Scalar propagators on adS space

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Abstract

This is intended to be a pedagogical review of scalar field propagator on adS space in view of adS/CFT correspondence. There no new results but instead it gathers ideas from the literature and present them in a consistent way, with insights on some difficult points [1].

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[1]This version is still a draft and may contain some mistakes, and some parts need more developments.

Contents

1 Equations of motion

In this section we consider a scalar field $\phi(X)$ on $\mathcal{M} = \mathrm{adS}_{d+1}$ background.

The action for a scalar is

$$S = -\frac{1}{2} \int \mathrm{d}^{d+1}X \, \sqrt{g} \, (g^{AB} \partial_A \phi \, \partial_B \phi + m^2 \phi^2). \tag{1.1}$$

Partial derivatives can be replaced by covariant derivatives D_μ because they are acting on scalar fields. An integration by part gives

$$S = -\frac{1}{2} \int \mathrm{d}^{d+1}X \, \sqrt{g} \, \phi(-\Delta + m^2)\phi + \frac{1}{2} \int \mathrm{d}^{d+1}X \, \partial_A(\sqrt{g} g^{AB} \phi \, \partial_B \phi), \tag{1.2}$$

where Δ is the laplacian on adS_{d+1}. The second term can be rewritten as a surface integral [7, app. E]:

$$S_\mathrm{b} = \int_\mathcal{M} \mathrm{d}^{d+1}x \, \partial_A(\sqrt{g} g^{AB} \phi \, \partial_B \phi) = \int_{\partial\mathcal{M}} \mathrm{d}^d y \, \sqrt{\gamma} \, \phi \, n^A \partial_A \phi, \tag{1.3}$$

where n^A is the vector normal to $\partial\mathcal{M}$, γ is the induced metric and y the coordinates on the surface.

The first term in (1.2) gives the Klein–Gordon equation:

$$(-\Delta + m^2)\phi = 0. \tag{1.4}$$

Using the upper-half plane metric (B.17) one computes the laplacian:

$$\begin{aligned}
\Delta &= \frac{1}{\sqrt{g}} \partial_A(\sqrt{g} g^{AB} \partial_B) \\
&= \frac{z^{d+1}}{L^{d+1}} \left[\partial_z \left(\frac{L^{d+1}}{z^{d+1}} \, \frac{z^2}{L^2} \, \partial_z \right) + \frac{L^{d+1}}{z^{d+1}} \, \frac{z^2}{L^2} \, \partial_x^2 \right]
\end{aligned}$$

gives

$$\Delta = \frac{z^2}{L^2} \Big(z^{d-1} \partial_z (z^{-d+1} \partial_z) + \partial_x^2 \Big) \tag{1.5a}$$

$$= \frac{z^2}{L^2} \Big(\partial_z^2 - (d-1) z^{-1} \partial_z + \partial_x^2 \Big). \tag{1.5b}$$

If ϕ satisfies the equation of motion the action (1.2) reduces to the surface term (1.3) only. Now let decompose the surface in two pieces with whose normal are in the z and x directions respectively, i.e. split the sum in x and z: the first term will vanish if we assume that the field vanishes for $x^\mu \to \pm\infty$ (as is done usually because we don't need boundary data for Minkowski space – we refer to litterature for a proper handling of this), and only the z boundary contributes:

$$S_\mathrm{b} = \int_{\partial\mathcal{M}} \mathrm{d}^d x \, \sqrt{\gamma} \, \phi \, n^z \partial_z \phi \Big|_{z=\varepsilon}^{z=\infty}$$

where $\partial\mathcal{M}$ is just Minkowski. We introduced a cut-off because the induced metric diverges for $z = 0$ [2]. Note that if the space is bound for some reason (e.g. if there is a black hole in the interior) then the integration should go from z_min to z_max [32].

[2] Note that the precise value of the cut-off $z_\text{cut-off}$ is not important since the adS metric is left unchanged by a rescaling of coordinates, which might bring any value $z_\text{cut-off}$ the ε [17, p. 17]. We will thus do the computation with ε and remove it at the end.

