\phi - \hat{e}^\nu_b\eta_{cd}\hat{e}^c_\mu\hat{e}^d_\nu\,\hat{e}^{a\rho}\,\partial_\rho\phi \\ &= \hat{\omega}_\mu{}^a{}_b + \hat{e}^a_\mu\hat{e}^\nu_b\,\partial_\nu\phi - \hat{e}_{b\mu}\hat{e}^{a\rho}\,\partial_\rho\phi. \end{aligned}$$

For the curvature, we have:

$$
\begin{aligned}
R_{\mu\nu}{}^{\rho}{}_{\sigma} &= \hat{R}_{\mu\nu}{}^{\rho}{}_{\sigma} + \partial_\mu\big(\delta^\rho_\sigma\,\partial_\nu\phi + \delta^\rho_\nu\,\partial_\sigma\phi - \hat{g}_{\nu\sigma}\,\partial^\rho\phi\big) - \partial_\nu\big(\delta^\rho_\sigma\,\partial_\mu\phi + \delta^\rho_\mu\,\partial_\sigma\phi - \hat{g}_{\mu\sigma}\,\partial^\rho\phi\big) \\
&\quad + \hat{\Gamma}^\rho_{\mu\tau}\big(\delta^\tau_\sigma\,\partial_\nu\phi + \delta^\tau_\nu\,\partial_\sigma\phi - \hat{g}_{\nu\sigma}\,\partial^\tau\phi\big) + \hat{\Gamma}^\tau_{\nu\sigma}\big(\delta^\rho_\tau\,\partial_\mu\phi + \delta^\rho_\mu\,\partial_\tau\phi - \hat{g}_{\mu\tau}\,\partial^\rho\phi\big) \\
&\quad + \big(\delta^\tau_\sigma\,\partial_\nu\phi + \delta^\tau_\nu\,\partial_\sigma\phi - \hat{g}_{\nu\sigma}\,\partial^\tau\phi\big)\big(\delta^\rho_\tau\,\partial_\mu\phi + \delta^\rho_\mu\,\partial_\tau\phi - \hat{g}_{\mu\tau}\,\partial^\rho\phi\big) \\
&\quad - \hat{\Gamma}^\rho_{\nu\tau}\big(\delta^\tau_\sigma\,\partial_\mu\phi + \delta^\tau_\mu\,\partial_\sigma\phi - \hat{g}_{\mu\sigma}\,\partial^\tau\phi\big) - \hat{\Gamma}^\tau_{\mu\sigma}\big(\delta^\rho_\tau\,\partial_\nu\phi + \delta^\rho_\nu\,\partial_\tau\phi - \hat{g}_{\nu\tau}\,\partial^\rho\phi\big) \\
&\quad - \big(\delta^\tau_\sigma\,\partial_\mu\phi + \delta^\tau_\mu\,\partial_\sigma\phi - \hat{g}_{\mu\sigma}\,\partial^\tau\phi\big)\big(\delta^\rho_\tau\,\partial_\nu\phi + \delta^\rho_\nu\,\partial_\tau\phi - \hat{g}_{\nu\tau}\,\partial^\rho\phi\big) \\
&= \hat{R}_{\mu\nu}{}^{\rho}{}_{\sigma} + \cancel{\delta^\rho_\sigma\,\partial_\mu\partial_\nu\phi} + \delta^\rho_\nu\,\partial_\mu\partial_\sigma\phi - \hat{g}_{\nu\sigma}\,\partial_\mu\partial^\rho\phi - \cancel{\delta^\rho_\sigma\,\partial_\mu\partial_\nu\phi} - \delta^\rho_\mu\,\partial_\nu\partial_\sigma\phi + \hat{g}_{\mu\sigma}\,\partial_\nu\partial^\rho\phi \\
&\quad + \big(\partial_\nu\hat{g}_{\mu\sigma} - \partial_\mu\hat{g}_{\nu\sigma}\big)\partial^\rho\phi \\
&\quad + \cancel{\hat{\Gamma}^\rho_{\mu\sigma}\,\partial_\nu\phi} + \cancel{\hat{\Gamma}^\rho_{\mu\nu}\,\partial_\sigma\phi} - \hat{g}_{\nu\sigma}\hat{\Gamma}^\rho_{\mu\tau}\,\partial^\tau\phi + \cancel{\hat{\Gamma}^\rho_{\nu\sigma}\,\partial_\mu\phi} + \delta^\rho_\mu\hat{\Gamma}^\tau_{\nu\sigma}\,\partial_\tau\phi - \hat{g}_{\mu\tau}\hat{\Gamma}^\tau_{\nu\sigma}\,\partial^\rho\phi \\
&\quad - \cancel{\hat{\Gamma}^\rho_{\nu\sigma}\,\partial_\mu\phi} - \cancel{\hat{\Gamma}^\rho_{\nu\mu}\,\partial_\sigma\phi} + \hat{g}_{\mu\sigma}\hat{\Gamma}^\rho_{\nu\tau}\,\partial^\tau\phi - \cancel{\hat{\Gamma}^\rho_{\mu\sigma}\,\partial_\nu\phi} - \delta^\rho_\nu\hat{\Gamma}^\tau_{\mu\sigma}\,\partial_\tau\phi + \hat{g}_{\nu\tau}\hat{\Gamma}^\tau_{\mu\sigma}\,\partial^\rho\phi \\
&\quad + \cancel{\delta^\rho_\sigma\,\partial_\mu\phi\partial_\nu\phi} + \cancel{\delta^\rho_\nu\,\partial_\mu\phi\partial_\sigma\phi} - \cancel{\hat{g}_{\nu\sigma}\,\partial_\mu\phi\partial^\rho\phi} + \cancel{\delta^\rho_\mu\,\partial_\nu\phi\partial_\sigma\phi} + \delta^\rho_\mu\,\partial_\nu\phi\partial_\sigma\phi \\
&\quad - \delta^\rho_\mu\hat{g}_{\nu\sigma}\,(\partial\phi)^2 - \cancel{\hat{g}_{\mu\sigma}\,\partial_\nu\phi\partial^\rho\phi} - \cancel{\hat{g}_{\mu\nu}\,\partial^\rho\phi\partial_\sigma\phi} + \cancel{\hat{g}_{\nu\sigma}\,\partial_\mu\phi\partial^\rho\phi} \\
&\quad - \cancel{\delta^\rho_\sigma\,\partial_\mu\phi\partial_\nu\phi} - \cancel{\delta^\rho_\mu\,\partial_\nu\phi\partial_\sigma\phi} + \cancel{\hat{g}_{\mu\sigma}\,\partial_\nu\phi\partial^\rho\phi} - \cancel{\delta^\rho_\nu\,\partial_\mu\phi\partial_\sigma\phi} - \delta^\rho_\nu\,\partial_\mu\phi\partial_\sigma\phi \\
&\quad + \delta^\rho_\nu\hat{g}_{\mu\sigma}\,(\partial\phi)^2 + \hat{g}_{\nu\sigma}\,\partial_\mu\phi\partial^\rho\phi + \cancel{\hat{g}_{\mu\nu}\,\partial^\rho\phi\partial_\sigma\phi} - \hat{g}_{\mu\sigma}\,\partial_\nu\phi\partial^\rho\phi \\
&= \hat{R}_{\mu\nu}{}^{\rho}{}_{\sigma} + \delta^\rho_\nu\,\partial_\mu\partial_\sigma\phi - \hat{g}_{\nu\sigma}\,\partial_\mu\partial^\rho\phi - \delta^\rho_\mu\,\partial_\nu\partial_\sigma\phi + \hat{g}_{\mu\sigma}\,\partial_\nu\partial^\rho\phi \\
&\quad + \delta^\rho_\mu\,\partial_\nu\phi\partial_\sigma\phi - \delta^\rho_\nu\,\partial_\mu\phi\partial_\sigma\phi + \hat{g}_{\nu\sigma}\,\partial_\mu\phi\partial^\rho\phi - \hat{g}_{\mu\sigma}\,\partial_\nu\phi\partial^\rho\phi \\
&\quad + \delta^\rho_\nu\hat{g}_{\mu\sigma}(\partial\phi)^2 - \delta^\rho_\mu\hat{g}_{\nu\sigma}(\partial\phi)^2 + (\hat{g}_{\mu\sigma}\hat{\Gamma}^\rho_{\nu\tau} - \hat{g}_{\nu\sigma}\hat{\Gamma}^\rho_{\mu\tau})\partial^\tau\phi + (\delta^\rho_\mu\hat{\Gamma}^\tau_{\nu\sigma} - \delta^\rho_\nu\hat{\Gamma}^\tau_{\mu\sigma})\partial_\tau\phi \\
&\quad + (\partial_\nu\hat{g}_{\mu\sigma} + \hat{\Gamma}^\tau_{\mu\sigma}\hat{g}_{\nu\tau} - \partial_\mu\hat{g}_{\nu\sigma} - \hat{\Gamma}^\tau_{\nu\sigma}\hat{g}_{\mu\tau})\partial^\rho\phi \\
&= \hat{R}_{\mu\nu}{}^{\rho}{}_{\sigma} + \delta^\rho_\nu\,\hat{\nabla}_\mu\partial_\sigma\phi - \hat{g}_{\nu\sigma}\,\hat{\nabla}_\mu\partial^\rho\phi - \delta^\rho_\mu\,\hat{\nabla}_\nu\partial_\sigma\phi + \hat{g}_{\mu\sigma}\,\hat{\nabla}_\nu\partial^\rho\phi \\
&\quad + \delta^\rho_\mu\,\partial_\nu\phi\partial_\sigma\phi - \delta^\rho_\nu\,\partial_\mu\phi\partial_\sigma\phi + \hat{g}_{\nu\sigma}\,\partial_\mu\phi\partial^\rho\phi - \hat{g}_{\mu\sigma}\,\partial_\nu\phi\partial^\rho\phi \\
&\quad + \delta^\rho_\nu\hat{g}_{\mu\sigma}(\partial\phi)^2 - \delta^\rho_\mu\hat{g}_{\nu\sigma}(\partial\phi)^2.
\end{aligned}
$$

The last line at the penultimate step vanishes because:

$$\hat{g}_{\nu\tau}\hat{\Gamma}^\tau_{\mu\sigma} - \hat{g}_{\mu\tau}\hat{\Gamma}^\tau_{\nu\sigma} = \partial_\mu\hat{g}_{\nu\sigma} - \partial_\nu\hat{g}_{\mu\sigma}.$$

Contracting ρ and μ, we get:

$$R_{\nu\sigma} = \hat{R}_{\nu\sigma} + (2-d)\hat{\nabla}_\nu\partial_\sigma\phi - \hat{g}_{\nu\sigma}\,\hat{\Delta}\phi + (d-2)\,\partial_\nu\phi\partial_\sigma\phi + \hat{g}_{\nu\sigma}(\partial\phi)^2 + (1-d)\hat{g}_{\nu\sigma}\,(\partial\phi)^2.$$

Finally, we compute the Einstein tensor transformation:

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R \\
&= \hat{R}_{\mu\nu} + (d-2)\Big(\partial_\mu\phi\partial_\nu\phi - \hat{g}_{\mu\nu}(\partial\phi)^2 - \hat{\nabla}_\mu\partial_\nu\phi\Big) - \hat{g}_{\mu\nu}\hat{\Delta}\phi \\
&\quad - \frac{1}{2}\,\hat{g}_{\mu\nu}\Big(\hat{R} - (d-1)(d-2)(\partial\phi)^2 - 2(d-1)\hat{\Delta}\phi\Big) \\
&= \hat{G}_{\mu\nu} + (d-2)(\partial_\mu\phi\partial_\nu\phi - \hat{\nabla}_\mu\partial_\nu\phi) - (d-2)\hat{g}_{\mu\nu}(\partial\phi)^2 - \hat{g}_{\mu\nu}\hat{\Delta}\phi \\
&\quad + \frac{(d-1)}{2}(d-2)\hat{g}_{\mu\nu}(\partial\phi)^2 + (d-1)\hat{g}_{\mu\nu}\hat{\Delta}\phi \\
&= \hat{G}_{\mu\nu} + (d-2)(\partial_\mu\phi\partial_\nu\phi - \hat{\nabla}_\mu\partial_\nu\phi) + (d-2)\hat{g}_{\mu\nu}\hat{\Delta}\phi + \frac{(d-3)}{2}(d-2)\hat{g}_{\mu\nu}(\partial\phi)^2.
\end{aligned}
$$

Covariant derivatives of fields transform as [11, sec. 4]:

$$\nabla_\mu f = \hat{\nabla}_\mu f, \tag{A.50a}$$

$$\nabla_\mu A^\rho = \hat{\nabla}_\mu A^\rho + \delta^\rho_\mu A^\nu \partial_\nu \phi + A^\rho \partial_\mu \phi - A_\mu \partial^\rho \phi, \tag{A.50b}$$

$$\nabla_\mu A_\nu = \hat{\nabla}_\mu A_\nu - A_\mu \partial_\nu \phi - A_\nu \partial_\mu \phi + \hat{g}_{\mu\nu} A_\rho \partial^\rho \phi, \tag{A.50c}$$

$$\nabla_\mu \psi = \hat{\nabla}_\mu \psi + \frac{1}{2} \hat{\gamma}_{\mu\nu} (\partial^\nu \phi) \psi, \tag{A.50d}$$

$$\begin{aligned} \not{\nabla} \psi &= \mathrm{e}^{-\phi} \left(\hat{\not{\nabla}} + \frac{d-1}{2} (\hat{\not{\partial}} \phi) \right) \psi \\ &= \mathrm{e}^{-\frac{d+1}{2}\phi} \hat{\not{\nabla}} (\mathrm{e}^{\frac{d-1}{2}\phi} \psi). \end{aligned} \tag{A.50e}$$

The transformations for the Laplacians are [11, sec. 4]:

$$\Delta f = \mathrm{e}^{-2\phi} \left(\hat{\Delta} + (d-2)\, \partial_\mu \phi \partial^\mu \right) f, \tag{A.51a}$$

$$\begin{aligned} \Delta_c f &= \mathrm{e}^{-2\phi} \left[\hat{\Delta}_c + \frac{d-2}{4} (4\, \partial \phi \cdot \partial + (d-2)\, (\partial \phi)^2 + 2\, \hat{\Delta} \phi) \right] f \\ &= \mathrm{e}^{-\frac{d+2}{2}} \hat{\Delta}_c (\mathrm{e}^{\frac{d-2}{2}} f) \end{aligned} \tag{A.51b}$$

$$\begin{aligned} \Delta \psi &= \mathrm{e}^{-2\phi} \left[\hat{\Delta} + (d-2)(\partial^\mu \phi) \hat{\nabla}_\mu + \hat{\gamma}^{\mu\nu} (\partial_\nu \phi) \hat{\nabla}_\mu - \frac{d-1}{4} (\partial \phi)^2 \right] \psi \\ &= \mathrm{e}^{-2\phi} \left[\hat{\Delta} + (d-1)(\partial^\mu \phi) \hat{\nabla}_\mu - (\hat{\not{\partial}} \phi) \hat{\not{\nabla}}_\mu - \frac{d-1}{4} (\partial \phi)^2 \right] \psi, \end{aligned} \tag{A.51c}$$

$$\not{\nabla}^2 \psi = \mathrm{e}^{-2\phi} \left[\hat{\not{\nabla}}^2 + (d-1)(\partial^\mu \phi) \hat{\nabla}_\mu - (\hat{\not{\partial}} \phi) \hat{\not{\nabla}} + \frac{(d-1)(d-3)}{4} (\partial \phi)^2 + \frac{d-1}{2} \hat{\Delta} \phi \right] \psi. \tag{A.51d}$$

For a vector field, we have:

$$\begin{aligned} \nabla_\mu A^\rho &= \partial_\mu A^\rho + \Gamma_{\mu\nu}{}^\rho A^\nu \\ &= \partial_\mu A^\rho + \left(\hat{\Gamma}_{\mu\nu}{}^\rho + \delta^\rho_\mu \partial_\nu \phi + \delta^\rho_\nu \partial_\mu \phi - \hat{g}_{\mu\nu} \partial^\rho \phi \right) A^\nu, \\ &= \hat{\nabla}_\mu A^\rho + \delta^\rho_\mu A^\nu \partial_\nu \phi + A^\rho \partial_\mu \phi - A_\mu \partial^\rho \phi. \end{aligned}$$

and similarly for a 1-form:

$$\begin{aligned} \nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma_{\mu\nu}{}^\rho A_\rho \\ &= \partial_\mu A_\nu + \left(\hat{\Gamma}_{\mu\nu}{}^\rho - \delta^\rho_\mu \partial_\nu \phi - \delta^\rho_\nu \partial_\mu \phi + \hat{g}_{\mu\nu} \partial^\rho \phi \right) A_\rho \\ &= \hat{\nabla}_\mu A_\nu - A_\mu \partial_\nu \phi - A_\nu \partial_\mu \phi + \hat{g}_{\mu\nu} A_\rho \partial^\rho \phi. \end{aligned}$$

For a spinor field, we have:

$$\begin{aligned} \nabla_\mu \psi &= \partial_\mu \psi + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \psi \\ &= \partial_\mu \psi + \frac{1}{4} (\hat{\omega}_{\mu ab} + \hat{e}_{\mu a} \hat{e}^\nu_b \partial_\nu \phi - \hat{e}_{\mu b} \hat{e}^\nu_a \partial_\nu \phi) \gamma^{ab} \psi \\ &= \hat{\nabla}_\mu \psi + \frac{1}{2} \hat{\gamma}_{\mu\nu} (\partial^\nu \phi) \psi. \end{aligned}$$

Contracting with γ^μ, we get [11, sec. 4, 37, sec. 1.4]:

$$\begin{aligned}
\nabla\!\!\!/\,\psi &= \partial_\mu\psi + \frac{1}{4}\,\omega_{\mu ab}\gamma^{ab}\psi \\
&= \partial_\mu\psi + \frac{1}{4}\,(\hat\omega_{\mu ab} + \hat e_{\mu a}\hat e^\nu_b\partial_\nu\phi - \hat e_{\mu b}\hat e^\nu_a\partial_\nu\phi)\gamma^{ab}\psi \\
&= \mathrm{e}^{-\phi}\hat{\nabla\!\!\!/}\psi + \frac{\mathrm{e}^{-\phi}}{2}\,\hat\gamma^\mu\hat\gamma_{\mu\nu}(\partial^\nu\phi)\psi \\
&= \mathrm{e}^{-\phi}\hat{\nabla\!\!\!/}\psi + \frac{d-1}{2}\,\mathrm{e}^{-\phi}(\hat{\partial\!\!\!/}\phi)\psi \\
&= \mathrm{e}^{-\frac{d+1}{2}\phi}\hat{\nabla\!\!\!/}(\mathrm{e}^{\frac{d-1}{2}\phi}\psi).
\end{aligned}$$

where we used:

$$\gamma^\mu\gamma_{\mu\nu} = \gamma^\mu(\gamma_\mu\gamma_\nu - \eta_{\mu\nu}) = (d-1)\gamma_\nu.$$

The scalar Laplacian transforms as:

$$\begin{aligned}
\Delta f &= g^{\mu\nu}\,\nabla_\mu\,\partial_\nu f \\
&= \mathrm{e}^{-2\phi}\,\hat g^{\mu\nu}(\hat\nabla_\mu\partial_\nu f - \partial_\nu\phi\partial_\mu f - \partial_\mu\phi\partial_\nu f + \hat g_{\mu\nu}\,\partial^\rho\phi\partial_\rho f) \\
&= \mathrm{e}^{-2\phi}(\hat\Delta f + (d-2)\,\partial\phi\cdot\partial f).
\end{aligned}$$

The transformation of the conformal Laplacian follows immediately:

$$\begin{aligned}
\Delta_c f &= \Delta f - \frac{d-2}{4(d-1)}\,Rf \\
&= \mathrm{e}^{-2\phi}\left[\hat\Delta + (d-2)\,\partial\phi\cdot\partial - \frac{d-2}{4(d-1)}\Big(\hat R - (d-1)(d-2)(\partial\phi)^2 - 2(d-1)\hat\Delta\phi\Big)\right]f \\
&= \mathrm{e}^{-2\phi}\left[\hat\Delta_c + \frac{d-2}{4}(4\,\partial\phi\cdot\partial + (d-2)\,(\partial\phi)^2 + 2\,\hat\Delta\phi)\right]f \\
&= \mathrm{e}^{-\frac{d+2}{2}}\,\hat\Delta_c(\mathrm{e}^{\frac{d-2}{2}}f),
\end{aligned}$$

using (A.26).

This fermion Laplacian transforms as:

$$
\begin{aligned}
\Delta\psi &= g^{\mu\nu}\,\nabla_\mu\left(\hat{\nabla}_\nu\psi + \frac{1}{2}\,\hat{\gamma}_{\nu\rho}(\partial^\rho\phi)\psi\right) \\
&= g^{\mu\nu}\left(\nabla_\mu\,\hat{\nabla}_\nu\psi + \frac{1}{2}\,\hat{\gamma}_{\nu\rho}(\nabla_\mu\,\partial^\rho\phi)\psi + \frac{1}{2}\,\hat{\gamma}_{\nu\rho}(\partial^\rho\phi)\,\nabla_\mu\,\psi\right) \\
&= g^{\mu\nu}\Big[\Big(\hat{\nabla}_\mu\hat{\nabla}_\nu\psi - (\partial_\nu\phi)\hat{\nabla}_\mu\psi - (\partial_\mu\phi)\hat{\nabla}_\nu\psi + \hat{g}_{\mu\nu}\,(\partial^\rho\phi)\hat{\nabla}_\rho\psi + \frac{1}{2}\,\hat{\gamma}_{\mu\rho}(\partial^\rho\phi)\hat{\nabla}_\nu\psi\Big) \\
&\qquad + \frac{1}{2}\,\hat{\gamma}_{\nu\rho}\Big(\hat{\nabla}_\mu\partial^\rho\phi + \delta^\rho_\mu\,\partial_\sigma\phi\partial^\sigma\phi + \cancel{\partial_\mu\phi\partial^\rho\phi} - \cancel{\partial_\mu\phi\partial^\rho\phi}\Big)\psi \\
&\qquad + \frac{1}{2}\,\hat{\gamma}_{\nu\rho}(\partial^\rho\phi)\left(\hat{\nabla}_\mu\psi + \frac{1}{2}\,\hat{\gamma}_{\mu\sigma}(\partial^\sigma\phi)\psi\right)\Big] \\
&= \mathrm{e}^{-2\phi}\Big[\hat{\Delta}\psi + (d-2)(\partial^\mu\phi)\hat{\nabla}_\mu\psi + \frac{1}{2}\,\hat{\gamma}^{\mu\nu}(\partial_\nu\phi)\hat{\nabla}_\mu\psi \\
&\qquad + \frac{1}{2}\,\Big(\cancel{\hat{\gamma}^{\mu\nu}\hat{\nabla}_\mu\partial_\nu\phi} - \cancel{\hat{g}_{\mu\nu}\hat{\gamma}^{\mu\nu}\,(\partial\phi)^2}\Big)\psi \\
&\qquad + \frac{1}{2}\,\hat{\gamma}^{\mu\nu}(\partial_\nu\phi)\hat{\nabla}_\mu\psi + \frac{1}{4}\,\hat{g}_{\mu\nu}\hat{\gamma}^{\nu\rho}\hat{\gamma}^{\mu\sigma}(\partial^\rho\phi)(\partial^\sigma\phi)\psi\Big] \\
&= \mathrm{e}^{-2\phi}\Big[\hat{\Delta}\psi + (d-2)(\partial^\mu\phi)\hat{\nabla}_\mu\psi + \hat{\gamma}^{\mu\nu}(\partial_\nu\phi)\hat{\nabla}_\mu\psi \\
&\qquad - \frac{1}{4}\,((d-2)\gamma^\rho\gamma^\sigma + \eta^{\rho\sigma})(\partial^\rho\phi)(\partial^\sigma\phi)\psi\Big] \\
&= \mathrm{e}^{-2\phi}\left[\hat{\Delta}\psi + (d-2)(\partial^\mu\phi)\hat{\nabla}_\mu\psi + \hat{\gamma}^{\mu\nu}(\partial_\nu\phi)\hat{\nabla}_\mu\psi - \frac{d-2}{4}\,(\not{\partial}\phi)^2\psi - \frac{1}{4}(\partial\phi)^2\psi\right] \\
&= \mathrm{e}^{-2\phi}\left[\hat{\Delta}\psi + (d-2)(\partial^\mu\phi)\hat{\nabla}_\mu\psi + \hat{\gamma}^{\mu\nu}(\partial_\nu\phi)\hat{\nabla}_\mu\psi - \frac{d-1}{4}\,(\partial\phi)^2\psi\right].
\end{aligned}
$$

