Janis–Newman algorithm

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2nd December 2016

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1 Comments

1.1 Justifications

Many attempts have been made to justify this algorithm, and we quickly review few of them:

- Intuitions and explanations can also be found in [51, 59]; see also [31, 58]. The algorithm can be applied to Kerr–Schild metrics and for spacetime which are of Petrov type II or D [1, sec. 2.3].
- In various papers, Newman shows that the imaginary part of complex coordinates may be interpreted as an angular momentum, and there are similar correspondences for other charges (magnetic...) [48, 50, 52].
- Quevedo [54] decomposes the Riemann tensor in irreducible representations of $\mathrm{SO}(3, \mathbb{C}) \sim \mathrm{SO}(3, 1)$ and then uses the symmetry group to generate new solutions. This approach is more general but it is difficult to get the metric from the curvature (he addresses this question in another paper [55]). Moreover the argument can not be generalized to other dimensions because there are no homomorphism between $\mathrm{SO}(d-1, 1)$ and $\mathrm{SO}(n, \mathbb{C})$ for any n, despite the fact that the algorithm is successful also in other dimensions [1].
- Demiański [17] does not try to complexify the function appearing in the metric: he finds their expressions by solving Einstein equations. But when we have the functions he does not appear to be possible to get them from the complexification of some static functions.
- Drake and Szekeres [18] get the complexified functions in terms of the function $g(r)$ and $h(r)$ that are used to transform the metric to Boyer–Lindquist coordinates. Then they solve Einstein equations in different cases for these two functions, and they get some unicity theorems, but their work is really restricted.

1.2 (Gauged) supergravity

Other examples could be found from [5, 13, 14, 30].

1.3 Higher dimensions

It may be possible to find examples in [23] for $d = 5$, and in [16, 40, 43].

1.3.1 Why $d > 5$ is harder

As explained in [34], there are significant differences for $d \leq 5$ and $d > 5$ black holes. For example in the last case there are no inner horizon (there is only one solution to $\Delta = 0$). Moreover the singularity is spacelike.

1.4 Comments on other works

1.4.1 Incorrect works

It is easy to arrive at wrong solutions using the Janis–Newman algorithm. The two key points to check are:

[1] It is also possible to check that there are no homomorphisms with the groups $\mathrm{SO}(n, \mathbb{Q})$ and $\mathrm{SO}(n, \mathbb{O})$.

- that the transformation to Boyer–Lindquist is well defined (i.e. there is no θ dependence);
- the metric and the associated matter fields solve the equations of motion (which can indicate that the chosen complexification is wrong or just that it is impossible).

Mallett [44] applies the JNA on solution with cosmological constant but he does not give the derivation, and Xu shows that his solution does not solve the equations [66].

The solution given in eq. (24) of [26] cannot come from JNA since we know it can not be applied to solution with cosmological constant (for example it is impossible to get $g_{\theta\theta} \neq \rho^2$). So if they got the solution somewhere it is not from JNA (especially they are quoting Xu [66]).

The solutions [35] looks like subcases of [44], while [25] is itself a subcase of [35]: are they correct? It looks like [25] just reproduces a known solution (Carmeli) but it incorporates an arbitrary energy–momentum tensor on the RHS. In [35] no comment is made for the equations of motion.

The rotating black hole coupled to Yang–Mills fields derived in [27] can not be trusted: they do not give the gauge fields and they do not check Einstein–Yang–Mills equations either.

The rotating braneworld black hole in [64, sec. 5.4.2] are not solutions of Einstein equations: I think the reason is that the BL transformation is not valid (because there are square roots and factor ρ^4). It becomes a solution only in the slow-rotation limit.

The configuration [11, 12, 24, 64, sec. 5.4.2] are not valid because the function g and h of the BL transformation depend on θ [6, 8].

Finally the solutions [16] do not solve the equations of motion.

Are the works [35, 42, 61] correct?

Authors of [32] argues that the JNA can not be applied in modified theories of gravity. Similarly the paper [53] argues against application to Brans–Dicke black hole; but looking at [33, 67] it seems that it works only if it can be obtain from string theory/KK reduction.

1.4.2 Understanding and generalizations

As recalled in [19, 31] a complex change of coordinate is allowed for metric of the Kerr–Schild form. Also there are understanding of why it works for vacuum metric; but there is no Kerr–Schild form for the energy tensor of a perfect fluid and so we don't have any a priori idea of what we could get. In few words the condition is that it exists a coordinate system where the pseudo energy tensor vanishes (or the Einstein equations are linear). But Whisker [64, p. 94] note that this is neither sufficient (dS-Schwarschild can be put in Kerr–Schild form) nor proven that it is necessary.

The author of [7] generalizes the approach of Drake and Szekeres [18].

It is believed that the JNA generates always type D spacetime, but Demiański shows that the spacetime is of type II for $c \neq 0$ [17].

The transformation with a NUT charge (i.e. $c = 0$ from Demiański) has been found by Newman and Demiański [10, 57, chap. 1].

The JNA may work only for linear theories. Moreover the transformation of the r coordinates translates into a translation of z as $z \to z + ia$.

1.5 Properties of black holes

1.5.1 Higher-dimensional black holes

Myers–Perry black hole can be written in Kerr–Schild form [46, sec. 1.2.5].

It is possible to have ultra-spinning black holes for $d \geq 6$.

See [20, sec. 7.2] for properties of CCLP and BMPV. In particular from CCLP (with six charges: m, a, b and three electric charges) one get a four-charges solution because the extremal limit does not coincide with the BPS limit: extremality is a necessary but not sufficient condition for a BPS black hole [22, sec. 1].

Moreover BMPV is the only asymptotically flat, topologically spherical, supersymmetric black holes in five dimensional ungauged supergravity [39]. It seems that BPS stationary asymptotically flat black holes are known only for $d = 4, 5$ [20, sec. 7.2.2, 56]. See also [36] for info on BMPV.

BMPV is half-BPS (but full supersymmetry is restored at the horizon) [22].

1.5.2 Near-horizon geometries

For generic rotating black holes in d dimensions the near-horizon geometry is $\mathrm{AdS}_2 \times S^{d-2}$ [46, p. 15].

Any supersymmetric solution in $5d$ near-horizon geometries is locally isomorphic to [20, sec. 7.2.2, 8.5, 56]

$$\mathbb{R}^5, \qquad \mathrm{AdS}_3 \times S^2, \qquad \mathrm{BMPV}, \tag{1.1}$$

with $\mathrm{AdS}_2 \times S^3 \subset \mathrm{BMPV}$. The corresponding horizons are

$$T^3, \qquad S^1 \times S^2, \qquad \text{(squashed) } S^3. \tag{1.2}$$

In particular (under some assumptions) BMPV black hole is the only BPS black hole with the corresponding near-horizon geometry. Note that BMPV horizon is non-rotating (due to supersymmetry) [28].

Black rings in $5d$ have horizons $S^1 \times S^2$.

1.6 Modified gravity

Consider the dilatonic Einstein–Maxwell action [33]

$$S = \frac{1}{2} \int \sqrt{-g}\, \big(-R + (\partial\Phi)^2 + \mathrm{e}^{-2\alpha\Phi} F^2\big)\,. \tag{1.3}$$

After rescaling the metric and doing a change of variable

$$g \longrightarrow \mathrm{e}^{4\alpha\Phi} g, \qquad \phi = \frac{1}{\alpha}\, \mathrm{e}^{\alpha\Phi} \tag{1.4}$$

one obtains

$$S = \frac{1}{2} \int \sqrt{-g}\, \big(- \alpha^2 \phi^2 R + (\partial\phi)^2 + F^2\big). \tag{1.5}$$

1.7 NUT charge

The condition for regularity of a nut in Euclidean signature is

$$m = \nu \left(\kappa + \frac{4\Lambda}{3}\, \nu^2 \right) \tag{1.6}$$

where ν is the analytical continuation of the nut charge. This amounts to taking $m' = 0$ in the formula for the complexification of the mass.

The Weyl scalar of Taub–NUT in flat space reads

$$\Psi_2 = - \frac{m + in}{(r + in)^3}. \tag{1.7}$$

2 General properties

2.1 More on the tetrads

General review can be found in [1, sec. 2.1].

We recall the definition of the tetrads

$$\ell^\mu = \delta^\mu_r, \qquad n^\mu = \delta^\mu_u - \frac{f}{2}\,\delta^\mu_r, \qquad m^\mu = \frac{1}{\sqrt{2}r}\left(\delta^\mu_\theta + \frac{i}{\sin\theta}\delta^\mu_\phi\right). \tag{2.1}$$

grouped as

$$Z^\mu_a = \{\ell^\mu, n^\mu, m^\mu, \bar m^\mu\}. \tag{2.2}$$

They form a null basis

$$\ell^2 = n^2 = m^2 = 0, \qquad \ell_\mu n^\mu = -m_\mu \bar m^\mu = -1, \qquad \ell_\mu m^\mu = n_\mu m^\mu = 0. \tag{2.3}$$

The metric is obtained by

$$g^{\mu\nu} = -\ell^\mu n^\nu - \ell^\nu n^\mu + m^\mu \bar m^\nu + m^\nu \bar m^\mu. \tag{2.4}$$

The orthogonality conditions (2.3) implies that

$$g^{\mu\nu} = \eta^{ab} Z^\mu_a Z^\nu_b \tag{2.5}$$

where

$$\eta^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.6}$$

The rotating tetrads are

$$\begin{gathered} \ell'^\mu = \delta^\mu_r, \qquad n'^\mu = \delta^\mu_u - \frac{\tilde f}{2}\,\delta^\mu_r, \\ m'^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(\delta^\mu_\theta + \frac{i}{\sin\theta}\delta^\mu_\phi - ia\sin\theta\,(\delta^\mu_u - \delta^\mu_r)\right). \end{gathered} \tag{2.7}$$

We define the Newman–Penrose (NP) coefficients by

$$\phi_0 = -F_{\mu\nu}\ell^\mu m^\nu, \qquad \phi_1 = -\frac{1}{2}\,F_{\mu\nu}(\ell^\mu n^\nu - \bar m^\mu m^\nu), \qquad \phi_2 = F_{\mu\nu}\bar m^\mu m^\nu. \tag{2.8}$$

For Kerr–Newman NP coefficients read [49]

$$\phi_0 = 0, \qquad \phi_1 = \frac{Q}{\sqrt{2}(r - ia\cos\theta)^2}, \qquad \phi_2 = \frac{iQa\sin\theta}{(r - ia\cos\theta)^3}. \tag{2.9}$$

while for Reissner–Nordstrøm they are

$$\phi_0 = \phi_2 = 0, \qquad \phi_1 = \frac{Q}{2r^2} \tag{2.10}$$

Note that NP coefficients transform as scalars under coordinate transformation, but not under null Lorentz rotation [1, 41].

Look also at [21].