Assuming [3] an exponential decay for ϕ at $z \to \infty$ gives finally

$$S_\mathrm{b} = \int_{\partial\mathcal{M}} \mathrm{d}^d x \, \sqrt{\gamma_\varepsilon}\, \phi\, n^z \partial_z \phi \Big|_{z=\varepsilon}. \tag{1.6}$$

The induced metric and the normal vector are given in section B:

$$n^z = \frac{z}{L}, \qquad \gamma_\varepsilon = \frac{L^2}{\varepsilon^2}\eta. \tag{1.7}$$

2 Solutions

The solutions to the Klein–Gordon equation (1.4) are discussed (for example) in [3], [1, sec. 4.4], [32, app. A.1].

2.1 Separation of variables and solution in x-direction

Now we will look for a solution to Klein–Gordon equation (1.4): write the fields ϕ as

$$\phi(z, x) = f(z)\Phi(x) \tag{2.1}$$

since we can hope to separate variables because of translation invariance in the x direction [19]. Then (1.4) becomes

$$-\frac{z^2}{L^2}\Big(z^{d-1}\partial_z(z^{-d+1}f')\Phi) + f\partial^2\Phi\Big) + m^2 f\Phi = 0$$

where we noted $\partial^2 = \partial_x^2 = \Delta_{(d)}$ (laplacian on Minkowski), and by dividing with $f\Phi$ one can separate variables:

$$-\frac{z^{d-1}}{f}\,\partial_z(z^{-d+1}f') + \frac{m^2L^2}{z^2} = \frac{\partial^2\Phi}{\Phi} = -k^2 \tag{2.2}$$

where k^2 is the norm of a d-dimensional vector k^μ. This gives the two equations:

$$(-\partial_x^2 - k^2)\Phi_k = 0, \tag{2.3a}$$

$$\big[-z^{d+1}\partial_z(z^{-d+1}\partial_z) + m^2L^2 + k^2z^2\big]f_k = 0, \tag{2.3b}$$

and we have added a k subscript since solutions now depends of this parameter. According to the discussion of Wick rotation (appendix A.2), the expression in Euclidean space of the first equation is

$$(-\partial_E^2 - k^2)\Phi_k = 0 \tag{2.4}$$

The only difference here is that $k^2 \geq 0$ which justify the sign of the right hand side in (2.2). Notice also that this equation is different from the one we got in the appendix (opposite "mass"), which allow plane wave-like solutions: in euclidean space, plane waves are possible for at most $d-1$ directions.

Since one get modes depending on a parameter k, the full solution will be obtained by superposing all of them [4]:

$$\phi(z, x) = \int \mathrm{d}^d k\; f_k(z)\Phi_k(x). \tag{2.5}$$

Before solving explicitly the equations, let's summary what are the consequences of the k^2 sign, since solutions will depend on it [34, p. 16], [19, lec. 15], [29]:

[3] This will be shown in next section.

[4] One could decide to write an overall coefficient $\phi_0(k)$ for each mode, as it is done in [32, app. A], but we prefer to include it in f_k.

- $k^2 = \mu^2 > 0$ (Euclidean): this will lead to the Euclidean Green function. In the z direction this gives real exponentials.
- $k^2 = -\mu^2 < 0$ (timelike Minkowskian): the momentum satisfies the usual on-shell mass condition and so μ would be the mass in x-space. The z-equation leads to imaginary exponentials, corresponding to retarded/advanced Green functions.
- $k^2 = \mu^2 > 0$ (spacelike Minkowskian): here the momentum is off-shell. The z solution are again real exponentials.

If we wanted to interpret μ^2 as the mass of a particle in d-dimensional space in the first two cases [5], then the second equation (2.3b) should give discrete value for k^2 (i.e. a dispersion relation). Sadly we will see that it is not the case, so the parameter μ can not be interpreted as a mass [9, sec. 2.1.1] [6].

We will work mostly in euclidean space but there is no great difference with Minkowski, as explained for example in [27].

In (2.3a) (or its Euclidean version) one recognizes the Klein–Gordon equation in d-dimensional spacetime, whose solutions are plane-waves:

$$\Phi(x) = \frac{e^{ikx}}{(2\pi)^d}. \tag{2.6}$$

So let's superpose modes using equation (2.5):

$$\phi(z, x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, f_k(z)\, \mathrm{e}^{ikx} \tag{2.7}$$

and we get that $\phi(z, x)$ is the Fourier transform of f_k, which is understable due to the translation invariance in x direction. By inverting the transformation we see that $f_k(z)$ is the solution in momentum space.