We have used:

$$
\begin{aligned}
\gamma_{\mu\rho}\gamma^{\mu\sigma} &= -\gamma_{\rho\mu}\gamma^{\mu\sigma} = -(\gamma_\rho\gamma_\mu - \eta_{\mu\rho})\gamma^{\mu\sigma} \\
&= -(d-1)\gamma_\rho\gamma^\sigma + \eta_{\mu\rho}(\gamma^\mu\gamma^\sigma - \eta^{\mu\sigma}) \\
&= -(d-1)\gamma_\rho\gamma^\sigma + \gamma_\rho\gamma^\sigma - \delta^\sigma_\rho \\
&= -(d-2)\gamma_\rho\gamma^\sigma - \delta^\sigma_\rho,
\end{aligned}
$$

and $(\not{\partial}\phi)^2 = (\partial\phi)^2$.

We can then consider the square of the Dirac operator:

$$
\begin{aligned}
\nabla\!\!\!/^2 \psi &= \left(\Delta - \frac{R}{4}\right)\psi \\
&= \mathrm{e}^{-2\phi}\left[\hat\Delta\psi + (d-1)(\partial^\mu\phi)\hat\nabla_\mu\psi - (\hat{\partial\!\!\!/}\phi)\hat{\nabla\!\!\!/}_\mu\psi - \frac{d-1}{4}\,(\partial\phi)^2\psi\right] \\
&\quad - \frac{\mathrm{e}^{-2\phi}}{4}\Big(\hat R - (d-1)(d-2)(\partial\phi)^2 - 2(d-1)\hat\Delta\phi\Big)\psi \\
&= \mathrm{e}^{-2\phi}\hat{\nabla\!\!\!/}^2\psi + \mathrm{e}^{-2\phi}\bigg[(d-1)(\partial^\mu\phi)\hat\nabla_\mu\psi - (\hat{\partial\!\!\!/}\phi)\hat{\nabla\!\!\!/}\psi - \frac{d-1}{4}\,(\partial\phi)^2\psi \\
&\qquad\qquad + \frac{(d-1)(d-2)}{4}\,(\partial\phi)^2\psi + \frac{d-1}{2}\,(\hat\Delta\phi)\psi\bigg] \\
&= \mathrm{e}^{-2\phi}\hat{\nabla\!\!\!/}^2\psi + \mathrm{e}^{-2\phi}\bigg[(d-1)(\partial^\mu\phi)\hat\nabla_\mu\psi - (\hat{\partial\!\!\!/}\phi)\hat{\nabla\!\!\!/}\psi \\
&\qquad\qquad + \frac{(d-1)(d-3)}{4}\,(\partial\phi)^2\psi + \frac{d-1}{2}\,(\hat\Delta\phi)\psi\bigg].
\end{aligned}
$$

A.6 Geometrical functionals

The metric $g_{\mu\nu}$, background metric $\hat g_{\mu\nu}$ and Liouville mode (conformal factor) ϕ are related as:

$$g_{\mu\nu} = \mathrm{e}^{2\phi}\hat g_{\mu\nu}. \tag{A.52}$$

The Kähler potential K is related to the Liouville field as [54, sec. 2.1]:

$$K(x) := -\frac{2}{A}\int \mathrm{d}^2x\sqrt{g}\,\hat G(x,y) = -\frac{2}{A}\int \mathrm{d}^2x\sqrt{\hat g}\,\hat G(x,y)\mathrm{e}^{2\phi(y)}. \tag{A.53}$$

It can also be defined as the unique solution to the equation [35]

$$\mathrm{e}^{2\phi} = \frac{A}{\hat A}\left(1 + \frac{\hat A}{2}\,\hat\Delta K\right) \tag{A.54}$$

with the condition

$$\int \mathrm{d}^2x\sqrt{\hat g}\,K(x) = 0. \tag{A.55}$$

The necessity of the first term in the RHS of (A.54) can be seen by integrating over the surface. [HE: Maybe we can replace the bosonic $G(x,y)$ by the fermionic $\operatorname{tr} G(x,y)$.] ⇐ 16

We have the relation [54, sec. 3]:

$$\int \mathrm{d}^2x\sqrt{\hat g}\,K\,\Delta\,K = \frac{4}{A^2}\int \mathrm{d}^2x\sqrt{g}\int \mathrm{d}^2x'\sqrt{g}\,\hat G(x,x'). \tag{A.56}$$

[HE: Prove it.] ⇐ 17

The variation of the Liouville mode can be written as:

$$\delta\phi = \frac{\delta A}{2A} + \frac{A}{4}\,\Delta\,\delta K, \qquad \delta\left(\frac{\mathrm{e}^{2\phi}}{A}\right) = -\frac{1}{2}\,\hat\Delta K. \tag{A.57}$$

The variations of the Liouville, Mabuchi and Aubin–Yau functionals read:

$$\delta S_{\mathrm{L}} = 2\pi\chi \, \frac{\delta A}{A} - \frac{A}{4} \int \mathrm{d}^2x \sqrt{g} \, \Delta R \, \delta K, \tag{A.58a}$$

$$\delta S_{\mathrm{M}} = \frac{2\delta A}{A} - \frac{A}{4} \int \mathrm{d}^2x \sqrt{g} \, \delta K \left(R - \frac{4\pi\chi}{A} \right), \tag{A.58b}$$

$$\delta S_{\mathrm{AY}} = \frac{1}{A} \int \mathrm{d}^2x \sqrt{g} \, \delta K. \tag{A.58c}$$

Let's guess another normalization for K:

$$\mathrm{e}^{2\phi} = \frac{A}{N_0} \, \mathrm{tr}_D \, P(x) + \frac{A}{2} \, \hat{\Delta} K. \tag{A.59}$$

Multiplying the RHS by $\sqrt{\hat{g}}$ and integrating over x gives A as necessary. Then we have:

$$\begin{aligned} 2\delta\phi \, \mathrm{e}^{2\phi} &= \delta A \left(\frac{1}{N_0} \, \mathrm{tr}_D \, P(x) + \frac{1}{2} \, \hat{\Delta} K \right) + \frac{A}{N_0} \, \mathrm{tr}_D \, \delta P(x) + \frac{A}{2} \, \hat{\Delta} \delta K \\ &= \mathrm{e}^{2\phi} \frac{\delta A}{A} + \frac{A}{N_0} \, \mathrm{tr}_D \, \delta P(x) + \frac{A}{2} \, \hat{\Delta} \delta K \end{aligned}$$

such that:

$$\delta\phi = \frac{\delta A}{2A} + \frac{A}{2N_0} \, \mathrm{e}^{-2\phi} \, \mathrm{tr}_D \, \delta P(x) + \frac{A}{2} \, \Delta \, \delta K. \tag{A.60}$$

[HE: This does not seem very useful.] $\Leftarrow$ 18

A.7 Heat kernel regularization

Given a matrix M and an elliptic differential operator D with eigenvalues $\lambda_n \geq 0$ and eigenfunctions ψ_n, the zeta function is defined by:

$$\zeta_{D,M}(s) := \mathrm{tr}'(M D^{-s}) := \sum_{n>0} \frac{1}{\lambda_n^s} \langle \psi_n | \, M \, | \psi_n \rangle \, . \tag{A.61}$$

The zero-modes are denoted by $\psi_{0,i}$ ($i = 1, \ldots, k$) and corresponds to the eigenvalue λ_0. When $M = 1$, the index is omitted. When there is no confusion, D is also omitted. For references, see for example [11].

The determinant of D is then defined by

$$\det D = \exp\big(- \zeta_D'(0) \big). \tag{A.62}$$

The interest of this formula is that it always yield a finite result.

Similarly, one can define a bi-local zeta function

$$\zeta_{D,M}(s, x, y) := \sum_{n>0} \frac{1}{\lambda_n^s} \, \psi_n(x) \psi_n(y)^\dagger. \tag{A.63}$$

The zeta function is then given by the trace:

$$\zeta_{D,M}(s) = \int \mathrm{d}^2x \sqrt{g} \, \zeta_{D,M}(s, x, x). \tag{A.64}$$

One can rewrite the zeta function thanks to a Mellin transform

$$\zeta_{D,M}(s) = \frac{1}{\Gamma(s)} \sum_{n>0} \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-\lambda_n t} \langle \psi_n | \, M \, | \psi_n \rangle \tag{A.65a}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \, \mathrm{tr}' \left(M \mathrm{e}^{-tD} \right) \tag{A.65b}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \Big[\mathrm{tr} \left(M \mathrm{e}^{-tD} \right) - \mathrm{tr} \left(M P \right) \Big]. \tag{A.65c}$$

The convergence of the series is given by Weyl's asymptotic law:

$$\lambda_n \sim C_d \left(\frac{n}{\mathrm{Vol}} \right)^{2/d}. \tag{A.66}$$

Hence, the the series converges for $s > 2/d$ and can be analytically extended: the function then has simple poles at $s = d/2, d/2 - 1, \ldots, 1$, and in particular it is analytic at $s = 0$ (for odd dimensions it has also zeros at $s \in -\mathbb{N}$).