2.2 Transformation in Newman–Penrose language

In [18, p. 5] the authors rewrite the algorithm ina nice way. After we have introduced the tetrads $Z_a^\mu(x)$ we can introduce a new set of coordinates

$$z^\mu = x^\mu + i\, y^\mu(x) \tag{2.11}$$

where y^μ are analytic function of x^μ. On the other hand the tetrad transform as

$$Z_a^\mu(x) \longrightarrow \tilde{Z}_a^\mu(z, \bar{z}). \tag{2.12}$$

We first want to get back the old tetrad if $\bar{z} = z = x$

$$Z_a^\mu(z, z) = Z_a^\mu(x), \tag{2.13}$$

and also we want that the components of the new metric $\tilde{g}^{\mu\nu}$ be real function of the complex variables $(z, \bar{z})$.

We do the change of coordinates

$$z = z' + i\gamma(x) \tag{2.14}$$

and the tetrads transform following

$$Z_a'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\, Z_a^\nu. \tag{2.15}$$

2.3 Aligning principal null directions

In [37] it is explained that it is not possible to get the Maxwell field because we have $\phi_2 = 0$ for RN and $\phi_2 \neq 0$ for KN. The solution would be to rotate the tetrad by a Lorentz transformation in order to align the repeated principal null directions. Since by definition a Lorentz transformation preserves the tetrad metric, the original metric is not modified. For a comment and an application to the 3d BTZ black hole see [41, app. B].

The null Lorentz transformation reads

$$\hat{\ell} = \ell', \qquad \hat{n} = n' + |\alpha|^2 \ell' + \alpha \bar{m}' + \bar{\alpha} m', \qquad \hat{m} = m' + \alpha \ell' \tag{2.16}$$

and the parameter α is given by

$$\alpha = \frac{ia \sin\theta}{\sqrt{2}(r + ia\cos\theta)}. \tag{2.17}$$

Note that the JN transformation and the null rotation are commuting. In principle the Newman–Penrose scalars transform under a null Lorentz transformation [41, app. B.2.2], and $F_{\mu\nu}$ would be invariant in this case, which is not what we want: we thus keep the coefficients invariant (in fact we could proceed the other way around).

At the end this should be equivalent to the gauge transformation we are doing (which may explain why the NP coefficients are not transformed).

The new tetrads are given explicitly by

$$\begin{aligned} \hat{n}^\mu &= \frac{1}{\rho^2}\left((r^2 + a^2)\delta_u^\mu - \frac{\Delta}{2}\, \delta_r^\mu + a \delta_\phi^\mu \right), \\ \hat{m}^\mu &= \frac{1}{\sqrt{2}(r + ia\cos\theta)} \left(\delta_\theta^\mu + \frac{i}{\sin\theta} \delta_\phi^\mu - ia\sin\theta\, \delta_u^\mu \right). \end{aligned} \tag{2.18}$$

We now want to apply this rotation on the Kerr–Newman gauge field

$$A = f_A \,\mathrm{d}t = f_A(\mathrm{d}u + f^{-1}\mathrm{d}r), \qquad f_A(r) = \frac{q}{r}. \tag{2.19}$$

For this we need to obtain its contravariant form, and then to write it in terms of the tetrads. We have

$$A_u = f_A, \qquad A_r = f_A f^{-1}. \tag{2.20}$$

The (u, r) part of the metric and its inverse are

$$g_{u,r} = \begin{pmatrix} -f & -1 \\ -1 & 0 \end{pmatrix}, \qquad g^{-1}_{u,r} = \begin{pmatrix} 0 & -1 \\ -1 & f \end{pmatrix}. \tag{2.21}$$

Finally we recall that

$$\ell^\mu = \delta^\mu_r, \qquad n^\mu = \delta^\mu_u - \frac{f}{2}\,\delta^\mu_r. \tag{2.22}$$

Using this one has

$$A^u = g^{ur} A_r = -f_A f^{-1}, \qquad A^r = g^{rr} A_r + g^{ru} A_u = 0 \tag{2.23}$$

or written differently

$$A^\mu = -f_A f^{-1}\,\delta^\mu_u. \tag{2.24}$$

The goal is to write it in terms of the static tetrads, and then replace them with Keane's tetrads. The combination

$$A^\mu = -f_A f^{-1}\left(n^\mu + \frac{f}{2}\,\ell^\mu\right) \tag{2.25}$$

is not successful.

It may be useful to write everything in terms of 1-forms [1, p. 5].

3 Five-dimensional JNA

It may be possible to find examples in [23].

3.1 Algorithm with quaternions

Why did we always proceed step-by-step when doing the transformation? This is because we would have obtained cross products that are not desirable. For example look at the transformation of r^2

$$\varrho^2 = r^2 + \big(a\cos\theta + b\sin\theta\big)^2 \tag{3.1}$$

under the transformation

$$r = r' - ia\cos\theta - ib\sin\theta. \tag{3.2}$$

Compare this with the correct formula

$$\varrho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta. \tag{3.3}$$

The problem is that the transformation has no way to know how to assign each angular momentum to each plane since the expression is totally symmetric. We thus need a way to disentangle the two contributions. A tempting idea is to introduce quaternions: we would be able to separate the two contributions by assigning each of them to a different element of the basis.

Recall that a quaternion is a number of dimension 4, written as

$$A = a_1 + i a_2 + j a_3 + k a_4 \tag{3.4}$$

where

$$i^2 = j^2 = k^2 = -1, \qquad ij = k. \tag{3.5}$$

We can the conjugate

$$A^* = a_1 - i a_2 - j a_3 - k a_4 \tag{3.6}$$

and also a norm

$$|A|^2 = AA^* = a_1^2 + a_2^2 + a_3^2 + a_4^2. \tag{3.7}$$

This is all what we need for our proposal. The transformation reads

$$r = r' - ia\cos\theta - jb\sin\theta, \qquad u = u' + ia\cos\theta + jb\sin\theta. \tag{3.8}$$

The natural ansatz to make are

$$i\,\mathrm{d}\theta = \sin\theta\,\mathrm{d}\phi, \qquad j\,\mathrm{d}\theta = -\cos\theta\,\mathrm{d}\psi, \tag{3.9}$$

and we see that again using two quaternions allow us to separate these two directions.

We look at the various term. The first transforms directly as

$$(1-f)\mathrm{d}u^2 \longrightarrow (1-\tilde{f}^{\{1,2\}})\,(\mathrm{d}u' - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi)^2. \tag{3.10}$$

The second is a bit harder

$$\begin{aligned}
\mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r) &\longrightarrow (\mathrm{d}u - ia\sin\theta\,\mathrm{d}\theta + jb\cos\theta\,\mathrm{d}\theta)(\mathrm{d}u + 2\mathrm{d}r + ia\sin\theta\,\mathrm{d}\theta - jb\cos\theta\,\mathrm{d}\theta) \\
&= (\mathrm{d}u - ia\sin\theta\,\mathrm{d}\theta)(\mathrm{d}u + 2\mathrm{d}r + ia\sin\theta\,\mathrm{d}\theta) \\
&\quad + (\mathrm{d}u + jb\cos\theta\,\mathrm{d}\theta)(\mathrm{d}u + 2\mathrm{d}r - jb\cos\theta\,\mathrm{d}\theta) - 2k\,ab\cos\theta\sin\theta\,\mathrm{d}\theta^2 \\
&= \mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r) + 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi + a^2\sin^4\theta\,\mathrm{d}\phi^2 \\
&\quad + 2b\cos^2\theta\,\mathrm{d}r\mathrm{d}\psi + b^2\cos^4\theta\,\mathrm{d}\psi^2.
\end{aligned}$$

To arrive at the final line we had to make the new ansatz

$$k\,\mathrm{d}\theta = 0. \tag{3.11}$$

In one sense it is natural because we don't have a third sphere, but on the other hand we always used the ansatz before taking product.

The angular term does not transform very well and we have two terms that should not be present

$$\begin{aligned}
r^2\mathrm{d}\Omega_3^2 &\longrightarrow (r^2 + a^2\cos^2\theta + b^2\sin^2\theta)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \cos^2\theta\,\mathrm{d}\psi^2) \\
&= (r^2 + a^2\cos^2\theta + b^2\sin^2\theta)\mathrm{d}\theta^2 + \big(r^2 + a^2(1-\sin^2\theta)\big)\sin^2\theta\,\mathrm{d}\phi^2 \\
&\quad + \big(r^2 + b^2(1-\cos^2\theta)\big)\cos^2\theta\,\mathrm{d}\phi^2 + a^2\cos^4\theta\,\mathrm{d}\psi^2 + b^2\sin^4\theta\,\mathrm{d}\phi^2.
\end{aligned}$$

When we will add all the pieces, the two last terms will remain whereas they should not be here, and there is no direct argument to remove them. It seems that, at the end, quaternions can not work.

When we will turn to higher dimensions we might hope that quaternions will be helpful, but they can only add three angular momenta in one transformation. Then arise the question of using octonions to add until seven angular momenta. But this kind of generalization are not totally satisfying: we would hope to have not limit on the number of dimensions that we can manage with our algorithm, so there is no reason that some special numbers distinguish themselves. Moreover we will see that we have trouble to generalize this to higher dimensions, and we will not even be able to explain this fact by using quaternions since we did not ran out of the possibility they give.

3.2 Myers–Perry in five dimensions

In five dimensions we have

$$1 - f = \frac{m}{r^2} \tag{3.12}$$

which we complexify first as

$$1 - \tilde{f}^{\{1\}} = \frac{m}{|r_1|^2} = \frac{m}{r^2 + a^2(1-\mu^2)} \tag{3.13}$$

We get the correct expression for one angular momentum.

The second time gives

$$1 - \tilde{f}^{\{1,2\}} = \frac{m}{|r_2|^2 + a^2(1-\mu^2)} = \frac{m}{r^2 + a^2(1-\mu^2) + b^2(1-\nu^2)}. \tag{3.14}$$

Let's denote the denominator by ρ^2/r^2 and compute

$$\begin{aligned}\frac{\rho^2}{r^2} &= r^2 + a^2(1-\mu^2) + b^2(1-\nu^2) = (\mu^2+\nu^2)r^2 + \nu^2 a^2 + \mu^2 b^2 \\ &= \mu^2(r^2+b^2) + \nu^2(r^2+a^2) = (r^2+b^2)(r^2+a^2)\left(\frac{\mu^2}{r^2+a^2} + \frac{\nu^2}{r^2+b^2}\right).\end{aligned}$$

and thus

$$\rho^2 = \Pi F. \tag{3.15}$$

Plugging this into $\tilde{f}^{\{1,2\}}$ we have

$$1 - \tilde{f}^{\{1,2\}} = \frac{mr^2}{\Pi F} \tag{3.16}$$

which is the correct expression.

Note that despite the similitude the derivation does not extend to $d = 6$.