2.2 Solution for the radial direction ($k^2 > 0$) and scalings

Results are identical for spacelike Minkowskian and Euclidean cases, writing k for both [32, app. A.2.1].

The z equation can be written

$$z^2 f_k'' - (d-1) z\, f_k' - (m^2 L^2 + k^2 z^2) f_k = 0, \tag{2.8}$$

and it is almost the (modified) Bessel equation (D.6): let's do the change of variable $f_k = z^{d/2} g_k$ to get [7] [1, sec. 4.4]

$$z^2 g_k'' + z g_k' - \left(\frac{d^2}{4} + k^2 z^2 + m^2 L^2\right) g_k = 0. \tag{2.9}$$

Then one does not takes g_k as a function of z, but as a function of kz to get (due to derivatives) [8]

$$(kz)^2 g_k'' + (kz) g_k' - \left(\frac{d^2}{4} + m^2 L^2 + k^2 z^2\right) g_k = 0. \tag{2.10}$$

[5] Indeed if $k^2 = -\mu^2 < 0$, then the previous condition says that Φ satisfies $(-\Delta + \mu^2)\Phi = 0$ in Minkowskian space.

[6] If z is bound by above in the interior, then conformal symmetry is broken and μ will take discrete values [27].

[7] To find the power of z, write z^α and choose α in order to have z as coefficient for g_k'.

[8] Here we note $k = |k| = \mu$, which should not be confused with the vector k^μ itself; this should be clear from the context.

And from the appendix D.3 on modified Bessel functions, one reads the solution for g_k:

$$g_k(kz) = a_k K_\nu(kz) + b_k I_\nu(kz), \tag{2.11}$$

where we have defined the parameter ν as

$$\nu = \sqrt{\frac{d^2}{4} + m^2 L^2}. \tag{2.12}$$

Finally the solution for f_k is

$$f_k(z) = a_k\,(kz)^{d/2} K_\nu(kz) + b_k\,(kz)^{d/2} I_\nu(kz). \tag{2.13}$$

We have to impose that solutions are regular everywhere in the interior, and more specifically for $z \to \infty$. Using asymptotic form (D.7b) for the Bessel functions

$$I_\nu(z) \sim \mathrm{e}^{kz}, \qquad K_\nu(z) \sim \mathrm{e}^{-kz}, \tag{2.14}$$

one sees that I_ν diverges so $b_k = 0$ and

$$f_k(z) = a_k\,(kz)^{d/2} K_\nu(kz). \tag{2.15}$$

Using the asymptotic forms of the Bessel functions (D.10), one finds that near the boundary $z \approx 0$ the solution behaves like

$$f_k(z) \approx a_k\,(kz)^{d/2} \left[\frac{\Gamma(\nu)}{2}\left(\frac{2}{kz}\right)^\nu + \frac{\Gamma(-\nu)}{2}\left(\frac{kz}{2}\right)^\nu\right]$$

and after simplication

$$f_k(z) \approx \phi_0(k) z^{\Delta_-} + \phi_1(k) z^{\Delta_+} \tag{2.16}$$

where we have defined

$$\phi_0(k) = a_k\, 2^{\nu-1}\Gamma(\nu)\, k^{\Delta_-}, \quad \phi_1(k) = a_k\, 2^{-(\nu+1)}\Gamma(-\nu)\, k^{\Delta_+}. \tag{2.17}$$

The scaling exponents $\Delta_\pm$ are defined as

$$\Delta_\pm = \frac{d}{2} \pm \nu = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2}. \tag{2.18}$$

Note that the positivity of the square-root gives the Breitenlohner–Freedman bound [34, p. 16]

$$m^2 L^2 > -\frac{d^2}{4}. \tag{2.19}$$

For future reference note that we have

$$\frac{\phi_1(k)}{\phi_0(k)} = \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{k}{2}\right)^{2\nu} = \frac{\phi_1(-k)}{\phi_0(-k)} \tag{2.20}$$

In position space (2.7) the asymptotic (2.16) becomes

$$\phi(z,x) \approx \phi_0(x) z^{\Delta_-} + \phi_1(x) z^{\Delta_+} \tag{2.21}$$

where $\phi_0(x)$ and $\phi_1(x)$ are the Fourier transform of $\phi_0(k)$ and $\phi_1(k)$.