The heat kernel is defined by:

$$K_D(t, x, y) := \mathrm{e}^{-tD}(x, y) \tag{A.67}$$

where t is called the time. This kernel satisfies the diffusion equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + D \right) K_D(t, x, y) = 0 \tag{A.68}$$

with the boundary condition

$$\lim_{t \to 0} K_D(t, x, y) = \frac{\delta(x - y)}{\sqrt{g}}. \tag{A.69}$$

The mode expansion is

$$K_D(t, x, y) = \sum_{n>0} \mathrm{e}^{-\lambda_n t} \psi_n(x) \psi_n(y)^\dagger. \tag{A.70}$$

The integrated heat kernel reads

$$K_D(t) = \int \mathrm{d}^2 x \sqrt{g}\, K_D(t, x, y) = \sum_{n>0} \mathrm{e}^{-\lambda_n t}. \tag{A.71}$$

Moreover, the integration over the time yields the Green function

$$G(x, y) = \int_0^\infty \mathrm{d}t\, K_D(t, x, y). \tag{A.72}$$

The short-distance singularity is logarithmic in two dimensions. The heat kernel is related to the zeta function by

$$K_D(t, x, y) = \int_0^\infty \mathrm{d}t\, t^{s-1} \zeta_D(t, x, y), \qquad K_D(t) = \int_0^\infty \mathrm{d}t\, t^{s-1} \zeta_D(x). \tag{A.73}$$

This kernel has a small time expansion

$$K_D(t, x, y) = \frac{1}{(4\pi t)^{d/2}} \, \mathrm{e}^{-\ell(x,y)/4t} \left(\sum_{n=0}^{N} a_n(x, y) \, t^n + O(t^{N+1}) \right) \tag{A.74}$$

where $\ell(x, y)$ is the geodesic distance between x and y. If the operator D is diagonal in position then one can evaluate the previous series at $x = y$

$$K_D(t, x) = \frac{1}{(4\pi t)^{d/2}} \left(\sum_{n=0}^{N} a_n(x)\, t^n + O(t^{N+1}) \right) \tag{A.75}$$

This implies in particular that

$$\zeta_D(0, x, y) = \frac{1}{4\pi}\, a_1(x, y), \qquad \operatorname{Res} \zeta_D(1, x, y) = \frac{1}{4\pi}\, a_0(x, y). \tag{A.76}$$

Note that we have

$$a_0(x) = 1. \tag{A.77}$$

In order to find the first coefficient $a_1(x)$ for a spinor, one can use the following argument [11, pp. 1473–74]: from dimensional analysis, the coefficient is

$$a_1(-\nabla\!\!\!/^2) = k\, R \tag{A.78}$$

where k is a constant and R is the curvature. First, one can use that

$$a_1(-\nabla\!\!\!/^2) = -a_1(\Delta_{1/2}) + \frac{R}{4} \tag{A.79}$$

and then the fact that in two dimensions the spinor Laplacian $\Delta_{1/2}$ is locally related to the scalar Laplacian Δ_0 by a similarity transformation

$$a_1(\Delta_{1/2}) = a_1(\Delta_0) + \frac{R}{6}. \tag{A.80}$$

This implies that

$$a_1(-\nabla\!\!\!/^2) = -\frac{R}{12}. \tag{A.81}$$

We can introduce a one-parameter operator [11]:

$$D_\tau^{(0)} = \frac{\sqrt{g_0}}{\sqrt{g_\tau}}\, e^{\tau f^\dagger} D_0^{(0)} e^{\tau f}, \tag{A.82}$$

where $f(x)$ is a matrix-valued function and g_τ is the corresponding metric. Two standard cases are:

- chiral transformation: $f = \gamma_*$, $g_\tau = g_0$;
- Weyl transformation: $f = \phi/2$, $g_\tau = e^{2\tau\phi} g_0$, see (A.50e).

Note that such relations cannot be written in the massive case. It is obvious that if $\psi_{0,i}$ are zero-modes of $D_\tau^{(0)}$, then

$$\psi_{i,\tau} = e^{-\tau f} \psi_{i,0} \tag{A.83}$$

are zero-modes of $D_\tau^{(0)}$, and as consequence their number is independent of τ (note that we use the subscript zero to indicate two things at the same time: that it is a zero-mode, and that it is for $\tau = 0$). The zero-modes obtained in this way are not orthonormal and we define

$$\begin{aligned}\kappa_{\tau,ij} := \langle (\psi_{i,\tau})^\dagger | \psi_{j,\tau} \rangle &= \int \mathrm{d}^2 x \sqrt{g_\tau}\, \psi_{i,\tau}(x)^\dagger \psi_{j,\tau}(x) \\ &= \int \mathrm{d}^2 x \sqrt{g_\tau}\, \psi_{i,0}(x)^\dagger \psi_{j,0}(x) \mathrm{e}^{-\tau(f+f^\dagger)}.\end{aligned} \tag{A.84}$$

Note that the derivative is

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \kappa_{\tau,ij} = - \int \mathrm{d}^2 x \sqrt{g_\tau} \psi_{i,\tau}(x)^\dagger F_\tau(x) \psi_{j,\tau}(x) \tag{A.85}$$

where

$$F_\tau(x) = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau} \ln g_\tau + f(x) + f(x)^\dagger. \tag{A.86}$$

The inverse of this matrix will be denoted with upper indices κ_τ^{ij}. The projection can then be written as

$$P_\tau(x, y) = \sum_{i,j=1}^{N_0} \psi_{i,\tau}(x) \kappa_\tau^{ij} \psi_{j,\tau}(y)^\dagger. \tag{A.87}$$

One finds the relation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \ln \det \kappa_{\tau,ij} = - \int \mathrm{d}^2 x \sqrt{g_\tau}\, F_\tau(x) P_\tau(x). \tag{A.88}$$

For $D = D_\tau^{(0)}$ defined as (A.82) one has (we removed the index τ to have cleaner expressions)

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \zeta_{D^{(0)}}(s) = -2s\, \zeta_{D^{(0)2},F}(s) \tag{A.89}$$

where F was defined in (A.86). Then this implies that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \ln \mathrm{det}' D^{(0)} = \zeta_{D^{(0)2},F}(0) = \int \mathrm{d}^2 x \sqrt{g}\, F(x) \left[\frac{1}{(4\pi)^{d/2}} a_{d/2}(x, D^{(0)2}) - P(x) \right]. \tag{A.90}$$

Combining this with (A.88)

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau} \ln \frac{\det' D^{(0)}}{\det \kappa} = \frac{1}{4\pi} \int \mathrm{d}^2 x \sqrt{g}\, F(x) a_{d/2}(x, D^{(0)2}).} \tag{A.91}$$

[HE: in [11, app. B] another regularization for the Green function is given.] ⇐ 19

A.8 Comments

In isothermal coordinates $g_{\mu\nu} = \mathrm{e}^{2\phi(z)} |\mathrm{d}z|^2$, the geodesic distance satisfies [54, sec. 2]:

$$\ell_g(z, z') = \ln |z - z'| + O(1). \tag{A.92}$$

We have:

$$\ell(x, x) = 1. \tag{A.93}$$

[48, sec. 5]

$$G_\zeta(x) := \lim_{x \to y} \left[G(x, y) - \frac{a_0(x, y)}{4\pi^2 \ell(x, y)^2} \right]. \tag{A.94}$$

[48, sec. 5]

$$\delta\left(\frac{a_0(x,y)}{\ell(x,y)^2}\right) = -(\delta\phi(x)+\delta\phi(y))\,\frac{a_0(x,y)}{\ell(x,y)^2} + \frac{1}{2}\,\Delta\delta\phi + O(\ell). \tag{A.95}$$

This implies

$$\delta G_\zeta(x) = -2\delta\phi\, G_\zeta - \frac{1}{48\pi}\,\Delta\delta\phi. \tag{A.96}$$

For a discussion of spin structure and global aspects for fermions, see [47, sec. 5.6.4].

Given the scalar Green function, $G(x,y)$, the Robin mass $m(x)$ is defined as the "constant" term in the expansion of the Green function for $\ell(x,y)\to 0$ [41, 42, 44, 45]:

$$G(x,y) = \frac{1}{2\pi}\,\ln \ell(x,y)^2 + m(x) + O\big(\ell(x,y)^2\big). \tag{A.97}$$

It's integral over the surface is a spectral invariant:

$$M := \int \mathrm{d}^2x \sqrt{g}\, m(x). \tag{A.98}$$

The Robin mass changes as [44]:

$$m(x) = \hat{m}(x) + \frac{\phi(x)}{2\pi} - \frac{2}{A}\,(G\mathrm{e}^{2\phi})(x) - \frac{1}{A^2}\int \mathrm{d}^2y\sqrt{y}\mathrm{e}^{2\phi}(G\mathrm{e}^{2\phi})(x) \tag{A.99}$$

Note that the regularized Green function at coincident point precisely equal the Robin mass. It is related to the regularized ζ-function up to a constant [41, 42, 44, 45].

In [45, thm. 4, p. 160], there is a general formula for the variation of $\zeta(1)$ associated to an operator D with N_0 zero-modes:

$$\widetilde{\zeta}_{R,g}(1) - \widetilde{\zeta}_{R,\hat{g}}(1) = \int_0^\infty \mathrm{d}t\, (\mathrm{e}^{2\phi}-1)\,\mathrm{tr}\, K_g(t) + \frac{2}{A_{S^2}}\int \mathrm{d}^2x\sqrt{\hat{g}}\,\phi\mathrm{e}^{2\phi} - \frac{N_0}{A}\int \mathrm{d}^2x\sqrt{\hat{g}}\,(\mathrm{e}^{2\phi}-1)D^{-1}\mathrm{e}^{2\phi} - 1, \tag{A.100}$$

where $K(t)$ is the heat kernel for the operator D.

B Fermions

References on Dirac matrices are [19, app. 7.5, 8.5, 24, chap. 2, 3, 55] (see also [15, 16, 33, 56, 57]). In this appendix, we consider the groups $\mathrm{SO}(p,q)$ with metric $\eta = \mathrm{diag}(-1_q, 1_p)$ such that $d = p+q = 2$.

B.1 Gamma matrices

The Dirac matrices γ^μ satisfy a Clifford algebra associated to the group $\mathrm{SO}(p,q)$:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{B.1}$$

Matrices squaring to $+1$ (-1) are called spacelike (timelike): the latter can be obtained by multiplying with i the corresponding spacelike matrices from the case $\mathrm{SO}(p+q)$.

The only non-trivial antisymmetric product of Dirac matrices is:

$$\gamma^{\mu\nu} := \frac{1}{2}\,[\gamma^\mu, \gamma^\nu]. \tag{B.2}$$

It is traceless. Its commutator with γ^μ is:

$$[\gamma^{\mu\nu}, \gamma^\rho] = \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu. \tag{B.3}$$

All together the 4 matrices $\{1, \gamma^\mu, \gamma^{\mu\nu}\}$ generate a finite group. Every representation of a finite group can be made unitary through a similarity transformation. It follows from the Clifford algebra that all matrices squaring to $+1$ (resp. -1) are Hermitian (resp. anti-Hermitian). We define the matrix $\hat\gamma$ such that

$$(\gamma^\mu)^\dagger =: (-1)^q\,\hat\gamma\gamma^\mu\hat\gamma^{-1}. \tag{B.4}$$

We deduce that:

$$(\gamma^{\mu\nu})^\dagger = -\hat\gamma\gamma^{\mu\nu}\hat\gamma^{-1}. \tag{B.5}$$

The matrix satisfies the following properties:

$$\hat\gamma^{-1} = \hat\gamma^\dagger = (-1)^{q(q+1)/2}\,\hat\gamma. \tag{B.6}$$

Lorentz group The matrix $\gamma^{\mu\nu}$ correspond to the generators

$$M^{\mu\nu} := \frac{\alpha}{2}\,\gamma^{\mu\nu}. \tag{B.7}$$

of the $\mathrm{SO}(p, q)$ group

$$[M^{\mu\nu}, M^{\rho\sigma}] = \alpha\,(\eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma}\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}) \tag{B.8}$$

where α is some normalization ($\alpha = -\mathrm{i}$ in [19], $\alpha = 1$ in [24]). According to (B.3), γ^μ transforms appropriately as a vector.