Other manipulations Let's denote by A the denominator of $1 - f$, we get

$$A = r^2\, r^2 \left(\frac{\mu^2}{r^2} + \frac{\nu^2}{r^2}\right). \tag{3.17}$$

which is a very symmetric expression and generalizable to any dimension: we may make the rotation totally symmetric with this formula: the transformation amounts to replace the r^2 in the denominator and one r^2 on the left by $r^2 + a^2$.

Also for one angular momentum we should find

$$\frac{\rho^2}{r^2} = (r^2+a^2)r^2\left(\frac{\mu^2}{r^2+a^2} + \frac{\nu^2}{r^2}\right) \tag{3.18}$$

$$= \frac{1}{r^2}(r^2+a^2)r^2\left(1 - \frac{a^2\mu^2}{r^2+a^2}\right). \tag{3.19}$$

With two angular momenta we have

$$\frac{\rho^2}{r^2} = (r^2+a^2)(r^2+b^2)\left(\frac{\mu^2}{r^2+a^2} + \frac{\nu^2}{r^2+b^2}\right) \tag{3.20}$$

$$= \frac{1}{r^2}(r^2+a^2)(r^2+b^2)\left(1 - \frac{a^2\mu^2}{r^2+a^2} - \frac{b^2\nu^2}{r^2+b^2}\right). \tag{3.21}$$

3.3 CCLP black hole

3.3.1 Result

In [15] was given the most general charged black hole with two independent angular momenta from the minimal $d = 5$ gauged supergravity. We will consider the case of vanishing cosmological constant which corresponds to ungauged supergravity (the gauge coupling is also set to zero). The metric reads

$$\begin{aligned} \mathrm{d}s^2 = &-\mathrm{d}t^2 + (1-\tilde{f})(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi)^2 + \frac{r^2\varrho^2}{\Delta_r}\,\mathrm{d}r^2 \\ &+ \varrho^2\mathrm{d}\theta^2 + (r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2 + (r^2+b^2)\cos^2\theta\,\mathrm{d}\psi^2 \\ &- \frac{2Q}{\varrho^2}(b\sin^2\theta\,\mathrm{d}\phi + a\cos^2\theta\,\mathrm{d}\psi)(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi) \end{aligned} \tag{3.22}$$

where the function are given by

$$\tilde{f} = 1 - \frac{2m}{\varrho^2} + \frac{Q^2}{\varrho^4}, \qquad \Delta_r = \Pi + 2abQ + Q^2 - 2mr^2. \tag{3.23}$$

where we recall that

$$\varrho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta. \tag{3.24}$$

The gauge field is given by

$$A = \frac{\sqrt{3}}{2\lambda}\,\frac{Q}{\varrho^2}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi) \tag{3.25}$$

and we already derived it for the BMPV black hole.

This metric is also given in [3, sec. 2], but its form is not so nice; among other things it has

$$\tilde{f} = \frac{1}{F} - \frac{2m}{\varrho^2} + \frac{Q^2}{\varrho^4} \tag{3.26}$$

and I do not see directly where the F^{-1} will come from. In the paper it is given as

$$\frac{1}{F} = \frac{\Pi}{r^2\varrho^2} = \frac{\Pi}{\rho^2}. \tag{3.27}$$

In [4, 47] it is shown that the CCLP solution can be written as an extended Kerr–Schild metric. There is an additional term (proportional to a spacelike vector), and this new term is the one that can not be obtained from the JNA.

3.3.2 Transformation

We should start with the $5d$ RN black hole [65, sec. 3]

$$f = 1 - \frac{2m}{r^2} + \frac{Q^2}{r^4}. \tag{3.28}$$

As we can see if $Q = 0$ then we have the correct solution, but there is no obvious way to obtain the extra piece proportional to Q in the metric (we would guess that the Q should always appear in the function $\tilde{f}$). But understanding the presence of these extra piece is not difficult: the function

$$\Delta = \tilde{f}\rho^2 + \Pi(1-F) \tag{3.29}$$

is different from the Δ_r in the metric, and the reason is that our Δ depends on θ since $\tilde{f} \sim Q^2/\varrho^4$ and then $\tilde{f}\rho^2 \sim \varrho^2$. Note also that the angular momenta are reversed in the first parenthesis of the Q metric term.

3.3.3 Slowly rotating black hole

Keeping only linear terms in the rotation parameters (i. e. considering slow rotation) we are able to find the charged and rotating solution from [2, sec. 4].

3.4 General seed metric

We consider the more general 5-dimensional seed metric

$$\mathrm{d}s^2 = -f_t\,\mathrm{d}t^2 + f_r\,\mathrm{d}r^2 + f_\Omega\, r^2\,\mathrm{d}\Omega_3^2. \tag{3.30}$$

All functions depend only on r. In Eddington–Finkelstein coordinates the metric reads

$$\mathrm{d}s^2 = -f_t\,\mathrm{d}u^2 - 2\sqrt{f_t f_r}\,\mathrm{d}u\mathrm{d}r + f_\Omega\, r^2\,\mathrm{d}\Omega_3^2. \tag{3.31}$$

We complexify the seed metric (3.31) as

$$\mathrm{d}s^2 = (1-\tilde{f}_t)\,\mathrm{d}u^2 - \mathrm{d}u\Big(\mathrm{d}u + 2\sqrt{\tilde{f}_t\tilde{f}_r}\,\mathrm{d}r\Big) + \tilde{f}_\Omega\,\Big(r^2(\mathrm{d}\mu^2 + \mu^2\,\mathrm{d}\phi^2 + \mathrm{d}\nu^2) + R^2\,\nu^2\,\mathrm{d}\psi^2\Big). \tag{3.32}$$

We complexified the function $\tilde{f}_\Omega$ everywhere since it appears as a warp factor.

We now do the transformation on the (u,r) part

$$\begin{aligned}
\mathrm{d}s^2_{u,r} &= (1-\tilde{f}_t)\,(\mathrm{d}u - a\mu^2\,\mathrm{d}\phi)^2 - (\mathrm{d}u - a\mu^2\,\mathrm{d}\phi)\,\Big(\mathrm{d}u + 2\sqrt{\tilde{f}_t\tilde{f}_r}\,\mathrm{d}r + a\mu^2\Big(2\sqrt{\tilde{f}_t\tilde{f}_r} - 1\Big)\,\mathrm{d}\phi\Big)\\
&= (1-\tilde{f}_t)\,(\mathrm{d}u - a\mu^2\,\mathrm{d}\phi)^2 - \mathrm{d}u\Big(\mathrm{d}u + 2\sqrt{\tilde{f}_t\tilde{f}_r}\,\mathrm{d}r\Big) + 2\Big(1 - \sqrt{\tilde{f}_t\tilde{f}_r}\Big)\mathrm{d}u\mathrm{d}\phi\\
&\qquad + 2a\mu^2\sqrt{\tilde{f}_t\tilde{f}_r}\,\mathrm{d}r\mathrm{d}\phi + a^2\mu^4\Big(2\sqrt{\tilde{f}_t\tilde{f}_r} - 1\Big)\,\mathrm{d}\phi^2.
\end{aligned}$$

Looking at the (μ,ϕ) part gives (omitting the overall function $\tilde{f}_\Omega$)

$$\begin{aligned}
\mathrm{d}s^2_{\mu,\phi} &= \big(r^2 + a^2(1-\mu^2)\big)(\mathrm{d}\mu^2 + \mu^2\,\mathrm{d}\phi^2 + \mathrm{d}\nu^2)\\
&= (r^2 + a^2)(\mathrm{d}\mu^2 + \mu^2\mathrm{d}\phi^2) + r^2\mathrm{d}\nu^2 + a^2\big(-\mu^2\mathrm{d}\mu^2 + (1-\mu^2)\mathrm{d}\nu^2\big) - a^2\mu^4\mathrm{d}\phi^2\\
&= (r^2 + a^2)(\mathrm{d}\mu^2 + \mu^2\mathrm{d}\phi^2) + r^2\mathrm{d}\nu^2 - a^2\mu^4\mathrm{d}\phi^2.
\end{aligned}$$

We want to show that the third term vanishes: the derivative of the constraint reads

Gluing the two parts of the metric we get finally (setting $R = r$)

$$\begin{aligned}
\mathrm{d}s^2 &= (1-\tilde{f}_t)\,(\mathrm{d}u - a\mu^2\,\mathrm{d}\phi)^2 - \mathrm{d}u\Big(\mathrm{d}u + 2\sqrt{\tilde{f}_t\tilde{f}_r}\,\mathrm{d}r\Big) + 2\Big(1 - \sqrt{\tilde{f}_t\tilde{f}_r}\Big)\mathrm{d}u\mathrm{d}\phi\\
&\qquad + 2a\mu^2\sqrt{\tilde{f}_t\tilde{f}_r}\,\mathrm{d}r\mathrm{d}\phi + a^2\mu^4\Big(2\sqrt{\tilde{f}_t\tilde{f}_r} - 1 - \tilde{f}_\Omega\Big)\,\mathrm{d}\phi^2\\
&\qquad + \tilde{f}_\Omega\Big((r^2 + a^2)(\mathrm{d}\mu^2 + \mu^2\mathrm{d}\phi^2) + r^2(\mathrm{d}\nu^2 + \nu^2\mathrm{d}\psi^2)\Big).
\end{aligned} \tag{3.33}$$

4 Extension to higher dimensions with maximal number of momenta

4.1 Seed metric

We consider the d-dimensional static metric

$$\mathrm{d}s^2 = -f\,\mathrm{d}t^2 + f^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{d-2}^2 \tag{4.1}$$

where $\mathrm{d}\Omega_{d-2}^2$ is the metric on S^{d-2}

$$\mathrm{d}\Omega_{d-2}^2 = \mathrm{d}\theta_{d-2} + \sin^2\theta_{d-2}\,\mathrm{d}\Omega_{d-3}^2 = \sum_i \left(\mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{d}\phi_i^2\right). \tag{4.2}$$

We recall the notations: the indices are

$$i = 1, \ldots, n, \qquad n = \left\lfloor \frac{d-1}{2} \right\rfloor, \tag{4.3}$$

and we defined

$$\varepsilon = \begin{cases} 0 & d \text{ even} \\ 1 & d \text{ odd} \end{cases} \qquad \varepsilon' = 1 - \varepsilon. \tag{4.4}$$

After transforming to Eddington–Finkelstein coordinates and using direction cosines we get

$$\mathrm{d}s^2 = -f\,\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 \sum_i \left(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2\right) + \varepsilon'\,r^2\mathrm{d}\alpha^2 \tag{4.5a}$$

$$= (1-f)\,\mathrm{d}u^2 - \mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r) + r^2 \sum_i \left(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2\right) + \varepsilon'\,r^2\mathrm{d}\alpha^2, \tag{4.5b}$$

4.2 Rotating solution

We take the results from [45] where we replaced

$$\mu r^{1+\varepsilon} = (1-\tilde{f})\rho^2 \tag{4.6}$$

everywhere (see below for the definition of ρ^2) in order to get the general solution in term of the function $\tilde{f}$: in fact we see that, as in $4d$, the specific form of $\tilde{f}$ is needed only for the coefficient of $\mathrm{d}t^2$. We omit the term $\varepsilon'\,r^2\mathrm{d}\alpha^2$.