Another way to deduce these scalings are to plug the ansatz $f(z) = z^\Delta$ into (2.3b) [27]:

$$\big(-\Delta(\Delta - d) + m^2 L^2 + k^2 z^2\big) z^\Delta = 0. \tag{2.22}$$

Close to the boundary $z \approx 0$ and one can ignore the term in z^2 which otherwise spoils the power-law solution, and then one finds again the two roots (2.18).

2.3 Solution for the radial direction ($k^2 < 0$) and scalings

For a timelike Minkowskian momentum all the previous results are found through the replacement $\mu \to i\mu$ [34, p. 16]. So the differential equation becomes the one for Bessel functions (cf appendix D.2):

$$(kz)^2 g_k'' + (kz) g_k' - \left(\frac{d^2}{4} + m^2 L^2 - k^2 z^2\right) g_k = 0. \tag{2.23}$$

Note that here $k = i\mu$ (doing the replacement does not change any sign in the first two terms).

Solutions are $J_{\pm\nu}(kz) \sim K_{\pm\nu}(ikz)$ if ν is not an integer, $J_\nu(kz)$ and $Y_\nu(kz)$ otherwise [3] [9]. Using the previous solution one sees that

$$K_{\pm\nu}(ikz) \sim \mathrm{e}^{\pm ikz} \tag{2.24}$$

so both solutions are regular at $z = \infty$ and we have to impose in-falling or outgoing boundary solution, which are related to retarded/advanced Green functions [29], [32, sec. 3.2].

2.4 Asymptotic behavior and boundary field ($k^2 > 0$)

Now study in more details the asymptotic form (2.16) of the solution:

$$f_k(z) \approx \phi_0(k) z^{\Delta_-} + \phi_1(k) z^{\Delta_+}. \tag{2.25}$$

Since $\Delta_+ > 0$ the solution $z^{\Delta_+} \xrightarrow[z\to 0]{} 0$ is a normalizable solution and corresponds to a bulk excitation which decays at the boundary [16, sec. 5.1] and [14, p. 6]. But the other solution does not decay since $\Delta_- < 0$ and it is said to be non-normalizable: it defines a field on the boundary [8, p. 54] [10]

$$\phi_0(k) = \lim_{z\to 0} z^{-\Delta_-} f_k(z), \tag{2.26}$$

or written in position space through (2.7)

$$\phi_0(x) = \lim_{z\to 0} z^{-\Delta_-} \phi(z, x) \tag{2.27}$$

where

$$\phi_0(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \phi_0(k)\, \mathrm{e}^{ikx}. \tag{2.28}$$

Note that usually one defines the boundary field through (2.27) by using the trick at the end of the subsection 2.2, just saying that since the divergences is only a power-law in z, then there should be a x-field. Then later one identifies this boundary field with the one we got in our development (2.16) [11].

In principle one asks only for normalizable modes in order to construct the Hilbert space of the theory, since they corresponds to the physical modes which propagates in the bulk [3, 4], [10, sec. 6]. On the other side, non-normalizable modes are necessary to specify boundary conditions, and then these modes do not fluctuate and they provides the classical

[9] Hankel functions – the linear combinations $J_\nu \pm iY_\nu$ – can be more suitable for some purposes, since they describe in- and outgoing waves at $z = \infty$ [29, 32].

[10] Due to the fact that m^2 can be negative there is some subtleties with the normalizable states which are not reviewed here.

[11] Moreover sometimes one uses the solution in terms of Bessel function (2.13) but before asking to regularity and this leads to errors in the analysis since one uses only a part of the asymptotic of Bessel functions, namely $K_\nu \sim z^{\Delta_-}$ and $I_\nu \sim z^{\Delta_+}$.

background on which the normalizable modes propagate. Thus we should not throw away the non-normalizable modes.

In the first case it is then necessary to obtain the full solution by using the method of Green function, since the source gives a response.