Chirality matrix In even dimensions, one can also define a chirality matrix γ_*:

$$\gamma_* := \eta_*\,\mathrm{i}^{1-q}\,\gamma_0\gamma_1 = \eta_*\,\mathrm{i}^{q+1}\,\gamma^0\gamma^1, \tag{B.9}$$

where $\eta_* = \pm 1$ is a normalization sign. For example, in [24, chap. 3], $\eta_* = (-1)^{d/2} = -1$. It has the following properties:

$$\gamma_*^2 = 1, \qquad (\gamma_*)^\dagger = \gamma_*, \qquad \{\gamma_*, \gamma^\mu\} = 0. \tag{B.10}$$

The factor is chosen such that it squares to 1 (the other properties follow from the definition of the gamma matrices).

We have:

$$\gamma^{\mu\nu} = -\eta_*\mathrm{i}^{q+1}\,\epsilon^{\mu\nu}\gamma_*. \tag{B.11}$$

using $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = (-1)^q 2$.

The projectors on left (plus) and right (minus) chiralities are defined by:

$$P_\pm := \frac{1}{2}(1 \pm \gamma_*). \tag{B.12}$$

Equivalent representations Two representations γ^μ and γ'^μ of the Clifford algebra (i.e. an explicit matrix representation for the gamma matrices) are said to be equivalent if there is an intertwiner matrix U such that

$$\gamma'^\mu = U\gamma^\mu U^{-1}. \tag{B.13}$$

In order to preserve the unitarity of the representation (along with the Hermiticity condition), U must be unitary. It is straightforward to check that $-\gamma^\mu$, $\pm(\gamma^\mu)^t$ and $\pm(\gamma^\mu)^*$ also satisfy the Clifford algebra (B.1) and the question is whether those are equivalent to γ^μ. First, the representations $-\gamma^\mu$ and γ^μ are equivalent only in even dimensions

$$-\gamma^\mu = \gamma_*\gamma^\mu\gamma_* \tag{B.14}$$

owing to the existence of the chirality matrix.

Charge conjugation matrix The charge conjugation matrices $C_\pm$ are defined as the interwiners for the transpose matrices

$$(\gamma^\mu)^t =: -\eta\, C_\eta\gamma^\mu C_\eta^{-1}, \qquad \eta = \pm 1. \tag{B.15}$$

Taking a second time the transpose leads to the condition

$$C^t = -\epsilon\, C, \qquad \epsilon = \pm 1. \tag{B.16a}$$

As expected, the matrix C is unitary:

$$C^\dagger = C^{-1}. \tag{B.16b}$$

One finds also

$$C^* = -\epsilon\, C^{-1}. \tag{B.16c}$$

Note that in general $C^2 \neq 1$ but it is possible to find a representation where this is true.

One can see that

$$(\gamma_*)^t = -C_\eta\gamma_* C_\eta^{-1} \tag{B.17}$$

which implies

$$C_{-\eta} = C_\eta\gamma_*. \tag{B.18}$$

We also have

$$\hat\gamma^t = \eta^q\, C_\eta\hat\gamma^{-1}C_\eta^{-1} = (-1)^{q(q+1)/2}\eta^q\, C_\eta\hat\gamma C_\eta^{-1}. \tag{B.19}$$

More generally one has that each combination $C\gamma^{[r]}$ is symmetric or antisymmetric

$$(C\gamma^{[r]})^t = -t_r\, C\gamma^{[r]}, \tag{B.20}$$

where the signs are:

$$t_0 = \epsilon, \qquad t_1 = -\epsilon\,\eta, \qquad t_2 = -t_0. \tag{B.21}$$

In particular for $r = 2$ one finds

$$(C\gamma^{\mu\nu})^t = t_0\, C\gamma^{\mu\nu}$$

Due to the fact that $t_0 t_1 = \pm 1$, in even dimensions one possibility is when t_0 and t_1 have the same sign, another one is when they are opposite. The possible signs are:

$$\begin{cases} t_0 = -1, & t_1 = -1, \\ t_0 = 1, & t_1 = -1, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \epsilon = -1, & \eta = -1, \\ \epsilon = 1, & \eta = 1. \end{cases} \tag{B.22}$$

Reality matrix Similarly, we define the matrices B_ζ to be the interwiner for the conjugate matrices

$$(\gamma^\mu)^* =: \zeta\, B_\zeta \gamma^\mu B_\zeta^{-1}, \qquad \zeta = \pm 1. \tag{B.23}$$

Taking the transpose and requiring equality with (B.4) give the conditions:

$$\zeta = -(-1)^q\, \eta, \qquad B_\zeta^t = \xi\, C_\eta \hat{\gamma}^{-1}. \tag{B.24}$$

The number ξ is a normalization such that $|\xi| = 1$ (this condition is necessary for having a unitary B) that is not determined: generic choices are $\xi = 1$ [19, 55] and $\xi = \mathrm{i}$ [24]. Then, one can take the transpose of the above relation to find

$$B_\zeta = -\xi \epsilon \eta^q\, C_\eta \hat{\gamma}. \tag{B.25}$$

This also shows that B is unitary since C and $\hat{\gamma}$ are unitary. The expressions for $\hat{\gamma}^*$ and $(\gamma_*)^*$ can be deduced by using their Hermiticity properties:

$$\hat{\gamma}^* = (-1)^{q(q+1)/2} \hat{\gamma}^t, \qquad (\gamma_*)^* = \gamma_*^t. \tag{B.26}$$

Finally, we find that

$$B_\zeta B_\zeta^* = -(-1)^{q(q+1)/2}\, \epsilon\, \eta^q. \tag{B.27}$$

Note that if the basis is changed according to (B.13) then

$$C' = U^{-1t} C U^{-1}, \qquad B' = U^{-1t} B U^{-1}. \tag{B.28}$$

B.2 Spinors

In this section the indices on the B and C matrices are removed if not necessary.

Co-spinor A Dirac spinor Ψ is a (often reducible) 4-dimensional complex representation of the Clifford algebra which transforms as a vector under $\mathrm{SO}(p, q)$

$$\Psi \in \mathbb{C}^2, \qquad \delta\Psi = -\frac{1}{2\alpha}\, \omega_{\mu\nu} M^{\mu\nu} \Psi = -\frac{1}{4}\, \omega_{\mu\nu} \gamma^{\mu\nu} \Psi, \tag{B.29}$$

the $M^{\mu\nu}$ being the generators (B.7) of the Clifford algebra and $\omega_{\mu\nu}$ the parameters of the transformation. The components of the spinor are denoted by Ψ_a. Note that the components of different spinors anticommute. Moreover, one should decide if taking the complex conjugation of a product reverses the order or not

$$(\Psi_a X_b)^* = \pm X_b^* \Psi_a^*, \tag{B.30}$$

the sign $+$ is chosen in [24].

Complex conjugation doesn't act naturally on spinors. For this reason, it is convenient to introduce the charge conjugation

$$\Psi^c := B^{-1} \Psi^* \tag{B.31}$$

which also acts on matrices by

$$M^c := B^{-1} M^* B. \tag{B.32}$$

In particular, Ψ^c transforms in the same way as Ψ

$$\delta\Psi^c = -\frac{1}{4}\,\omega_{\mu\nu}\gamma^{\mu\nu}\Psi^c. \tag{B.33}$$

Of particular importance are the relations

$$(\gamma^\mu)^c = -\eta(-1)^q\gamma^\mu, \qquad \gamma_*^c = (-1)^{q+1}\,\gamma_*. \tag{B.34}$$

An important relation satisfied for two spinors Ψ_1 and Ψ_2 is the so-called Majorana flip relation

$$\widetilde{\Psi}_1\gamma^{\mu_1}\cdots\gamma^{\mu_n}\Psi_2 = -(-1)^n\eta^n\epsilon\,\widetilde{\Psi}_2\gamma^{\mu_n}\cdots\gamma^{\mu_1}\Psi_1. \tag{B.35}$$

By introducing co-spinors, it becomes possible to construct bilinears which results from the contraction of a co-spinor and a spinor, with possible insertions of gamma matrices. These bilinears will transform as (pseudo)tensors of the group $\mathrm{SO}(p,q)$.

The simplest co-spinor can be obtained using the Hermitian conjugation: the Dirac conjugate $\bar{\Psi}$ of a fermion Ψ is defined by

$$\bar{\Psi} := \Psi^\dagger\hat{\gamma}. \tag{B.36}$$

The Dirac conjugate transforms as

$$\delta\bar{\Psi} = \frac{1}{4}\,\omega_{\mu\nu}\,\bar{\Psi}\gamma^{\mu\nu}. \tag{B.37}$$

This implies that $\bar{\Psi}_1\Psi_2$ transforms as a scalar

$$\delta(\bar{\Psi}_1\Psi_2) = 0. \tag{B.38}$$

This leads to the definition of the Majorana conjugate as

$$\widetilde{\Psi} := \Psi^t C. \tag{B.39}$$

One can note that it is proportional to the charge conjugate of the Dirac conjugate

$$\bar{\Psi}^c = -\xi\epsilon\eta^q(-1)^{q(q+1)/2}\,\Psi^t C = -\xi\epsilon\eta^q(-1)^{q(q+1)/2}\,\widetilde{\Psi}. \tag{B.40}$$

One can check that it transforms in the same way as $\bar{\Psi}$

$$\delta\widetilde{\Psi} = \frac{1}{4}\,\widetilde{\Psi}\gamma^{\mu\nu}. \tag{B.41}$$

Next one can look for irreducible representations: one can apply different projections on the spinors using the chirality and charge conjugation matrix. In table 1 we summarize the different irreducible representations discussed below.

Chiral components and Weyl (or chiral) spinors We define

$$\Psi^A := C^{AB}\Psi_B = \begin{pmatrix}\psi^\alpha \\ \psi_{\dot{\alpha}}\end{pmatrix}. \tag{B.42}$$

where C^{AB} are the components of C^t. Components are lowered with C^{-1} of components C_{AB}:

$$\Psi_A := \Psi^B C_{BA}. \tag{B.43}$$

d \ q	0 (Euclidean)		1 (Minkowskian)		2	
	type	#	type	#	type	#
2	M_-	2	MW	1	M_+	2

Table 1: Irreducible representations of the $\mathrm{SO}(p, q)$ Clifford algebra. M stands for Majorana, MW for Majorana–Weyl. A sign indicates if only one of the charge conjugation matrix $C_\pm$ is allowed.

The left-handed (positive chirality) Ψ_+ and right-handed (negative chirality) Ψ_- parts of a spinor Ψ are defined by

$$\Psi_\pm := P_\pm \Psi. \tag{B.44}$$

The chiral parts satisfy

$$\gamma_* \Psi_\pm = \pm \Psi_\pm, \qquad P_\mp \Psi_\pm = 0. \tag{B.45}$$

A Dirac spinor can be written as the sum of its positive and negative helicities

$$\Psi = \Psi_+ + \Psi- = P_+ \Psi + P_- \Psi. \tag{B.46}$$

The *Weyl representation* is a basis where

$$\Psi_+ = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}, \qquad \Psi_- = \begin{pmatrix} 0 \\ \psi_- \end{pmatrix}, \tag{B.47}$$

which implies that γ_* is diagonal with

$$\gamma_* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.48}$$

The chiral components of the Dirac conjugate are obtained by

$$\bar\Psi_\pm = \bar\Psi P_\mp. \tag{B.49}$$

Similarly, the chiral components of the Majorana conjugate are

$$\Psi^c_\pm = \Psi^c P_\mp. \tag{B.50}$$

A Weyl spinor is a Dirac spinor which is restricted to one of the two chiralities and has two time less components:

$$\Psi = \Psi_\pm, \qquad \Psi_\mp = 0. \tag{B.51}$$

Note that chirality is preserved by Lorentz transformations since $[\gamma_*, \gamma^{\mu\nu}] = 0$.