In what we call non-BL coordinates, the solution reads [2]

$$\begin{aligned}\mathrm{d}s^2 = &- \mathrm{d}t^2 + \mathrm{d}r^2 + \sum_i (r^2 + a_i^2)(\mathrm{d}\mu_i^2 + \mu_i^2\mathrm{d}\phi_i^2) \pm 2\sum_i a_i\mu_i^2\,\mathrm{d}r\mathrm{d}\phi_i \\ &+ (1-\tilde{f})\left(\mathrm{d}t \pm \mathrm{d}r + \sum_i a_i\mu_i^2\mathrm{d}\phi_i\right)^2.\end{aligned} \tag{4.7}$$

We want to switch to Eddington–Finkelstein coordinates

$$\mathrm{d}u = \mathrm{d}t - \mathrm{d}r. \tag{4.8}$$

The resulting metric is

$$\begin{aligned}\mathrm{d}s^2 = &- \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \sum_i (r^2 + a_i^2)(\mathrm{d}\mu_i^2 + \mu_i^2\mathrm{d}\phi_i^2) - 2\sum_i a_i\mu_i^2\,\mathrm{d}r\mathrm{d}\phi_i \\ &+ (1-\tilde{f})\left(\mathrm{d}u + \sum_i a_i\mu_i^2\mathrm{d}\phi_i\right)^2.\end{aligned} \tag{4.9}$$

In BL coordinates we have

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (1-\tilde{f})\left(\mathrm{d}t + \sum_i a_i\mu_i^2\mathrm{d}\phi_i\right)^2 + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \sum_i (r^2 + a_i^2)\left(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2\right), \tag{4.10}$$

[2] Two sets of coordinates are presented, see p. 314 and 326. The one with minus sign seems to fit better.

where

$$\Pi = \prod_i (r^2 + a_i^2), \qquad F = 1 - \sum_i \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} = r^2 \sum_i \frac{\mu_i^2}{r^2 + a_i^2}, \tag{4.11a}$$

$$\rho^2 = \Pi F, \qquad \Delta = \tilde{f}\rho^2 + \Pi(1 - F). \tag{4.11b}$$

4.3 One non-vanishing angular momentum

Several authors applied the Janis–Newman algorithm with one angular momentum to various higher-dimensional black holes [2, 27, 65].

4.3.1 General result

To shorten the notation we define

$$\theta = \theta_{d-2}, \qquad \phi = \theta_{d-3}. \tag{4.12}$$

The transformation in spherical coordinates reads [65, sec. 4, 2, sec. 2]

$$r = r' - ia\cos\theta, \qquad u = u' + ia\cos\theta. \tag{4.13}$$

The resulting metric reads in Boyer–Lindquist coordinates

$$\begin{aligned} ds^2 = &- \tilde{f}\,dt^2 + 2a(1 - \tilde{f})\sin^2\theta\,dt d\phi + \frac{r^{d-3}\rho^2}{\Delta}\,dr^2 + \rho^2 d\theta^2 \\ &+ \frac{\Sigma^2}{\rho^2}\sin^2\theta\,d\phi^2 + r^2\cos^2\theta^2\,d\Omega_{d-4}^2 \end{aligned} \tag{4.14}$$

where we defined as usual

$$\rho^2 = r^2 + a^2\cos^2\theta \qquad \Delta = \tilde{f}\rho^2 + a^2\sin^2\theta, \qquad \frac{\Sigma^2}{\rho^2} = r^2 + a^2 + a g_{t\phi}. \tag{4.15}$$

Note that we redefined

$$\sin^2\theta\sin^2\phi = \cos^2\theta. \tag{4.16}$$

The two functions to go to Boyer–Lindquist are

$$g = \frac{r^2 + a^2}{\Delta}, \qquad h = \frac{a}{\Delta}. \tag{4.17}$$

The result is very similar to the result in $4d$: the only difference is the factor r^{d-3} in the component g_{rr}, while the last term coming from the other directions of the sphere are passive. The reason is that we could restrict our analysis to the four dimensions (t, r, θ, ϕ) inside the total d dimensions and apply the algorithm: this is some kind of decoupling. Especially we see that the factor r^2 in front ofthe remaining directions is not transformed.

4.3.2 Examples

The charged Tangherlini metric reads [60]

$$f = 1 - \frac{\mu}{r^{d-3}} + \frac{Q^2}{r^{2(d-3)}} \tag{4.18}$$

while the gauge field is

$$A = \frac{Q}{r^{d-3}}\,dt. \tag{4.19}$$

The transformation of f is [2, sec. 2, 3, 65, sec. 4, 5]

$$\tilde{f} = 1 - \frac{1}{\rho^2} \left(\frac{\mu}{r^{d-5}} - \frac{Q^2}{r^{2(d-4)}} \right). \tag{4.20}$$

With this form Δ is independent of θ and the transformation to BL is well defined. The potential would be

$$A = \frac{Q}{r^{d-5}\rho^2} \, (\mathrm{d}t + a \sin^2\theta \, \mathrm{d}\phi). \tag{4.21}$$

We used a gauge transformation to remove the $A_r(r)$ component.

For $Q = 0$ we obtain the Myers–Perry metric with one angular momentum. If $Q \neq 0$ then the Einstein equation is not solved; this is linked to the fact that the trace of the EM stress–energy tensor does not vanish for $d \neq 4$. A solution can be obtained for slow rotation [2].

Another solution can be found from the rotating Yang–Mills black hole (for $d \geq 6$) [27]

$$f = 1 - \frac{\mu}{r^{d-3}} - \frac{Q^2}{r^2} \tag{4.22}$$

(note the minus sign in front of the charge term). In fact Q is proportional to $(d-5)^{-1}$ so this black hole is also valid for $d = 4$; then the charge term changes sign and we get Kerr–Newman with a magnetic charge Q. The complexification is

$$\tilde{f} = 1 - \frac{\mu}{r^{d-5}\rho^2} - \frac{Q^2}{\rho^2}. \tag{4.23}$$

Again Δ does not depend on θ. But this solution may not be trusted (see sec. 1.4).

4.4 Giampieri's approach

In this section we work with d odd, but generalization to even d is direct.

As written in (4.5) the metric looks like a 2-dimensional space (t, r) with a certain number of additional 2-spheres (μ_i, ϕ_i) which are independent from one another. We can thus imagine to put in rotation only one of these spheres. Then we will apply again and again the algorithm until all the spheres have angular momentum: the whole complexification will thus be a n-steps process.

We need to transform specifically the radial coordinate in front of each plane in ellipsoidal coordinates. For this reason we define a redundant set of radial coordinates $\{r_{i_1}, R\}$ such that

$$r_{i_1} = R = r, \tag{4.24}$$

where r_{i_1} is the one we want to transform, and R for the one which do not transform. We could want to write

$$r_{i_1}^2 (\mathrm{d}\mu_{i_1}^2 + \mu_{i_1}^2 \mathrm{d}\phi_{i_1}^2) + R^2 \sum_{i \neq i_1} \left(\mathrm{d}\mu_i^2 + \mu_i^2 \, \mathrm{d}\phi_i^2 \right) \tag{4.25}$$

but in this case the complexification will not work. The correct expression is

$$\mathrm{d}s^2 = (1-f)\,\mathrm{d}u^2 - \mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r_{i_1}) + r_{i_1}^2 (\mathrm{d}\mu_{i_1}^2 + \mu_{i_1}^2 \mathrm{d}\phi_{i_1}^2) + \sum_{i \neq i_1} \left(r_{i_1}^2 \mathrm{d}\mu_i^2 + R^2 \mu_i^2 \, \mathrm{d}\phi_i^2 \right). \tag{4.26}$$

Using inspiration from the $4d$ case, we apply the transformation

$$r_{i_1} = r'_{i_1} - i\, a_{i_1} \sqrt{1 - \mu_{i_1}^2}, \qquad u = u' + i\, a_{i_1} \sqrt{1 - \mu_{i_1}^2}. \tag{4.27}$$

Making the ansatz

$$i\,\frac{\mathrm{d}\mu_{i_1}}{\sqrt{1-\mu_{i_1}^2}} = \mu_{i_1}\,\mathrm{d}\phi_{i_1} \tag{4.28}$$

the differentials read

$$\mathrm{d}r_{i_1} = \mathrm{d}r'_{i_1} + a_{i_1}\mu_{i_1}^2\,\mathrm{d}\phi_{i_1}, \qquad \mathrm{d}u = \mathrm{d}u' - a_{i_1}\mu_{i_1}^2\,\mathrm{d}\phi_{i_1}. \tag{4.29}$$

We will write the complexified function as $\tilde{f}^{\{i_1\}}$: we need to keep track of the order in which we gave angular momentum since the function f will be transformed at each step.

We now replace in the seed metric (omitting the prime), beginning with (u,r) part

$$\begin{aligned}
\mathrm{d}s^2_{u,r} &= (1-\tilde{f}^{\{i_1\}})\left(\mathrm{d}u - a_{i_1}\mu_{i_1}^2\,\mathrm{d}\phi_{i_1}\right)^2 - \left(\mathrm{d}u - a_{i_1}\mu_{i_1}^2\,\mathrm{d}\phi_{i_1}\right)\left(\mathrm{d}u + 2\mathrm{d}r_{i_1} + a_{i_1}\mu_{i_1}^2\,\mathrm{d}\phi_{i_1}\right) \\
&= (1-\tilde{f}^{\{i_1\}})\left(\mathrm{d}u - a_{i_1}\mu_{i_1}^2\,\mathrm{d}\phi_{i_1}\right)^2 - \mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r_{i_1}) + 2a_{i_1}\mu_{i_1}^2\,\mathrm{d}r_{i_1}\mathrm{d}\phi_{i_1} + a_{i_1}^2\mu_{i_1}^4\,\mathrm{d}\phi_{i_1}^2.
\end{aligned}$$

We now turn to the $\{\mu_i,\phi_i\}$ part:

$$\begin{aligned}
\mathrm{d}s^2_{\mu,\phi} &= \big(r_{i_1}^2 + a_{i_1}^2(1-\mu_{i_1}^2)\big)(\mathrm{d}\mu_{i_1}^2 + \mu_{i_1}^2\mathrm{d}\phi_{i_1}^2) + \sum_{i\neq i_1}\Big(\big(r_{i_1}^2 + a_{i_1}^2(1-\mu_{i_1}^2)\big)\mathrm{d}\mu_i^2 + R^2\mu_i^2\,\mathrm{d}\phi_i^2\Big) \\
&= \big(r_{i_1}^2 + a_{i_1}^2\big)(\mathrm{d}\mu_{i_1}^2 + \mu_{i_1}^2\mathrm{d}\phi_{i_1}^2) + \sum_{i\neq i_1}\big(r_{i_1}^2\mathrm{d}\mu_i^2 + R^2\mu_i^2\,\mathrm{d}\phi_i^2\big) \\
&\quad + a_{i_1}^2\left[-\mu_{i_1}^2\mathrm{d}\mu_{i_1}^2 + (1-\mu_{i_1}^2)\sum_{i\neq i_1}\mathrm{d}\mu_i^2\right] - a_{i_1}^2\mu_{i_1}^4\,\mathrm{d}\phi_{i_1}^2.
\end{aligned}$$