Using a cut-off to remove the limit process, this relation can be inverted to give

$$\phi(\varepsilon, x) = \varepsilon^{\Delta_-} \phi_0(x), \quad f_k(\varepsilon) = \varepsilon^{\Delta_-} \phi_0(k). \tag{2.29}$$

This is simply what we have already written in the asymptotic expansion of f_k. This last relation is useful when one wants to compute on the boundary in order to avoid divergences. It is then possible to verify directly [25, p. 7][28, sec. 3.2] that ϕ_0 is an operator with conformal dimension Δ_-:

$$\begin{aligned}\phi_0(\lambda x) &= \lim_{z\to 0} z^{-\Delta_-} \phi(z, \lambda x) = \lambda^{-\Delta_-} \lim_{z\to 0} (\lambda^{-1} z)^{-\Delta_-} \phi(z, \lambda x) \\ &= \lambda^{-\Delta_-} \lim_{z'\to 0} (z')^{-\Delta_-} \phi(\lambda z', \lambda x) = \lambda^{-\Delta_-} \lim_{z'\to 0} (z')^{-\Delta_-} \phi(z', x)\end{aligned}$$

i.e.

$$\phi_0(x) = \lambda^{\Delta_-} \phi_0(\lambda x). \tag{2.30}$$

But now that there is on the boundary a field which acts as a source for the field in the bulk, we have to propagates it using Green function (more specifically the bulk-to-boundary propagator). This object will be computed in a later section and at this point we will come back to the full solution. This will be quivalent to the full solution we derived in next section (in fact we will identify the Green function using our solution, replacing the boundary field by a point-like source).

2.5 Complete (free) solution

Plugging the explicit solution (2.15) for f_k into the Fourier transform (2.7) gives the full solution in position space:

$$\phi(z, x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, a_k\, (kz)^{d/2} K_\nu(kz)\, \mathrm{e}^{ikx}. \tag{2.31}$$

The value of a_k is determined by boundary conditions at $z = \varepsilon$ (the cut-off is necessary to avoid divergences) [1, sec. 4.4] using the formula (2.26) [12]

$$\phi_0(k) = \varepsilon^{-\Delta_-} f_k(\varepsilon) = \varepsilon^{-\Delta_-} a_k\, (k\varepsilon)^{d/2} K_\nu(k\varepsilon). \tag{2.32}$$

which gives after inversion

$$a_k = \frac{\varepsilon^{\Delta_- - d/2}}{k^{d/2} K_\nu(k\varepsilon)}\, \phi_0(k). \tag{2.33}$$

Then f_k is given by

$$f_k(z) = \varepsilon^{\Delta_-} \left(\frac{z}{\varepsilon}\right)^{d/2} \frac{K_\nu(kz)}{K_\nu(k\varepsilon)}\, \phi_0(k). \tag{2.34}$$

Replacing $\phi_0(k)$ by (2.29) gives

$$f_k(z) = \left(\frac{z}{\varepsilon}\right)^{d/2} \frac{K_\nu(kz)}{K_\nu(k\varepsilon)}\, f_k(\varepsilon) \tag{2.35}$$

[12] Defining the boundary condition at $z = \varepsilon$ instead of $z = 0$ is more logical because it will avoid many ε factors in latter expression, but it is also necessary to get correct formula when evaluating the actions [4, app. A].

and one sees that taking $z = \varepsilon$ gives the right answer.

An approximation of a_k can also be found using (2.17):

$$a_k \approx \frac{\phi_0(k)}{2^{\nu-1}\Gamma(\nu) k^{\Delta_-}}. \tag{2.36}$$

Using (2.16) this also leads to the approximate form

$$f_k(z) \approx \frac{\phi_0(k) z^{\Delta_-} + \phi_1(k) z^{\Delta_+}}{\phi_0(k)\varepsilon^{\Delta_-} + \phi_1(k)\varepsilon^{\Delta_+}}. \tag{2.37}$$

The final solution in position space becomes

$$\phi(z,x) = \varepsilon^{\Delta_-} \left(\frac{z}{\varepsilon}\right)^{d/2} \int \mathrm{d}^d x' \, \frac{\mathrm{d}^d k}{(2\pi)^d} \, \frac{K_\nu(kz)}{K_\nu(k\varepsilon)} \, \phi_0(x') \, \mathrm{e}^{ik(x-x')} \tag{2.38}$$

where we recall that $\phi_0(x)$ is the Fourier transform of $\phi_0(k)$ (2.28):

$$\phi_0(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \phi_0(k) \, \mathrm{e}^{ikx}. \tag{2.39}$$

3 Another quantization scheme

There is a loophole in the previous discussion, where we considered only the case where z^{Δ_+} is normalizable. In the mass range

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1 \tag{3.1}$$

the action is also finite for z^{Δ_-} (with adapted boundary conditions) [18, 3, 22], such that it exists two different quantizations [13].