Majorana spinor A Majorana spinor ψ is a spinor with a reality condition which reduces the number of components by half:

$$\Psi^* = \alpha\, B \Psi, \tag{B.52}$$

where α is a constant of proportionality. This is equivalent to say that the Dirac and Majorana conjugates are proportional

$$\widetilde{\Psi} = \alpha^{-1} \bar{\Psi}. \tag{B.53}$$

The Majorana condition is consistent only if it is an involution, then iterating the previous equation leads to the condition

$$|\alpha|^2 BB^* = 1. \tag{B.54}$$

Using (B.27) this gives the two conditions

$$|\alpha| = 1, \qquad -(-1)^{q(q+1)/2} \, \epsilon \, \eta^q = 1, \tag{B.55}$$

the first one following because the modulus of the second part is already 1, so the factor can only be a phase. As a result one finds that the Majorana condition is possible in the following case

$$\begin{cases} p - q = 0 \\ p - q = 2 \mod 8 \quad -\eta = 1. \end{cases} \tag{B.56}$$

The Majorana representation is the one where the components of Ψ are real

$$\Psi^* = \Psi, \qquad C = -\epsilon \alpha \xi^{-1} \eta^q (-1)^{q(q+1)/2} \hat{\gamma}. \tag{B.57}$$

The Dirac matrices are then either real or purely imaginary depending on which $C_\pm$ is used. For example in Euclidean signature C_- (resp. C_+) leads to real (resp. imaginary) matrices, whereas in Lorentzian signature C_+ (resp. C_-) leads to real (resp. imaginary) matrices.

Majorana–Weyl spinor One can also try to impose both the Majorana and Weyl conditions. This is possible only if the chirality matrix is preserved by the charge conjugation, which leads to the condition

$$q = 1, \tag{B.58}$$

using (B.34).

B.2.1 Kinetic and mass terms

In this subsection, we consider Lagrangians.

Dirac terms The Dirac kinetic term is:

$$K_D = \mathrm{i}^{q+1} \, \bar{\Psi} \partial\!\!\!/ \Psi, \tag{B.59}$$

and it is Hermitian up to total derivative terms:

$$K_D^\dagger = K_D. \tag{B.60}$$

The Dirac mass terms is built from two pieces

$$M_D = \mathrm{i}^{q(q+1)/2} m \, \bar{\Psi}\Psi + \mathrm{i}^{q(q+3)/2} m' \, \bar{\Psi}\gamma_* \Psi. \tag{B.61}$$

The mass term is Hermitian:

$$M_D^\dagger = M_D. \tag{B.62}$$

Let's consider a Weyl spinor of positive chirality $\gamma_* \Psi = \Psi$, then one finds that

$$K_D = 0, \qquad q = 0 \mod 2 \tag{B.63}$$

since

$$\bar\Psi \partial\!\!\!/ \Psi = -(-1)^q \, \bar\Psi \partial\!\!\!/ \Psi.$$

Hence, one can have Weyl fermions only by introducing several of them and having crossed kinetic terms. Similarly, the mass term satisfies

$$M_D = 0, \qquad q = 1 \tag{B.64}$$

from

$$\bar\Psi \Psi = (-1)^q \, \bar\Psi \Psi.$$

Note that the kinetic and mass terms are trivial for Majorana fermions if there components are not anticommuting.

Majorana terms The Majorana kinetic term is (the integral is implicit)

$$K_M = \mathrm{i}^{q+1} \, \widetilde\Psi \partial\!\!\!/ \Psi. \tag{B.65}$$

This kinetic term has no specific reality property (except obviously for a Majorana fermion). Similarly the mass term is

$$M_M = m \, \widetilde\Psi \Psi + m' \, \widetilde\Psi \gamma_* \Psi. \tag{B.66}$$

Let's consider a Weyl spinor of positive chirality $\gamma_* \Psi = \Psi$, then one finds that

$$K_M \neq 0. \tag{B.67}$$

Hence, in the cases where the Dirac kinetic vanishes one can use a Majorana kinetic term instead. Similarly, the mass term satisfies

$$M_M = 0 \tag{B.68}$$

from

$$\widetilde\Psi \Psi = -\widetilde\Psi \partial\!\!\!/ \Psi.$$

B.3 Summary

The matrices $\hat\gamma$, γ_*, B and C are defined by the relations:

$$\gamma_* = \eta_* \, \mathrm{i}^{q+1} \, \gamma^0 \gamma^1, \qquad \eta_* = \pm 1, \tag{B.69a}$$
$$B_\zeta = -\xi \epsilon \eta^q \, C_\eta \hat\gamma, \qquad \xi \in \mathrm{U}(1), \tag{B.69b}$$
$$-\gamma^\mu =: \gamma_* \gamma^\mu \gamma_*, \tag{B.69c}$$
$$(\gamma^\mu)^\dagger =: (-1)^q \, \hat\gamma \gamma^\mu \hat\gamma^{-1}, \tag{B.69d}$$
$$(\gamma^\mu)^t =: -\eta \, C_\eta \gamma^\mu C_\eta^{-1}, \qquad \eta = \pm 1, \tag{B.69e}$$
$$(\gamma^\mu)^* =: \zeta \, B_\zeta \gamma^\mu B_\zeta^{-1}, \qquad \zeta = \pm 1, \tag{B.69f}$$

The relations between the different signs are:

$$t_0 = \epsilon, \qquad t_1 = -\epsilon\,\eta, \qquad t_2 = -t_0, \qquad \zeta = -(-1)^q \eta. \tag{B.70a}$$

The possible signs are:

$$\begin{cases} t_0 = -1, & t_1 = -1, \\ t_0 = 1, & t_1 = -1, \end{cases} \quad \Longrightarrow \quad \begin{cases} \epsilon = -1, & \eta = -1, \\ \epsilon = 1, & \eta = 1. \end{cases} \tag{B.71}$$

We consider only $t_1 = -1$ to allow for (pseudo-) Majorana spinors [24, p. 56].

Different relations are:

$$\gamma_*^2 = 1, \qquad \{\gamma_*, \gamma^\mu\} = 0, \qquad (\gamma_*)^\dagger = \gamma_*, \qquad (\gamma_*)^t = -C_\eta \gamma_* C_\eta^{-1}, \tag{B.72a}$$

$$\hat\gamma^{-1} = \hat\gamma^\dagger = (-1)^{q(q+1)/2}\,\hat\gamma, \tag{B.72b}$$

$$\hat\gamma^t = (-1)^{q(q+1)/2} \eta^q\, C_\eta \hat\gamma C_\eta^{-1}, \qquad \hat\gamma^* = (-1)^{q(q+1)/2} \hat\gamma^t, \tag{B.72c}$$

$$\gamma^{\mu\nu} = -\eta_* \mathrm{i}^{q+1}\,\epsilon^{\mu\nu}\gamma_*, \tag{B.72d}$$

$$(\gamma^{\mu\nu})^\dagger = -\hat\gamma \gamma^{\mu\nu} \hat\gamma^{-1}, \tag{B.72e}$$

$$B_\zeta B_\zeta^* = -(-1)^{q(q+1)/2}\,\epsilon\,\eta^q, \tag{B.72f}$$

$$C_{-\eta} = C_\eta \gamma_*, \tag{B.72g}$$

$$C^\dagger = C^{-1}, \qquad C^* = -\epsilon\, C^{-1}, \qquad C^t = -\epsilon\, C, \qquad \epsilon = \pm 1, \tag{B.72h}$$

$$C^t = -t_0\, C, \qquad (C\gamma^\mu)^t = -t_1\, C\gamma^{[r]}, \qquad (C\gamma^{\mu\nu})^t = t_0\, C\gamma^{\mu\nu}, \tag{B.72i}$$

$$(C\hat\gamma)^t = C\hat\gamma, \qquad (C\gamma_*)^t = t_0\, C\gamma_*. \tag{B.72j}$$

The definitions γ_* and $\hat\gamma$ in terms of γ^μ matrices are representation-independent: this is not the case of B and C. This explains why we don't use A for $\hat\gamma$.

C Temporary computations

C.1 Conformal variation of projector

$$P_{g,xy} = \mathrm{e}^{-\frac{1}{2}(\phi_x + \phi_y)} A_{g,xy} \tag{C.1}$$

$$\delta A_{xy} = -\int \mathrm{d}^2 z \sqrt{g}\, \delta\phi_z\, P_{xz} P_{zy}. \tag{C.2}$$

$$\begin{aligned} \int \mathrm{d}^2 z \sqrt{g}\, P_{g,xz} P_{g,zy} &= \mathrm{e}^{-\frac{1}{2}(\phi_x + \phi_y)} \int \mathrm{d}^2 z \sqrt{\hat g}\, \mathrm{e}^{\phi_z} A_{g,xz} A_{g,zy} \\ &= \mathrm{e}^{-\frac{1}{2}(\phi_x + \phi_y)} A_{g,xy} \end{aligned}$$

$$A_{g,xy} = \int \mathrm{d}^2 z \sqrt{\hat g}\, \mathrm{e}^{\phi_z} A_{g,xz} A_{g,zy} \tag{C.3}$$

$$\begin{aligned} P_{\hat g,xy} &= \mathrm{e}^{\frac{\phi_y}{2}} \int \mathrm{d}^2 z \sqrt{\hat g}\, \mathrm{e}^{\frac{3}{2}\phi_z} P_{\hat g,xz} \mathrm{e}^{-\frac{1}{2}(\phi_z + \phi_y)} A_{g,zy} \\ &= \int \mathrm{d}^2 z \sqrt{\hat g}\, \mathrm{e}^{\phi_z} P_{\hat g,xz} A_{g,zy} \end{aligned}$$

$$\int \mathrm{d}^2 z\sqrt{\hat g}\,\mathrm{e}^{\phi_z} P_{\hat g,xz}A_{g,zy} = \int \mathrm{d}^2 z\sqrt{\hat g}\,\mathrm{e}^{\phi_z}\int \mathrm{d}^2 w\sqrt{\hat g}\,\mathrm{e}^{\phi_w} P_{\hat g,xw}A_{g,wz}A_{g,zy}$$
$$= \int \mathrm{d}^2 z\sqrt{\hat g}\,\mathrm{e}^{\phi_z}\int \mathrm{d}^2 w\sqrt{\hat g}\,\mathrm{e}^{\phi_w} P_{\hat g,xz}A_{g,zw}A_{g,wy}$$

The variation of the projector (3.82) reads:

$$\boxed{\delta P_{xy} = -\frac{1}{2}(\delta\phi_x + \delta\phi_y)P_{xy} - \int \mathrm{d}^2 z\sqrt{g}\,\delta\phi_z\, P_{xz}P_{zy}.} \tag{C.4}$$