We want to show that the term in bracket vanishes. For this aim we use the relations

$$\sum_i \mu_i^2 = 1 \Longrightarrow \sum_i \mu_i\mathrm{d}\mu_i = 0 \tag{4.30}$$

which implies

$$\begin{aligned}
[\cdots] &= \mu_{i_1}^2\mathrm{d}\mu_{i_1}^2 - (1-\mu_{i_1}^2)\sum_{i\neq i_1}\mathrm{d}\mu_i^2 = \left(\sum_{i\neq i_1}\mu_i\mathrm{d}\mu_i\right)^2 - \sum_{j\neq i_1}\mu_j^2\sum_{i\neq i_1}\mathrm{d}\mu_i^2 \\
&= \sum_{i,j\neq i_1}\left(\mu_i\mu_j\mathrm{d}\mu_i\mathrm{d}\mu_j - \mu_j^2\mathrm{d}\mu_i^2\right) = \sum_{i,j\neq i_1}\mu_j\big(\mu_i\mathrm{d}\mu_j - \mu_j\mathrm{d}\mu_i\big)\mathrm{d}\mu_i = 0
\end{aligned}$$

by antisymmetry.

When we add $\mathrm{d}s^2_{u,r}$ and $\mathrm{d}s^2_{\mu,\phi}$ we see that the two last terms $a_{i_1}^2\mu_{i_1}^4\,\mathrm{d}\phi_{i_1}^2$ compensate, and we finally arrive at

$$\begin{aligned}
\mathrm{d}s^2 &= (1-\tilde{f}^{\{i_1\}})\left(\mathrm{d}u - a_{i_1}\mu_{i_1}^2\,\mathrm{d}\phi_{i_1}\right)^2 - \mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r_{i_1}) + 2a_{i_1}\mu_{i_1}^2\,\mathrm{d}r_{i_1}\mathrm{d}\phi_{i_1} \\
&\quad + \big(r_{i_1}^2 + a_{i_1}^2\big)(\mathrm{d}\mu_{i_1}^2 + \mu_{i_1}^2\mathrm{d}\phi_{i_1}^2) + \sum_{i\neq i_1}\big(r_{i_1}^2\mathrm{d}\mu_i^2 + R^2\mu_i^2\,\mathrm{d}\phi_i^2\big).
\end{aligned} \tag{4.31}$$

We can now set $r_{i_1} = R = r$ to get

$$\begin{aligned}
\mathrm{d}s^2 &= (1-\tilde{f}^{\{i_1\}})\left(\mathrm{d}u - a_{i_1}\mu_{i_1}^2\,\mathrm{d}\phi_{i_1}\right)^2 - \mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r) + 2a_{i_1}\mu_{i_1}^2\,\mathrm{d}r\mathrm{d}\phi_{i_1} \\
&\quad + \big(r^2 + a_{i_1}^2\big)(\mathrm{d}\mu_{i_1}^2 + \mu_{i_1}^2\mathrm{d}\phi_{i_1}^2) + \sum_{i\neq i_1} r^2\big(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2\big).
\end{aligned} \tag{4.32}$$

We see that we recover the same structure as the seed metric (with some extra terms, but they do not affect the other planes).

We should now split again r in terms of (r_{i_2}, R). Very similarly to the first time we have

$$\begin{aligned}\mathrm{d}s^2 = (1-\tilde{f}^{\{i_1\}}) \Big(\mathrm{d}u - a_{i_1}\mu_{i_1}^2\, \mathrm{d}\phi_{i_1}\Big)^2 - \mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r_{i_2}) + 2a_{i_1}\mu_{i_1}^2\, \mathrm{d}R\mathrm{d}\phi_{i_1} \\ + \big(r_{i_2}^2 + a_{i_1}^2\big)\mathrm{d}\mu_{i_1}^2 + \big(R^2 + a_{i_1}^2\big)\mu_{i_1}^2\mathrm{d}\phi_{i_1}^2 + r_{i_2}^2(\mathrm{d}\mu_{i_2}^2 + \mu_{i_2}^2\mathrm{d}\phi_{i_2}^2) \\ + \sum_{i\neq i_1,i_2} \Big(r_{i_2}^2\mathrm{d}\mu_i^2 + R^2\mu_i^2\, \mathrm{d}\phi_i^2\Big).\end{aligned} \tag{4.33}$$

We can now complexify as

$$r_{i_2} = r'_{i_2} - i a_{i_2}\sqrt{1-\mu_{i_2}^2}, \qquad u = u' + i\, a_{i_1}\sqrt{1-\mu_{i_2}^2}. \tag{4.34}$$

The steps are exactly the same as before, except that we have some inert terms. The complexified functions is now $\tilde{f}^{\{i_1,i_2\}}$.

Repeating the procedure n times we arrive at

$$\boxed{\begin{aligned}\mathrm{d}s^2 = -\,\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \sum_i (r^2 + a_i^2)(\mathrm{d}\mu_i^2 + \mu_i^2\mathrm{d}\phi_i^2) - 2\sum_i a_i\mu_i^2\, \mathrm{d}r\mathrm{d}\phi_i \\ + \Big(1 - \tilde{f}^{\{i_1,\ldots,i_n\}}\Big)\left(\mathrm{d}u + \sum_i a_i\mu_i^2\mathrm{d}\phi_i\right)^2.\end{aligned}} \tag{4.35}$$

This is the correct result [45].

Before ending this section, we comment the case of even dimensions: the term $\varepsilon'\, r^2\mathrm{d}\alpha^2$ is complexified as $\varepsilon'\, r_{i_1}^2\,\mathrm{d}\alpha^2$, since it contributes to the sum

$$\sum_i \mu_i^2 + \alpha^2 = 1. \tag{4.36}$$

This can be seen more clearly by defining $\mu_{n+1} = \alpha$ (we can also define $\phi_{n+1} = 0$), in which case the indice i runs from 1 to $n + \varepsilon$, and all the previous computations are still valid.

4.5 Tetrad approach

Recall the metric (4.5) (we focus on odd dimension)

$$\mathrm{d}s^2 = -f\,\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\sum_i \Big(\mathrm{d}\mu_i^2 + \mu_i^2\, \mathrm{d}\phi_i^2\Big). \tag{4.37}$$

Then the inverse metric is

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} = g^{\mu\nu}\partial_\mu\partial_\nu = f\,\partial_r^2 - 2\,\partial_u\partial_r + \frac{1}{r^2}\sum_i \left(\partial_{\mu_i}^2 + \frac{1}{\mu_i^2}\,\partial_{\phi_i}^2\right). \tag{4.38}$$

We can introduce a set of tetrads and write the metric as

$$g^{\mu\nu} = -\ell^\mu n^\nu - \ell^\nu n^\mu + \sum_i \big(m_i^\mu \bar{m}_i^\nu + m_i^\nu \bar{m}_i^\mu\big), \tag{4.39}$$

where the vectors are

$$\ell^\mu = \delta_r^\mu, \qquad n^\mu = \delta_t^\mu - \frac{f}{2}\,\delta_r^\mu, \qquad m_i^\mu = \frac{1}{\sqrt{2}\,r}\left(\delta_{\mu_i}^\mu + \frac{i}{\mu_i}\,\delta_{\phi_i}^\mu\right). \tag{4.40}$$

We do the first transformation in the i_1-plane

$$u = u' + i\, a_{i_1} \sqrt{1-\mu_{i_1}^2}, \qquad r = r' - i\, a_{i_1} \sqrt{1-\mu_{i_1}^2}. \tag{4.41}$$

We complexify the tetrad as

$$\begin{aligned} \ell^\mu &= \delta_r^\mu, \qquad n^\mu = \delta_t^\mu - \frac{\tilde{f}^{\{1\}}}{2}\, \delta_r^\mu, \\ m_{i_1}^\mu &= \frac{1}{\sqrt{2}\,\bar{r}} \left(\delta_{\mu_{i_1}}^\mu + \frac{i}{\mu_{i_1}}\, \delta_{\phi_{i_1}}^\mu \right), \\ m_i^\mu &= \frac{1}{\sqrt{2}\,\bar{r}}\, \delta_{\mu_i}^\mu + \frac{1}{\sqrt{2}\,R}\, \frac{i}{\mu_i}\, \delta_{\phi_i}^\mu, \qquad i \neq i_1. \end{aligned} \tag{4.42}$$

The vector ℓ^μ and n^μ do not change. For the two others we have

$$\begin{aligned} m_{i_1}^\mu &= \frac{1}{\sqrt{2}\big(r + i\, a_{i_1} \sqrt{1-\mu_{i_1}^2}\big)} \left(\delta_{\mu_{i_1}}^\mu + i\, \frac{a_{i_1}\mu_{i_1}}{\sqrt{1-\mu_{i_1}^2}}\, (\delta_u^\mu - \delta_r^\mu) + \frac{i}{\mu_{i_1}}\, \delta_{\phi_{i_1}}^\mu \right), \\ m_i^\mu &= \frac{1}{\sqrt{2}\big(r + i\, a_{i_1} \sqrt{1-\mu_{i_1}^2}\big)}\, \delta_{\mu_i}^\mu + \frac{1}{\sqrt{2}\, r}\, \frac{i}{\mu_i}\, \delta_{\phi_i}^\mu \end{aligned} \tag{4.43}$$

where again $i \neq i_1$.