We reproduce the analysis from [18]: consider a field whose asymptotics is

$$\phi \sim z^{\Delta} \phi_1(x) \tag{3.2}$$

where Δ is a solution of the equation

$$\Delta(d - \Delta) + m^2 = 0, \tag{3.3}$$

i.e.

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}. \tag{3.4}$$

We inject this form into the action (1.1)

$$S_1 = -\frac{1}{2} \int \mathrm{d}^{d+1} X \, \sqrt{g} \, (g^{\phi_1 B} \partial_A \phi \, \partial_B \phi + m^2 \phi^2) \tag{3.5}$$

whose integrand is

$$\begin{aligned} L_1 &= \sqrt{g} \left(g^{zz} (\partial_z \phi)^2 + \phi'^2 m^2 \phi^2 \right) \\ &= -z^{-d+1} \left((\partial_z \phi)^2 + \frac{m^2}{z^2} \, \phi^2 + z^{2\Delta} \phi_1'^2 \right) = -z^{-d+1} \left((\partial_z \phi)^2 + \frac{m^2}{z^2} \, \phi^2 + z^{2\Delta} \phi_1'^2 \right) \\ &= -z^{-d+1} \left(\Delta^2 z^{2\Delta-2} + m^2 z^{2\Delta-2} \right) \phi_1^2 + z^{2\Delta} \phi_1'^2 = -z^{2\Delta-d-1} \left(\Delta^2 + m^2 \right) \phi_1^2 + z^{2\Delta} \phi_1'^2, \end{aligned}$$

[13] Historically this was shown from energetic considerations, see e.g. [6].

(where $\phi' \partial_\mu \phi$ and the square hides a $g^{\mu\nu}$) and using the definition of Δ, the integrand is

$$L_1 = \phi_1^2\, \Delta(2\Delta - d) z^{2\Delta - d - 1} + z^{2\Delta} \phi_1'^2. \tag{3.6}$$

The last term is subleading and the integral will converge if

$$2\Delta - d - 1 > -1 \Longleftrightarrow \Delta > \frac{d}{2}. \tag{3.7}$$

In this range we have $2\Delta - d > 0$ and the coefficient of L can never vanish. We conclude that only Δ_+ is valid.

But we can also use the integrated form (1.2)

$$S_1 = -\frac{1}{2} \int \mathrm{d}^{d+1} X\, \sqrt{g}\, \phi(-\Delta + m^2)\phi, \tag{3.8}$$

then throwing the boundary term and using the expression for the laplacian (1.5) we have the integrand

$$\begin{aligned} L_2 &= z^{-d+1}\, \phi \left(-\partial_z^2 + \frac{d-1}{z}\, \partial_z - \partial_x^2 + \frac{m^2}{z^2} \right) \phi \\ &= z^{\Delta - d + 1} \left(-\Delta(\Delta - 1)\phi_1^2 z^{\Delta - 2} + \Delta \phi_1^2 \frac{d-1}{z}\, z^{\Delta - 1} + m^2 \phi_1^2 z^{\Delta - 2} - \phi_1'' z^{\Delta} \right) \\ &= z^{\Delta - d + 1} \left(\phi_1^2 \big[-\Delta(\Delta - 1) + \Delta(d-1) + m^2 \big] z^{\Delta - 2} - \phi_1'' z^{\Delta} \right). \end{aligned}$$

The coefficient in front of $z^{\Delta - 2}$ vanishes because of the definition of Δ, and thus

$$L_2 = \phi_1'' z^{2\Delta - d + 1}. \tag{3.9}$$

The integral is convergent if

$$2\Delta - d + 1 > -1 \Longleftrightarrow \Delta > \frac{d}{2} - 1. \tag{3.10}$$

The divergent piece comes from the boundary as can be verified. In this context both $\Delta_\pm$ are valid [14].