Let's make the following ansatz:

$$P_{g,xy} = \int \mathrm{d}^2 z\sqrt{\hat g}\,\mathrm{e}^{-\alpha_1\phi_x-\alpha_2\phi_y-\alpha_3\phi_z} P_{\hat g,xz}P_{\hat g,zy}. \tag{C.5}$$

$$\int \mathrm{d}^2 w\sqrt{g}\, P_{g,xw}P_{g,wy}$$
$$= \int \mathrm{d}^2 w\sqrt{\hat g}\,\mathrm{e}^{2\phi_w}\int \mathrm{d}^2 z\sqrt{\hat g}\int \mathrm{d}^2 z'\sqrt{\hat g}\,\mathrm{e}^{-\alpha_1\phi_x-\alpha_2\phi_w-\alpha_3\phi_z}\mathrm{e}^{-\alpha_2\phi_y-\alpha_1\phi_w-\alpha_3\phi_{z'}}$$
$$\times P_{\hat g,xz}P_{\hat g,zw}\,P_{\hat g,wz'}P_{\hat g,z'y}$$
$$= \mathrm{e}^{-\alpha_1\phi_x-\alpha_2\phi_y}\int \mathrm{d}^2 z\sqrt{\hat g}\int \mathrm{d}^2 z'\sqrt{\hat g}\,\mathrm{e}^{-\alpha_3\phi_z-\alpha_3\phi_{z'}}\int \mathrm{d}^2 w\sqrt{\hat g}\,\mathrm{e}^{(2-\alpha_1-\alpha_2)\phi_w}$$
$$\times P_{\hat g,xz}P_{\hat g,zw}\,P_{\hat g,wz'}P_{\hat g,z'y}$$

The integrated form of (3.82) is:

$$P_{g,xy} = \int \mathrm{d}^2 z\sqrt{\hat g}\,\mathrm{e}^{-\frac{\phi_x}{2}} P_{\hat g,xz}\,\mathrm{e}^{-\phi_z} P_{\hat g,zy}\,\mathrm{e}^{-\frac{\phi_y}{2}}. \tag{C.6}$$

The infinitesimal form is recovered after using (3.84) for $P_{\hat g}$. Let's prove that $P_g^2 = P_g$ if $P_{\hat g}^2 = P_{\hat g}$, see (3.84):

$$\int \mathrm{d}^2 w\sqrt{g}\, P_{g,xw}P_{g,wy}$$
$$= \int \mathrm{d}^2 w\sqrt{\hat g}\,\mathrm{e}^{2\phi_w}\int \mathrm{d}^2 z\sqrt{\hat g}\int \mathrm{d}^2 z'\sqrt{\hat g}\,\mathrm{e}^{-\frac{\phi_x}{2}-\frac{\phi_w}{2}-\phi_z} P_{\hat g,xz}P_{\hat g,zw}\,\mathrm{e}^{-\frac{\phi_y}{2}-\frac{\phi_w}{2}-\phi_{z'}} P_{\hat g,wz'}P_{\hat g,z'y}$$
$$= \mathrm{e}^{-\frac{\phi_x}{2}-\frac{\phi_y}{2}}\int \mathrm{d}^2 w\sqrt{\hat g}\,\mathrm{e}^{\phi_w}\int \mathrm{d}^2 z\sqrt{\hat g}\,\mathrm{e}^{-\phi_z}\int \mathrm{d}^2 z'\sqrt{\hat g}\,\mathrm{e}^{-\phi_{z'}}\, P_{\hat g,xz}P_{\hat g,zw}\,P_{\hat g,wz'}P_{\hat g,z'y}.$$

C.2 Scalar effective action

In [40, p. 11] and [35, sec. 3.2.5], they proceed by computing δG_ζ in terms of δK. For the scalar case, one finds the relation:

$$\widetilde{G}_g(x,y) - \widetilde{G}_{\hat g}(x,y) = \frac{1}{2}(K(x)+K(y)) - S_{\mathrm{AY}}[g,\hat g]. \tag{C.7}$$

This relation can be proved by check that G_g satisfies the Green equation if $G_{\hat g}$ does, and that it is correctly orthogonal to the zero-mode projector. This can also be obtained from the infinitesimal form:

$$\delta\widetilde{G}(x,y) = \frac{1}{2}(\delta K(x)+\delta K(y)) - \int \mathrm{d}^2 x\sqrt{g}\,\delta K, \tag{C.8}$$

which can be obtained by simplifying (4.32). This gives:

$$\widetilde{G}_{g,\zeta}(x) - \widetilde{G}_{\hat{g},\zeta}(x) = \frac{\phi(x)}{2\pi} + K(x) - S_{\text{AY}}[g,\hat{g}]. \tag{C.9}$$

Then, they obtain the difference for the integrated Green function with an additional factor of $1/A$:

$$\frac{\Psi_G[g]}{A} - \frac{\Psi_G[\hat{g}]}{\hat{A}} = \frac{1}{8\pi}\int \mathrm{d}^2x\sqrt{\hat{g}}\left[\frac{4}{A}\,\phi\,\mathrm{e}^{2\phi} - 2\pi\,K\hat{\Delta}K - 4\pi\,K\,\hat{\Delta}\widetilde{G}_{\hat{g},\zeta}(x)\right]. \tag{C.10}$$

The Laplacian of the Green function was found to be:

$$\Delta\, G_{g,\zeta} = \frac{R}{4\pi} - \frac{2}{A} + (2-\chi)\,\frac{\sqrt{g_c}}{g}, \tag{C.11}$$

where g_c is the canonical metric. This finally gives:

$$\frac{\Psi_G[g]}{A} - \frac{\Psi_G[\hat{g}]}{\hat{A}} = \frac{1}{8\pi}\,S_{\text{M}}[g,\hat{g}] + \left(1-\frac{\chi}{2}\right)\left(S_{\text{AY}}[g,\hat{g}] - \int \mathrm{d}^2x\sqrt{g_c}\,K(x)\right). \tag{C.12}$$

C.3 Trace of Green function

[HE: Equations below are wrong because assumed that Δ is proportionnal to the identity which is not true.] ⇐ 20

We want to understand how the trace of the Green function $\widetilde{G}(x,y)$ behaves. Let's define the normalized trace of the Green function and projector:

$$\boxed{\tilde{g}(x,y) := \frac{1}{2}\,\mathrm{tr}_D\,\widetilde{G}(x,y), \qquad p(x,y) := \frac{1}{2}\,\mathrm{tr}_D\,P(x,y).} \tag{C.13}$$

Taking the trace of (3.101b), we find:

$$\boxed{\left(-\Delta + \frac{R}{4} + m^2\right)\tilde{g}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} - p(x,y),} \tag{C.14}$$

where Δ is the scalar Laplacian. Denoting the Green function $\Gamma(x,y)$ for the scalar Laplacian with mass and the corresponding projector $\Pi(x,y)$ over its single-zero mode:

$$(-\Delta + m^2)\widetilde{\Gamma}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} - \Pi(x,y), \qquad \int \mathrm{d}^2z\sqrt{g}\,\Pi(x,z)\Gamma(z,y) = 0, \tag{C.15}$$

we can obtain an integral equation for $\tilde{g}(x,y)$:

$$\begin{aligned}\tilde{g}(x,y) &= \int \mathrm{d}^2y\sqrt{g}\,\Gamma(x,z)\left(\frac{\delta(z-y)}{\sqrt{g}} - \frac{R(z)}{4}\,\tilde{g}(z,y) - p(z,y)\right) + \int \mathrm{d}^2z\sqrt{g}\,\Pi(x,z)\tilde{g}(z,y)\\ &= \Gamma(x,y) - \int \mathrm{d}^2y\sqrt{g}\,\Gamma(x,z)\left(\frac{R(z)}{4}\,\tilde{g}(z,y) + p(z,y)\right) + \int \mathrm{d}^2z\sqrt{g}\,\Pi(x,z)\tilde{g}(z,y).\end{aligned}$$

[HE: Can we simplify this further? If not, we would have to study the properties of the Green functions for $-\Delta + \alpha R$ in general.] ⇐ 21

[HE: Can we compute $\Delta\tilde{g}_\zeta(x)$?] ⇐ 22

C.4 Gravitational action

C.4.1 Large mass expansion

In the large mass limit we can use (**??**) to rewrite the last term of (**??**) as

$$\int_0^\infty \frac{\mathrm{d}t}{t}\,(\tilde{K}_g^{(0)}(t) - \tilde{K}_{\hat{g}}^{(0)}(t)) = \tilde{\zeta}_g^{(0)}(0) - \tilde{\zeta}_{\hat{g}}^{(0)}(0) = -\frac{1}{24\pi}\int \mathrm{d}^2x(\sqrt{g}\,R - \sqrt{\hat{g}}\,\hat{R}). \tag{C.16}$$

This vanishes if the Liouville mode is smooth because the integral of R is a topological invariant.

The large mass limit of the Green function reads

$$m^2\tilde{G}_\zeta(x) = m^2 \lim_{s\to 1}\left(\mu^{2s-2}\tilde{\zeta}(s,x,x) - \frac{\mathrm{I}_2}{4\pi(s-1)}\right) \tag{C.17}$$

$$= m^2 \lim_{s\to 1}\left(\mu^{2s-2}\sum_{n\neq 0}\frac{\Psi_n(x)\Psi_n^\dagger(x)}{(m^2+\Lambda_n^{(0)})^s} - \frac{\mathrm{I}_2}{4\pi(s-1)}\right) \tag{C.18}$$

$$\sim \sum_{n\neq 0}\Psi_n(x)\Psi_n^\dagger(x) = \tilde{\zeta}(0,x,x) \tag{C.19}$$

since

$$\frac{1}{(m^2+\Lambda_n^{(0)})^s} = \frac{1}{m^{2s}}\frac{1}{(1+\Lambda_n^{(0)}/m^2)^s} \sim \frac{1}{m^{2s}}\left(1 - \frac{\Lambda_n^{(0)}}{m^2}\,s\right). \tag{C.20}$$

Then we find

$$\begin{aligned} m^2 \int \mathrm{d}^2x\;\mathrm{tr}_D\,(\sqrt{g}\,\tilde{G}_{g,\zeta}(x) - \sqrt{\hat{g}}\,\tilde{G}_{\hat{g},\zeta}(x)) \\ \sim \int \mathrm{d}^2x\;\mathrm{tr}_D\,(\sqrt{g}\,\tilde{\zeta}_g(0,x,x) - \sqrt{\hat{g}}\,\tilde{\zeta}_{\hat{g}}(0,x,x)) \\ = \tilde{\zeta}_g(0) - \tilde{\zeta}_{\hat{g}}(0) = \tilde{\zeta}_g^{(0)}(0) - \tilde{\zeta}_{\hat{g}}^{(0)}(0). \end{aligned} \tag{C.21}$$

This is the same result as the other term and it would vanish if the Liouville mode is smooth.

C.4.2 Torus with even spin structure

On the torus there are two real zero-modes (**??**). For a solution which is not normalised, the normalization matrix reads

$$\kappa[g] = \frac{1}{A}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{C.22}$$

such that

$$-\frac{1}{2}\ln\det\frac{\kappa[g]}{\kappa[\hat{g}]} = \ln\frac{A}{\hat{A}}. \tag{C.23}$$

References

[1] F. Langouche and H. Leutwyler. "Two-Dimensional Fermion Determinants as Wess-Zumino Actions." *Physics Letters B* 195.1 (Aug. 1987), pp. 56–59. DOI: 10.1016/0370-2693(87)90885-9.

[2] A. G. Sedrakyan and R. Stora. “Dirac and Weyl Fermions Coupled to Two-Dimensional Surfaces: Determinants.” *Physics Letters B* 188.4 (Apr. 1987), pp. 442–446. DOI: 10.1016/0370-2693(87)91645-5.

[3] F. Langouche. “Gravitational and Lorentz Anomalies in Two Dimensions.” *Physics Letters B* 148.1 (Nov. 1984), pp. 93–98. DOI: 10.1016/0370-2693(84)91617-4.