The product $m_{i_1}^\mu \bar{m}_{i_1}^\nu$ is

$$m_{i_1}^\mu \bar{m}_{i_1}^\nu \propto \partial_{\mu_{i_1}}^2 + \left(\frac{a_{i_1}\mu_{i_1}}{\sqrt{1-\mu_{i_1}^2}}\, (\delta_u^\mu - \delta_r^\mu) + \frac{1}{\mu_{i_1}}\, \delta_{\phi_{i_1}}^\mu \right) \left(\frac{a_{i_1}\mu_{i_1}}{\sqrt{1-\mu_{i_1}^2}}\, (\delta_u^\nu - \delta_r^\nu) + \frac{1}{\mu_{i_1}}\, \delta_{\phi_{i_1}}^\nu \right).$$

We reconstruct the contravariant metric

$$\begin{aligned} g^{\mu\nu}\partial_\mu\partial_\nu &= \tilde{f}^{\{1\}}\, \partial_r^2 - 2\, \partial_u\partial_r + \frac{1}{\rho_{i_1}^2} \sum_i \partial_{\mu_i}^2 + \frac{1}{r^2} \sum_{i\neq i_1} \frac{1}{\mu_i^2}\, \partial_{\phi_i}^2 \\ &\quad + \frac{1}{\rho_{i_1}^2} \left(\frac{a_{i_1}\mu_{i_1}}{\sqrt{1-\mu_{i_1}^2}}\, (\partial_u - \partial_r) + \frac{1}{\mu_{i_1}}\, \partial_{\phi_{i_1}} \right) \left(\frac{a_{i_1}\mu_{i_1}}{\sqrt{1-\mu_{i_1}^2}}\, (\partial_u - \partial_r) + \frac{1}{\mu_{i_1}}\, \partial_{\phi_{i_1}} \right) \\ &= \tilde{f}^{\{1\}}\, \partial_r^2 - 2\, \partial_u\partial_r + \frac{1}{\rho_{i_1}^2} \sum_i \partial_{\mu_i}^2 + \frac{1}{r^2} \sum_{i\neq i_1} \frac{1}{\mu_i^2}\, \partial_{\phi_i}^2 \\ &\quad + \frac{1}{\rho_{i_1}^2} \left(\frac{1}{\mu_{i_1}^2}\, \partial_{\phi_{i_1}}^2 + \frac{a_{i_1}^2\mu_{i_1}^2}{1-\mu_{i_1}^2}\, (\partial_u - \partial_r)^2 + \frac{2a_{i_1}}{\sqrt{1-\mu_{i_1}^2}}\, (\partial_u - \partial_r)\partial_{\phi_{i_1}} \right) \end{aligned}$$

where we defined

$$\rho_{i_1}^2 = r^2 + a_{i_1}^2 (1 - \mu_{i_1}^2). \tag{4.44}$$

4.6 Recovering Boyer–Lindquist coordinates

In this short section we just want to recall how to transform the metric (4.35) to Boyer–Lindquist coordinates, and to gather some useful formula; we will use [45] as a reference, just replacing

$$\mu r^{1+\varepsilon} = (1 - \tilde{f})\rho^2 \tag{4.45}$$

everywhere (see below for the definition of ρ^2).

The first step is to eliminate the light-cone coordinate u with

$$u = t - r. \tag{4.46}$$

The computation is straightforward and gives

$$\begin{aligned} \mathrm{d}s^2 = &- \mathrm{d}t^2 + \mathrm{d}r^2 + \sum_i (r^2 + a_i^2)(\mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{d}\phi_i^2) - 2 \sum_i a_i \mu_i^2 \, \mathrm{d}r \mathrm{d}\phi_i \\ &+ (1 - \tilde{f}^{\{i_1,\ldots,i_n\}}) \left(\mathrm{d}t - \mathrm{d}r + \sum_i a_i \mu_i^2 \mathrm{d}\phi_i \right)^2 . \end{aligned} \tag{4.47}$$

Arrived at this point we get the Boyer–Lindquist metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (1 - \tilde{f}) \left(\mathrm{d}t - \sum_i a_i \mu_i^2 \mathrm{d}\phi_i \right)^2 + \frac{\rho^2}{\Delta} \, \mathrm{d}r^2 + \sum_i (r^2 + a_i^2) \Big(\mathrm{d}\mu_i^2 + \mu_i^2 \, \mathrm{d}\phi_i^2 \Big) \tag{4.48}$$

from the transformation

$$g = \frac{\rho^2 (1 - \tilde{f})}{\Delta} = \frac{F(1 - \tilde{f})}{1 - F(1 - \tilde{f})}, \tag{4.49a}$$

$$h_i = \frac{\Pi}{\Delta} \, \frac{a_i}{r^2 + a_i^2} = \frac{1}{1 - F(1 - \tilde{f})} \, \frac{a_i}{r^2 + a_i^2}. \tag{4.49b}$$

The various quantities involved are

$$\begin{gathered} \Pi = \prod_i (r^2 + a_i^2), \qquad F = 1 - \sum_i \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} = r^2 \sum_i \frac{\mu_i^2}{r^2 + a_i^2}, \\ \rho^2 = \Pi F, \qquad \Delta = \tilde{f} \rho^2 + \Pi (1 - F). \end{gathered} \tag{4.50}$$

Note that we can also find a transformation directly from (u, r) coordinates to Boyer–Lindquist

$$g' = 1 + g = \frac{\Pi}{\Delta} = \frac{1}{1 - F(1 - \tilde{f})}. \tag{4.51}$$

4.7 Schwarzschild–Tangherlini to Myers–Perry

We consider the Schwarzschild–Tangherlini static metric

$$\mathrm{d}s^2 = -f \, \mathrm{d}t^2 + f^{-1} \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega_{d-2}^2, \qquad 1 - f = \frac{\mu}{r^{d-3}}. \tag{4.52}$$

We can rewrite the power of r:

$$d + \varepsilon = 2n + 2 \Longrightarrow d - 3 = 2n - (1 + \varepsilon). \tag{4.53}$$

As we have seen in section 4, the transformation

$$f \longrightarrow \tilde{f}^{\{i_1,\ldots,i_n\}} \tag{4.54}$$

is made in n steps. Here we consider that all the steps are symmetric so we do not have to distinguish the order and we note $\tilde{f} \equiv \tilde{f}^{\{i_1,\ldots,i_n\}}$.

We should find [45]

$$\tilde{f} = 1 - \frac{\mu r^{1+\varepsilon}}{\Pi F} = 1 - \frac{\mu r^{1+\varepsilon}}{\rho^2}, \qquad F = 1 - \sum_i \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}. \tag{4.55}$$

First approach In order to complexify f we need to rewrite it in terms if the $\{r_i, R\}$, and two ways come to our mind; as we will see both give the same result! The two possible splitting of r^{d-3} are

$$1 - f = \frac{\mu}{R^{d-n-3}} \prod_i \frac{1}{r_i} \tag{4.56a}$$

$$= \frac{\mu}{R^{d-2n-3}} \prod_i \frac{1}{r_i^2}. \tag{4.56b}$$

Let's complexify them:

1. The first one is

$$1 - \tilde{f} = \frac{\mu}{2^n R^{d-n-3}} \prod_i \left(\frac{1}{r_i} + \frac{1}{\bar{r}_i} \right) = \frac{\mu}{R^{d-n-3}} \prod_i \frac{\operatorname{Re} r_i}{|r_i|^2}$$

letting $\operatorname{Re} r_i = r'_i = R = r'$ gives

$$= \frac{\mu r'^n}{r^{d-n-3}} \prod_i \frac{1}{|r_i|^2} = \mu r'^{1+\varepsilon} \prod_i \frac{1}{|r_i|^2}$$

where we have used the relation

$$d + \varepsilon = 2n + 2 \tag{4.57}$$

2. The second is

$$1 - \tilde{f} = \frac{\mu}{R^{d-2n-3}} \prod_i \frac{1}{|r_i|^2} = \frac{\mu}{r'^{d-2n-3}} \prod_i \frac{1}{|r_i|^2} = \mu r'^{1+\varepsilon} \prod_i \frac{1}{|r_i|^2}$$

where we wrote $R = r'$.

In this context there are nothing arbitrary for the complexification as we have already shown for $4d$.

We can now rewrite the denominator as

$$\prod_i \left(r^2 + a_i^2 (1 - \mu_i^2) \right) = \prod_i (r^2 + a_i^2) \left(1 - \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \right) = \prod_i (r^2 + a_i^2) \prod_i \left(1 - \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \right).$$

Replacing the denominator gives

$$\boxed{\tilde{f} = 1 - \frac{\mu r^{1+\varepsilon}}{\Pi} \prod_i \frac{1}{\left(1 - \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \right)}.} \tag{4.58}$$

This is not correct if we compare with (4.55), but we see that the $\mu r^{1+\varepsilon}/\Pi$ factor agrees: we just need to find how to get F instead of the product. We can think that we fail because we tried to do all transformations at the same time; nonetheless the complexification of f given here works for $d = 3, 4$ or when only one angular momentum is not zero.

As special cases of this section, with $a_i = 0$ for $i \neq 1$, there are the black holes given in [27, 65].

Second approach We consider odd dimension and we focus on the denominator of $1-f$ (call it A), then

$$A = r^{d-3} = \frac{1}{r^2}\, r^{2n} = r^{2n} \sum_i \frac{\mu_i^2}{r^2} = \left(\prod_i r_i^2\right)\left(\sum_i \frac{\mu_i^2}{r_i^2}\right). \tag{4.59}$$

This last expression is perfectly symmetric under exchange 2-planes. Doing the replacements

$$r_i^2 \longrightarrow r_i^2 + a_i^2 \tag{4.60}$$

(followed by $r_i = r$) gives

$$A = \prod_i (r^2 + a_i^2)\left(\sum_i \frac{\mu_i^2}{r^2 + a_i^2}\right) = \frac{\Pi F}{r^2}. \tag{4.61}$$

This is the correct answer. So we were not able to really explain the rule that we gave, but it is natural when we look at the coefficient of the 2-sphere $\mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{d}\phi_i^2$ which goes from r^2 to $r^2 + a_i^2$. There the terms $-a_i^2\mu_i^2$ cancelled, so we can hope to see similar thing here.

4.8 Schwarzschild–Tangherlini to Myers–Perry with equal momenta

According to the formula (4.55) for $\tilde{f}$, for equal momenta $a_i = a$ we have (dor $d \geq 5$)

$$F = 1 - \frac{a^2}{r^2 + a^2} \Longrightarrow \tilde{f} = \frac{\mu}{(r^2 + a^2)^{n-1}}. \tag{4.62}$$

If we try to apply all the transformation to all the r we will obtain

$$\frac{1}{r^2} \longrightarrow \left(r^2 + a \sum_i (1 - \mu_i^2)\right)^{-1} = \frac{1}{r^2 + (n-1)a^2}. \tag{4.63}$$

This is in contradiction with the previous formula except for $n = 2$ (i.e. $d = 5$).

From here it appears that we would need a transformation

$$r_i \longrightarrow r_i - i a f_i(\mu_j) \tag{4.64}$$

such that

$$r_i^2 \longrightarrow r_i^2 + a_i^2. \tag{4.65}$$

But it is hard to see how this could agree with the transformation of the metric.

4.9 Other examples

An example of higher-dimensional black holes with a dilaton can be found in [40, 43].

In [16] there is another example for a rotating black hole. The complexification is quite strange. It is almost

$$\frac{1}{r^{d-3}} = \left(\frac{1}{r}\right)^{d-3} = \frac{r^{d-3}}{\rho^{2(d-3)}}. \tag{4.66}$$

5 Extremal rotating solutions

Sometimes when applying the Janis–Newman algorithm we are not able to go back to Boyer–Lindquist coordinates because the functions in the transformation get some θ dependence. Some limit may remove this dependence and give a correct transformation: since this process will reduce the parameter number we call the solution extremal, but we will have to show that this is equivalent to the usual notion of extremality.