In order to translate these bounds on the mass, we write

$$\begin{gathered} -\sqrt{\frac{d^2}{4} + m^2} = \left(\Delta_- - \frac{d}{2} \right) > -1 \\ \sqrt{\frac{d^2}{4} + m^2} < 1 \end{gathered}$$

and thus

$$m^2 < -\frac{d^2}{4} + 1 \Longleftrightarrow \nu < 1. \tag{3.11}$$

Above this limit we can not use the boundary condition with Δ_-.

In the rest of the text we focus on Δ_+ boundary condition. We found that the asymptotic field can be written

$$\phi = z^{\Delta_-} \phi_0 + z^{\Delta_+} \phi_1 \tag{3.12}$$

and that ϕ_0 is interpreted as a source on the boundary. But it is not necessary to do again all the computations: as explained in [18, p. 11–12][22], ϕ_0 and ϕ_1 are related by a canonical transformation and we can obtain the needed quantities by a Legendre transformation [15].

The link between boundary terms, boundary conditions and the two quantization schemes is further explored in [21].

[14] In view of the adS/CFT correspondence the bound $\Delta > d/2$ is strange because we know that the unitary bound for a scalar field in a CFT is $\Delta > d/2 - 1$, exactly what is found here.

[15] Using an hamiltonian analysis we can show that ϕ_1 is the canonical momentum associated to ϕ_0.

4 Bulk-to-bulk propagator

4.1 General properties

Now let's look for propagators. The bulk-to-bulk propagator $G(X;X')$ is defined by

$$(-\Delta + m^2)G = -\frac{i}{\sqrt{g}}\,\delta^{(d+1)}(X - X') \tag{4.1}$$

and so it is i times the Green function. The delta functions of the right hand side is defined such that

$$\int \mathrm{d}^{d+1}X\;\delta^{(d+1)}(X - X')\,\phi(X') = \phi(X) \tag{4.2}$$

(no metric determinant). And then Klein–Gordon equation with source J

$$(-\Delta_x + m^2)\phi = J \tag{4.3}$$

is solved by the convolution of G with the source

$$\phi(X) = \int \mathrm{d}^{d+1}X'\; G(X;X')J(X'), \tag{4.4}$$

since

$$\begin{aligned}(-\Delta_x + m^2)\phi &= (-\Delta_X + m^2)\int \mathrm{d}^{d+1}X'\;\sqrt{g}\,G(X;X')J(X') \\ &= \int \mathrm{d}^{d+1}X'\;\delta^{(d+1)}(X - X')\,G(X;X')J(X') = J(X).\end{aligned}$$

By doing a Wick rotation we get (where X^0 is now the euclidean time) [23]

$$(-\Delta + m^2)G = \frac{1}{\sqrt{g}}\,\delta^{(d+1)}(X - X') \tag{4.5}$$

because the delta function changes by $-i$ and the metric determinant by -1. Beginning from now we will use only euclidean time (and without writting differently quantities) [16].

Now for any solution $\phi(X)$ of the homogeneous equation (1.4) we have the relationship [5, p. 67]

$$\begin{aligned}\phi(X) = \int \mathrm{d}^{d+1}X'\;\sqrt{g}\,\Big(&\phi(X')(-\Delta_{X'} + m^2)G(X;X') \\ &- (-\Delta_{X'} + m^2)\phi(X')G(X;X')\Big).\end{aligned} \tag{4.6}$$

Integrating by part gives (variables are omitted)

$$\phi(X) = -\int_{\mathcal{M}} \mathrm{d}^{d+1}X'\;\sqrt{g}\,\partial^A(\phi\partial_A G - G\partial_A\phi) \tag{4.7a}$$

$$= -\int_{\partial\mathcal{M}} \mathrm{d}^d y'\;\sqrt{\gamma}\,(\phi\, n^A\partial_A G - G\, n^A\partial_A\phi). \tag{4.7b}$$

where again γ is the induced metric and y the coordinates on $\partial\mathcal{M}$. From this last form we can deduce that:

- if G vanishes on $\partial\mathcal{M}$ then $\phi(x)$ is given by Dirichlet conditions on $\phi(y)$;

[16] The computation for Minkowski space are mostly the same: there are some modifications due to the normalizable modes present in Minkowski space [4].