[4] H. Leutwyler. “Gravitational Anomalies: A Soluble Two-Dimensional Model.” *Physics Letters B* 153.1 (Mar. 1985), pp. 65–69. DOI: 10.1016/0370-2693(85)91443-1.

[5] H. Leutwyler and S. Mallik. “Gravitational Anomalies.” *Zeitschrift für Physik C Particles and Fields* 33.2 (June 1986), pp. 205–226. DOI: 10.1007/BF01411138.

[6] A. M. Polyakov. “Quantum Geometry of Fermionic Strings.” *Physics Letters B* 103.3 (July 1981), pp. 211–213. DOI: 10.1016/0370-2693(81)90744-9.

[7] K. Li. “Anomalous but Consistent Gravity in Two Dimensions.” *Physical Review D* 34.8 (Oct. 1986), pp. 2292–2297. DOI: 10.1103/PhysRevD.34.2292.

[8] T. Berger and I. Tsutsui. “Chiral Quantum Gravity in Two Dimensions.” *Nuclear Physics B* 335.1 (Apr. 1990), pp. 245–259. DOI: 10.1016/0550-3213(90)90179-H.

[9] E. S. Fradkin and A. A. Tseytlin. “Anomaly-Free Two-Dimensional Chiral Supergravity-Matter Models and Consistent String Theories.” *Physics Letters B* 162.4 (Nov. 1985), pp. 295–298. DOI: 10.1016/0370-2693(85)90926-8.

[10] E. Elizalde, S. Naftulin, and S. D. Odintsov. “Covariant Effective Action and One-Loop Renormalization of 2D Dilaton Gravity with Fermionic Matter.” *Physical Review D* 49.6 (Mar. 1994), pp. 2852–2861. DOI: 10.1103/PhysRevD.49.2852. arXiv: hep-th/9308020.

[11] S. K. Blau, M. Visser, and A. Wipf. “Determinants, Dirac Operators, and One-Loop Physics.” *International Journal of Modern Physics A* 04.06 (Apr. 1989), pp. 1467–1484. DOI: 10.1142/S0217751X89000625.

[12] E. D’Hoker and D. H. Phong. “Functional Determinants on Mandelstam Diagrams.” *Communications in Mathematical Physics* 124.4 (Dec. 1989), pp. 629–645. DOI: 10.1007/BF01218453.

[13] I. Sachs and A. Wipf. “Finite Temperature Schwinger Model.” *Helv. Phys. Acta* 65 (1992), pp. 652–678. arXiv: 1005.1822.

[14] F. Ferrari. “On the Schwinger Model on Riemann Surfaces.” *Nuclear Physics B* 439.3 (Apr. 1995), pp. 692–710. DOI: 10.1016/0550-3213(95)00068-4. arXiv: hep-th/9310055.

[15] A. Wipf. *Introduction to Supersymmetry*. 2016. URL: https://www.tpi.uni-jena.de/qfphysics/homepage/wipf/lectures/susy/susyhead.pdf.

[16] A. Dettki, I. Sachs, and A. Wipf. “Generalized Gauged Thirring Model on Curved Space-Times” (Aug. 1993). arXiv: hep-th/9308067.

[17] I. Sachs and A. Wipf. “Generalized Thirring Models.” *Annals of Physics* 249.2 (Aug. 1996), pp. 380–429. DOI: 10.1006/aphy.1996.0077. arXiv: hep-th/9508142.

[18] L. Griguolo and D. Seminara. "Non-Minimal Couplings in Two Dimensional Gravity: A Quantum Investigation." *Nuclear Physics B* 495.1-2 (June 1997), pp. 400–432. DOI: 10.1016/S0550-3213(97)00209-5. arXiv: hep-th/9612025.

[19] R. Blumenhagen, D. Lüst, and S. Theisen. *Basic Concepts of String Theory*. 2013 edition. Springer, Nov. 2014.

[20] J. B. Zuber and C. Itzykson. "Quantum Field Theory and the Two-Dimensional Ising Model." *Physical Review D* 15.10 (May 1977), pp. 2875–2884. DOI: 10.1103/PhysRevD.15.2875.

[21] P. Di Vecchia, B. Durhuus, P. Olesen, and J. L. Petersen. "Fermionic Strings with Boundary Terms." *Nuclear Physics B* 207.1 (Nov. 1982), pp. 77–95. DOI: 10.1016/0550-3213(82)90137-7.

[22] K. Becker, M. Becker, and J. H. Schwarz. *String Theory and M-Theory: A Modern Introduction*. 1st edition. Cambridge University Press, Dec. 2006.

[23] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. 2nd edition. Springer, Jan. 1999.

[24] D. Z. Freedman and A. Van Proeyen. *Supergravity*. Cambridge University Press, May 2012.

[25] F. Loran. "Chiral Fermions on 2D Curved Spacetimes." *International Journal of Modern Physics A* 32.16 (June 2017), p. 1750092. DOI: 10.1142/S0217751X17500920. arXiv: 1608.06899.

[26] D. Borthwick. "Euclidean Majorana Fermions, Fermionic Integrals, and Relative Pfaffians." *Journal of Mathematical Physics* 34.7 (July 1993), pp. 2691–2712. DOI: 10.1063/1.530413.

[27] M. Nakahara. *Geometry, Topology and Physics*. 2nd edition. Institute of Physics Publishing, June 2003.

[28] C. Itzykson and J.-M. Drouffe. *Statistical Field Theory: Volume 2, Strong Coupling, Monte Carlo Methods, Conformal Field Theory and Random Systems*. Cambridge University Press, Mar. 1991.

[29] Al. Zamolodchikov. "Scaling Lee-Yang Model on a Sphere. I. Partition Function." *Journal of High Energy Physics* 2002.07 (July 2002), pp. 029–029. DOI: 10.1088/1126-6708/2002/07/029. arXiv: hep-th/0109078.

[30] Al. B. Zamolodchikov and Y. Ishimoto. "Massive Majorana Fermion Coupled to Two-Dimensional Gravity and the Random-Lattice Ising Model." *Theoretical and Mathematical Physics* 147.3 (June 2006), pp. 755–776. DOI: 10.1007/s11232-006-0076-7.

[31] N. Seiberg. "Notes on Quantum Liouville Theory and Quantum Gravity." *Progress of Theoretical Physics Supplement* 102 (1990), pp. 319–349. DOI: 10.1143/PTPS.102.319.

[32] G. Mussardo. *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics*. Oxford University Press, 2009.

[33] P. van Nieuwenhuizen and A. Waldron. "On Euclidean Spinors and Wick Rotations." *Physics Letters B* 389.1 (Dec. 1996), pp. 29–36. DOI: 10.1016/S0370-2693(96)01251-8. arXiv: hep-th/9608174.

[34] H. Erbin. *Introduction to String Field Theory.* 2019. URL: http://www.lpthe.jussieu.fr/~erbin/files/reviews/string_theory.pdf.

[35] F. Ferrari, S. Klevtsov, and S. Zelditch. "Gravitational Actions in Two Dimensions and the Mabuchi Functional." *Nuclear Physics B* 859.3 (June 2012), pp. 341–369. DOI: 10.1016/j.nuclphysb.2012.02.003. arXiv: 1112.1352.

[36] N. Ginoux. *The Dirac Spectrum.* 2009th edition. Springer, May 2009.

[37] N. Hitchin. "Harmonic Spinors." *Advances in Mathematics* 14.1 (Sept. 1974), pp. 1–55. DOI: 10.1016/0001-8708(74)90021-8.

[38] E. D'Hoker and D. H. Phong. "The Geometry of String Perturbation Theory." *Reviews of Modern Physics* 60.4 (Oct. 1988), pp. 917–1065. DOI: 10.1103/RevModPhys.60.917.

[39] C. Bär and P. Schmutz. "Harmonic Spinors on Riemann Surfaces." *Annals of Global Analysis and Geometry* 10.3 (Jan. 1992), pp. 263–273. DOI: 10.1007/BF00136869.

[40] A. Bilal and L. Leduc. "2D Quantum Gravity on Compact Riemann Surfaces with Non-Conformal Matter." *Journal of High Energy Physics* 2017.01 (2017), p. 089. DOI: 10.1007/JHEP01(2017)089. arXiv: 1606.01901.

[41] J. Steiner. "A geometrical mass and its extremal properties for metrics on S^2." *Duke Mathematical Journal* 129.1 (July 2005), pp. 63–86. DOI: 10.1215/S0012-7094-04-12913-6.

[42] P. Doyle and J. Steiner. "Blowing Bubbles on the Torus" (Oct. 2017). arXiv: 1710.09865.

[43] K. Okikiolu. "Extremals for Logarithmic Hardy-Littlewood-Sobolev Inequalities on Compact Manifolds" (Nov. 2007). arXiv: math/0603717.

[44] K. Okikiolu. "A Negative Mass Theorem for Surfaces of Positive Genus." *Communications in Mathematical Physics* 290.3 (Sept. 2009), pp. 1025–1031. DOI: 10.1007/s00220-008-0722-z. arXiv: 0810.0724.

[45] C. Morpurgo. "The logarithmic Hardy-Littlewood-Sobolev inequality and extremals of zeta functions on S_n." *Geometric and Functional Analysis* 6.1 (Jan. 1996), pp. 146–171. DOI: 10.1007/BF02246771.

[46] S. Weinberg. *Lectures on Quantum Mechanics.* 2nd edition. Cambridge University Press, Sept. 2015.

[47] L. Parker and D. Toms. *Quantum Field Theory in Curved Spacetime.* 1st edition. Cambridge University Press, Sept. 2009.

[48] A. Bilal and F. Ferrari. "Multi-Loop Zeta Function Regularization and Spectral Cutoff in Curved Spacetime" (July 2013).

[49] J. Schnittger and U. Ellwanger. "Nonperturbative Conditions for Local Weyl Invariance on a Curved World Sheet." *Theoretical and Mathematical Physics* 95.2 (May 1993), pp. 643–662. DOI: 10.1007/BF01017149. arXiv: hep-th/9211139.

[50] L. O'Raifeartaigh, J. M. Pawlowski, and V. V. Sreedhar. "Duality in Quantum Liouville Theory." *Annals of Physics* 277.1 (Oct. 1999), pp. 117–143. DOI: 10.1006/aphy.1999.5951. arXiv: hep-th/9811090.

[51] T. Bautista and A. Dabholkar. "Quantum Cosmology Near Two Dimensions." *Physical Review D* 94.4 (2016), p. 044017. DOI: 10.1103/PhysRevD.94.044017. arXiv: 1511.07450.

[52] R. M. Wald. *General Relativity*. University of Chicago Press, 1984.

[53] S. M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Addison Wesley, 2004.

[54] H. Lacoin, R. Rhodes, and V. Vargas. "Path Integral for Quantum Mabuchi K-energy" (July 2018). arXiv: 1807.01758.

[55] A. Van Proeyen. "Tools for Supersymmetry" (Oct. 1999). arXiv: hep-th/9910030.

[56] P. B. Pal. "Dirac, Majorana and Weyl Fermions." *American Journal of Physics* 79.5 (May 2011), pp. 485–498. DOI: 10.1119/1.3549729. arXiv: 1006.1718.

[57] C. Wetterich. "Spinors in Euclidean Field Theory, Complex Structures and Discrete Symmetries." *Nuclear Physics B* 852.1 (Nov. 2011), pp. 174–234. DOI: 10.1016/j.nuclphysb.2011.06.013. arXiv: 1002.3556.