5.1 $\mathrm{adS}_2 \times S^2$ space

Let's consider the metric of $\mathrm{adS}_2 \times S^2$ with different radius $R_1 \neq R_2$:

$$\mathrm{d}s^2 = -\frac{r^2}{R_1^2}\,\mathrm{d}t^2 + \frac{R_1^2}{r^2}\,\mathrm{d}r^2 + R_2^2\,\mathrm{d}\Omega^2 \tag{5.1}$$

which can be rewritten as

$$\mathrm{d}s^2 = -\frac{r^2}{R_1^2}\,\mathrm{d}u^2 - 2\,\mathrm{d}u\mathrm{d}r + R_2^2\,\mathrm{d}\Omega^2$$

Using the Janis–Newman algorithm one has

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{\rho^2}{R_1^2}(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi)^2 - 2(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi)(\mathrm{d}r + a\sin^2\theta\,\mathrm{d}\phi) + R_2^2\,\mathrm{d}\Omega^2 \\
&= -\frac{\rho^2}{R_1^2}(\mathrm{d}u^2 + a^2\sin^4\theta\,\mathrm{d}\phi^2 - 2a\sin^2\theta\,\mathrm{d}u\mathrm{d}\phi) \\
&\quad - 2\big(\mathrm{d}u\mathrm{d}r - a^2\sin^4\theta\,\mathrm{d}\phi^2 + a\sin^2\theta\,\mathrm{d}\phi(\mathrm{d}u - \mathrm{d}r)\big) + R_2^2\,\mathrm{d}\Omega^2 \\
&= -\frac{\rho^2}{R_1^2}\,\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + a^2\sin^4\theta\left(2 - \frac{\rho^2}{R_1^2}\right)\mathrm{d}\phi^2 \\
&\quad + 2a\sin^2\theta\left(-1 + \frac{\rho^2}{R_1^2}\right)\mathrm{d}u\mathrm{d}\phi + 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi + R_2^2\,\mathrm{d}\Omega^2
\end{aligned}$$

(one does not use the formula derived previously because we have R_2^2 instead of r^2 in front of $\mathrm{d}\Omega^2$), i.e.

$$\begin{aligned}
\mathrm{d}s^2 = &-\frac{\rho^2}{R_1^2}\,\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2 + R_2^2\,\mathrm{d}\theta^2 + 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi \\
&- 2a\sin^2\theta\left(1 - \frac{\rho^2}{R_1^2}\right)\mathrm{d}u\mathrm{d}\phi
\end{aligned} \tag{5.2}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \frac{\Sigma^2}{\rho^2} = a^2\sin^2\theta\left(2 - \frac{\rho^2}{R_1^2}\right) + R_2^2. \tag{5.3}$$

Plugging the transformation

$$\mathrm{d}u = \mathrm{d}t - g\,\mathrm{d}r, \qquad \mathrm{d}\phi = \mathrm{d}\phi' - h\,\mathrm{d}r \tag{5.4}$$

in the metric gives

$$\begin{aligned}
\mathrm{d}s^2 = &-\frac{\rho^2}{R_1^2}\,(\mathrm{d}t - g\,\mathrm{d}r)^2 - 2(\mathrm{d}t - g\,\mathrm{d}r)\mathrm{d}r + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,(\mathrm{d}\phi - h\,\mathrm{d}r)^2 + R_2^2\,\mathrm{d}\theta^2 \\
&+ 2a\sin^2\theta\,\mathrm{d}r(\mathrm{d}\phi - h\,\mathrm{d}r) - 2a\sin^2\theta\left(1 - \frac{\rho^2}{R_1^2}\right)(\mathrm{d}t - g\,\mathrm{d}r)(\mathrm{d}\phi - h\,\mathrm{d}r).
\end{aligned} \tag{5.5}$$

The equations to cancel g_{tr} and $g^{r\phi}$ are

$$g_{tr} = -\frac{\rho^2}{R_1^2} \tag{5.6a}$$

$$(\sin^2\theta)^{-1}\,g^{r\phi} = \tag{5.6b}$$

These naive transformations are not allowed because we find that g and h depends on θ:

$$g = \frac{R_2^2 + a^2 \sin^2\theta}{\Delta}, \qquad h = \frac{a}{\Delta} \tag{5.7}$$

and we have defined

$$\Delta = \frac{R_1^2}{R_2^2}\,\rho^2 + a^2 \sin^2\theta. \tag{5.8}$$

When $R_1 = R_2$ the θ dependences vanishes in Δ

$$\Delta = r^2 + a^2, \tag{5.9}$$

but not in g: usually we have r^2 instead of R_2^2, and this transforms to $r^2 + a^2\cos^2\theta$, the second term adding with $a^2\sin^2\theta$ to a^2; but here R_2^2 do not transform.

5.1.1 Extremal limit

If we write Δ as

$$\begin{aligned}
\Delta &= \frac{R_1^2}{R_2^2}\,\rho^2 + a^2\sin^2\theta = \frac{R_1^2}{R_2^2}\,(r^2 + a^2\cos^2\theta) + a^2\sin^2\theta \\
&= \frac{R_1^2}{R_2^2}\,(r^2 + a^2) + a^2\sin^2\theta\left(1 - \frac{R_1^2}{R_2^2}\right) \\
&= \frac{R_1^2}{R_2^2}\,(r^2 + a^2) + \frac{a^2}{R_2^2}\sin^2\theta(R_2^2 - R_1^2)
\end{aligned}$$

then we see that we can remove the θ-dependence with the limit

$$\frac{a^2}{R_2^2} \longrightarrow 0, \tag{5.10}$$

or $a \ll R_2$. We get

$$\Delta \approx \frac{R_1^2}{R_2^2}\,(r^2 + a^2). \tag{5.11}$$

The same limit removes also the θ-dependence in the g denominator since

$$R_2^2 + a^2\sin^2\theta = R_2^2\left(1 + \frac{a^2}{R_2^2}\sin^2\theta\right) \approx R_2^2. \tag{5.12}$$

Then the two functions are explicitly

$$g \approx \frac{R_2^4}{R_1^2(r^2 + a^2)}, \qquad h \approx \frac{aR_2^2}{R_1^2(r^2 + a^2)}. \tag{5.13}$$

The only component which changes in the metric is

$$\frac{\Sigma^2}{\rho^2} = R_2^2\left[1 + \frac{a^2}{R_2^2}\sin^2\theta\left(2 - \frac{\rho^2}{R_1^2}\right)\right] \approx R_2^2. \tag{5.14}$$

6 Top-down solutions

6.1 From Einstein equations

6.1.1 One unknown function

Notice that we can write

$$2n = (\kappa\, F + G)'' + \frac{H'}{H}\,(\kappa\, F + G)' \tag{6.1}$$

by combining various equations.

It is also interesting to note that

$$\kappa\, F' + \kappa G' = -2n\,\frac{H'}{H} \tag{6.2}$$

which is the only piece for $\Lambda \neq 0$. Moreover this implies that for a and c the same functions appears with opposite signs in F and G (and multiplied by κ).

6.1.2 One unknown function in isotropic coordinates

This case seems very hard to solve, and it may be not very interesting: the only static solution is Minkowski spacetime.

6.1.3 Two unknown functions

The equations are really involved and can not be solved directly. But there is terms that look like in the Demiański case, so it may be a good approach to assume these equations holds and to use their solutions to simplify the others (using the equations themselves and not the solutions may help to keep prettier expressions).

6.1.4 Two unknown functions and electric field

6.2 From BPS equations

7 Solutions with cosmological constant

A huge problem is to apply the Janis–Newman algorithm in the presence of the cosmological constant. All standard trials to get (a)dS–Kerr–Newman failed [10, 17, 54, 66]. Nonetheless I still have some hopes, especially by modifying the prescription and taking inspiration from [54]. Note that Xu [65] says he derived the Kerr–dS solution somewhere else (but maybe not with the JNA), but I did not find the reference. There papers [29] that are arguing they found a solution so that we should check them.

The problem is that the angular momentum is coupling to the cosmological constant which generates new term, for example

$$g_{\theta\theta} = r^2 \longrightarrow \frac{\rho^2}{\Delta_\theta}, \qquad \Delta_\theta = 1 + \frac{1}{3}\,a^2\cos^2\theta, \tag{7.1}$$

$$f(r) = \cdots - \frac{\Lambda}{3}\,r^2 \longrightarrow \frac{\Lambda}{3}\,r^2(r^2 + a^2). \tag{7.2}$$

7.1 Applications in de Sitter spacetime

We define

$$\Lambda = \frac{3}{\ell^2}. \tag{7.3}$$

We write the generic static metric

$$\mathrm{d}s^2 = -f\,\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{7.4}$$

and for the rotating one

$$\mathrm{d}s^2 = \frac{\rho^2}{\Delta_r}\,\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\,\mathrm{d}\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2\Xi^2}\Big(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\Big)^2 - \frac{\Delta_r}{\rho^2\Xi^2}\Big(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\Big)^2. \tag{7.5}$$

7.2 (a)dS space

(Anti-) de Sitter metric is given by

$$f(r) = 1 - \frac{r^2}{\ell^2}. \tag{7.6}$$

7.2.1 Naive transformation

Its transformation could be

$$\tilde{f} = 1 - \frac{\rho^2}{\ell^2}. \tag{7.7}$$

The rotating version in Kerr coordinates reads

$$\begin{aligned}\mathrm{d}s^2 = &-\tilde{f}\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2 \\ &+ 2a\sin^2\theta(\tilde{f}-1)\mathrm{d}u\mathrm{d}\phi + 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi\end{aligned} \tag{7.8}$$

where we have defined [49]

$$\boxed{\begin{aligned}\frac{\Sigma^2}{\rho^2} &= \rho^2 + (2-\tilde{f})a^2\sin^2\theta && (7.9a)\\ &= r^2 + a^2 + (1-\tilde{f})\,a^2\sin^2\theta && (7.9b)\\ &= r^2 + a^2 + a g_{u\phi}. && (7.9c)\end{aligned}}$$

7.2.2 Rotating de Sitter spacetime

Using the known metric for Kerr–dS, we see that totating de Sitter should be given by

$$\begin{aligned}\rho^2 &= r^2 + a^2\cos^2\theta, &\qquad \Xi &= 1 + \frac{a^2}{\ell^2}, \\ \Delta_r &= (r^2+a^2)\left(1 - \frac{r^2}{\ell^2}\right), &\qquad \Delta_\theta &= 1 + \frac{a^2}{\ell^2}\,\cos^2\theta.\end{aligned} \tag{7.10}$$

The $g_{\theta\theta}$ coefficient might let think that r is complexified as

$$r' = \frac{r + ia\cos\theta}{1 + i\,a/\ell\cos\theta}. \tag{7.11}$$

7.3 Finding Kerr–dS

Schwarzschild–de Sitter black hole is [38, p. 4–5]

$$f(r) = 1 - \frac{r^2}{\ell^2} - \frac{2M}{r}, \tag{7.12}$$

whereas Kerr–de Sitter metric functions reads

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta, \qquad \Xi = 1 + \frac{a^2}{\ell^2}, \\ \Delta_r &= (r^2 + a^2)\left(1 - \frac{r^2}{\ell^2}\right) - 2Mr + Q^2, \qquad \Delta_\theta = 1 + \frac{a^2}{\ell^2}\cos^2\theta. \end{aligned} \tag{7.13}$$

A Rotating black holes

A.1 Coordinates systems

Consider the following metric in Kerr coordinates

$$\mathrm{d}s^2 = -\tilde{f}(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi)^2 - 2\,(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi)(\mathrm{d}r + a\sin^2\theta\,\mathrm{d}\phi) + \rho^2\mathrm{d}\Omega^2 \tag{A.1a}$$

$$\begin{aligned} &= -\,\tilde{f}\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2 \\ &\quad + 2a\sin^2\theta(\tilde{f} - 1)\mathrm{d}u\mathrm{d}\phi + 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi \end{aligned} \tag{A.1b}$$

$\tilde{f}$ being a function of r and a priori of θ, and we have defined ρ^2 as

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{A.2a}$$

and Σ^2 as

$$\begin{aligned} \frac{\Sigma^2}{\rho^2} &= \rho^2 + (2 - \tilde{f})a^2\sin^2\theta \\ &= r^2 + a^2 + (1 - \tilde{f})\,a^2\sin^2\theta \\ &= r^2 + a^2 + a g_{u\phi}. \end{aligned} \tag{A.2b}$$

We want to study in a systematic way some possible coordinate systems for our metric in order to choose the more convenient form (e.g. with less cross-terms). Since we should recover the formula for Kerr if $\tilde{f} = 1 - 2Mr/\rho^2$, we can compare our results with the review [62] (the Kerr system can already be found p. 5).

So let's do the transformation

$$\mathrm{d}u = \mathrm{d}T - g(r)\mathrm{d}r, \qquad \mathrm{d}\phi = \mathrm{d}\Phi - h(r)\mathrm{d}r. \tag{A.3}$$

where g and h are arbitrary functions. Inserting this in the metric (A.1), one gets

$$\begin{aligned} \mathrm{d}s^2 &= -\tilde{f}\Big(\mathrm{d}T - (g - ah\sin^2\theta)\mathrm{d}r - a\sin^2\theta\,\mathrm{d}\Phi\Big)^2 + \rho^2\Big(\mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\Phi - h\mathrm{d}r)^2\Big) \\ &\quad - 2\Big(\mathrm{d}T - (g - ah\sin^2\theta)\mathrm{d}r - a\sin^2\theta\,\mathrm{d}\Phi\Big)\Big((1 - ah\sin^2\theta)\mathrm{d}r + a\sin^2\theta\,\mathrm{d}\Phi\Big) \\ &= -\,\tilde{f}\mathrm{d}T^2 + \rho^2\mathrm{d}\theta^2 + \Big[-a^2\tilde{f}\sin^2\theta + 2a^2\sin^2\theta + \rho^2\Big]\mathrm{d}\Phi^2 \\ &\quad + \Big[-\tilde{f}(g - ah\sin^2\theta)^2 + 2(g - ah\sin^2\theta)(1 - ah\sin^2\theta) + \rho^2h^2\sin^2\theta\Big]\mathrm{d}r^2 \\ &\quad + 2a(\tilde{f} - 1)\sin^2\theta\,\mathrm{d}T\mathrm{d}\Phi + 2\Big[\tilde{f}(g - ah\sin^2\theta) - (1 - ah\sin^2\theta)\Big]\mathrm{d}T\mathrm{d}r \\ &\quad + 2a\left[-\tilde{f}(g - ah\sin^2\theta) + (g - ah\sin^2\theta) + (1 - ah\sin^2\theta) - \frac{\rho^2}{a}h\right]\sin^2\theta\,\mathrm{d}r\mathrm{d}\Phi. \end{aligned}$$

This can be rewritten as

$$\begin{aligned}
\mathrm{d}s^2 = &-\tilde{f}\mathrm{d}T^2 + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\Phi^2 + 2a(\tilde{f}-1)\sin^2\theta\,\mathrm{d}T\mathrm{d}\Phi \\
&+ \Big[-(g - ah\sin^2\theta)g_{Tr} + (g - ah\sin^2\theta)(1 - ah\sin^2\theta) + \rho^2h^2\sin^2\theta\Big]\mathrm{d}r^2 \\
&+ 2\Big[\tilde{f}(g - ah\sin^2\theta) - (1 - ah\sin^2\theta)\Big]\mathrm{d}T\mathrm{d}r \\
&+ 2\Big[-a\,g_{Tr} + a(g - ah\sin^2\theta) - \rho^2h\Big]\sin^2\theta\,\mathrm{d}r\mathrm{d}\Phi,
\end{aligned} \tag{A.4}$$

where Σ^2 was defined in (A.2b); since one has directly $g_{u\phi} = g_{T\Phi}$ the last formula stays correct

$$\frac{\Sigma^2}{\rho^2} = r^2 + a^2 + ag_{T\Phi}. \tag{A.5}$$

The terms $g_{TT}, g_{\theta\theta}, g_{\Phi\Phi}$ and $g_{T\Phi}$ are independent of g and h: it will be clearer that they encode informations about spacetime when we will study the Killing vectors.

There are three interesting cases:

1. A first choice is $h = 0, g = 1$ (u is viewed as "Minkowskian" light-cone coordinate) gives the metric [62, p. 13]

$$\begin{aligned}
\mathrm{d}s^2 = &-\tilde{f}\mathrm{d}t^2 + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2 + (2-\tilde{f})\mathrm{d}r^2 + 2(\tilde{f}-1)\mathrm{d}t\mathrm{d}r \\
&+ 2a(\tilde{f}-1)\sin^2\theta\,\mathrm{d}t\mathrm{d}\phi + 2a(2-\tilde{f})\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi,
\end{aligned} \tag{A.6}$$

   writing t instead of T. One has three non-diagonal terms.

2. If $h = 0$ – i.e. $\Phi = \phi$ – and $g = \tilde{f}^{-1}$, the transformation is very similar to the one which took us from (t, r) to (u, r). We denote the time by t and the metric is

$$\begin{aligned}
\mathrm{d}s^2 = &-\tilde{f}\mathrm{d}t^2 + \tilde{f}^{-1}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2 \\
&+ 2a(\tilde{f}-1)\sin^2\theta\,\mathrm{d}t\mathrm{d}\phi + 2a\tilde{f}^{-1}\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi.
\end{aligned} \tag{A.7}$$

   One has still two off-diagonal terms.

3. We can also choose g and h to remove two of the three off-diagonal terms (Boyer–Lindquist coordinates), so let's remove the $\mathrm{d}t\mathrm{d}r$ and $\mathrm{d}r\mathrm{d}\phi$ terms, i.e. we require that

$$\begin{cases} g_{Tr} = \tilde{f}(g - ah\sin^2\theta) - (1 - ah\sin^2\theta) = 0 \\ (\sin\theta)^{-2}\,g_{r\Phi} = -a\,g_{Tr} + a(g - ah\sin^2\theta) - \rho^2h = 0. \end{cases} \tag{A.8}$$

   First subract the second equation multiplied by $\tilde{f}$ to the first multiplied by a:

$$a(1 - ah\sin^2\theta) - \rho^2h\tilde{f} = 0 \Longrightarrow h = \frac{a}{\tilde{f}\rho^2 + a^2\sin^2\theta}.$$

   Then plugging this result in the second equation gives

$$g = \frac{h}{a}(\rho^2 + a^2\sin^2\theta).$$

   In summary the solutions are [18, p. 8] [3]

$$g = \frac{r^2 + a^2}{\Delta}, \qquad h = \frac{a}{\Delta} \tag{A.9}$$

[3] Note that in this reference the $\mathrm{d}r$ coefficients of equations (13) and (14) have been swapped.

(minus signs which differs from [18] are due to our different definition for g and h) and we defined

$$\boxed{\Delta = \tilde{f}\rho^2 + a^2 \sin^2\theta.} \tag{A.10}$$

The transformation (A.3) is valid only if both functions (A.9) do not depend on θ [9]. Now we have to compute the rr component of the metric:

$$\begin{aligned} g_{rr} &= -(g - ah\sin^2\theta)g_{Tr} + (g - ah\sin^2\theta)(1 - ah\sin^2\theta) + \rho^2 h^2 \sin^2\theta \\ &= \frac{\rho^2 h}{a}(1 - ah\sin^2\theta) + \rho^2 h^2 \sin^2\theta \end{aligned}$$

and replacing the first parenthesis thanks to (A.8),

$$= \frac{\rho^2 h}{a} = \frac{\rho^2}{\Delta}.$$

After computing the rr component (see appendix) the metric becomesc [18, p. 14]

$$\boxed{\mathrm{d}s^2 = -\tilde{f}\mathrm{d}t^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2 + 2a(\tilde{f} - 1)\sin^2\theta\,\mathrm{d}t\mathrm{d}\phi.} \tag{A.11}$$

This metric can also be written as [63]

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta - a^2\sin^2\theta}{\rho^2}\mathrm{d}t^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2 \\ &+ 2a\frac{\Delta - r^2 - a^2}{\rho^2}\sin^2\theta\,\mathrm{d}t\mathrm{d}\phi. \end{aligned} \tag{A.12}$$

Note that we can get Boyer–Lindquist coordinates by chaining 1) and the following transformation [18]

$$\mathrm{d}T = \mathrm{d}t - m(r)\mathrm{d}r, \qquad \mathrm{d}\Phi = \mathrm{d}\phi - h(r)\mathrm{d}r \tag{A.13}$$

where $h(r)$ is the same as in (A.9) and $m(r)$ is given by

$$m(r) = g(r) - 1 = \frac{\rho^2(1 - \tilde{f})}{\tilde{f}\rho^2 + a^2\sin^2\theta}. \tag{A.14}$$

A.2 Slowly rotating black holes

For slow rotation we consider only terms that are $O(a)$. Looking at Kerr–Newman metric, we see that the only effect is a modification of the non-diagonal term $g_{t\phi}$. Thus from a static metric g_{stat} we can obtain its slowy rotating form by [33, sec. 4]

$$\mathrm{d}s^2_{\mathrm{rot}} = \mathrm{d}s^2_{\mathrm{stat}} - 2aF(r)\sin^2\theta\,\mathrm{d}t\mathrm{d}\phi \tag{A.15}$$

where $F(r)$ is a function to be determined.

This may be useful since the JNA works much more often for slow rotation.

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