- if $n^A \partial_A G$ vanishes on $\partial\mathcal{M}$ then $\phi(x)$ is given by von Neumann conditions on $n^A \partial_A \phi(y)$;
- if none vanishes one has mixed boundary conditions.

Our interested will be in the first case since we found that for our solution ϕ approaches a constant ϕ_0 on the boundary (after a rescaling). If ϕ is given by Dirichlet data then the solution is unique if $\mu^2 > 0$ [5] – see also [33].

4.2 Explicit form

Since we know the modes of the scalar fields in (2.13)

$$f_k(z) = (kz)^{d/2} \big\{ K_\nu(kz), I_\nu(kz) \big\}, \tag{4.8}$$

we can use them to construct the Green function [23, 24]

$$\begin{aligned} G_0(X, X') = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, (zz')^{d/2} \, \mathrm{e}^{-ip(x-x')} \Big(&\theta(z - z') K_\nu(kz) I_\nu(kz') \\ &+ \theta(z' - z) I_\nu(kz) K_\nu(kz') \Big). \end{aligned} \tag{4.9}$$

Although one should note that the boundary condition is at $z = 0$ instead of $z = \varepsilon$.

Question 4.1 Why it is this specific order and why we have the non-normalizable modes?

This last expression can be integrated to give the result in term of the hypergeometric function [8, sec. 6.3]

$$G(X; X') = \frac{2C_{\Delta_+}}{\nu} \left(\frac{\xi}{2} \right)^{\Delta_+} F\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \nu + 1; \xi^2 \right) \tag{4.10}$$

where

$$\xi = \frac{2zz'}{z^2 + z'^2 + (x - x')^2} \tag{4.11}$$

(C_{Δ_+} will be given later, see (5.18)).

Now we want to fix the boundary at $z = \varepsilon$; we get

$$G_\varepsilon(X, X') = G_0(X, X') + \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, (zz')^{d/2} \, \mathrm{e}^{-ip(x-x')} \; K_\nu(kz) K_\nu(kz') \frac{I_\nu(k\varepsilon)}{K_\nu(k\varepsilon)}. \tag{4.12}$$

5 Boundary-to-bulk propagator

We will be interested in the boundary-to-bulk propagator $K(z, x; x')$ which happens when a point source is located on the boundary. Due to the fact that $\sqrt{\gamma} \propto z$, the right hand side of the Green equation vanishes, giving

$$(-\Delta + m^2) K(z, x; x') = 0. \tag{5.1}$$

The propagator should become a delta function when positions coincide and this will be proven later.

Here again general solution is given by convolution with a source $\phi_0(x)$:

$$\phi(z,x) = \int \mathrm{d}^d x' \; K(z,x;x')\phi_0(x'). \tag{5.2}$$

This convolution in position space becomes simply a product in momentum space:

$$f_k(z) = K_k(z)\phi_0(k). \tag{5.3}$$

5.1 Link to the bulk-to-bulk propagator

The formula (4.7) is helpful to get an expression depending of the bulk-to-bulk propagator, because if we take $\partial\mathcal{M}$ as the boundary of the adS, i.e. $z = 0$, then the $\phi(y)$ in the first term is located on the boundary, and so $\phi(z,x)$ is given by the convolution with the propagator. Using (5.2) and changing the notations to $y \to x', \phi(y) \to z^{\Delta_-}\phi_0(x')$ let us identify the boundary propagator to be

$$K(z,x;x') = \lim_{z'\to 0} \sqrt{\gamma_{z'}}\,(z')^{\Delta_-}\; n^{z'}\partial_{z'}G(z,x;z',x') \tag{5.4a}$$

$$= L^{d-1} \lim_{z'\to 0}(z')^{-\Delta_+}\; z'\partial_{z'}G(z,x;z',x'), \tag{5.4b}$$

using the normal vector (B.19), and the propagator vanishes for $z \to \infty$.

Now we wants to relate both propagators without derivatives using Green's theorem [19]

$$\begin{aligned}\int_{\mathcal{M}} \mathrm{d}^{d+1}X\; \sqrt{g}\,\Big(\phi(-\Delta+m^2)\psi - (-\Delta+m^2)\psi\;\phi\Big)&\\ = -\int_{\partial\mathcal{M}} \mathrm{d}^d y\; \sqrt{\gamma}\,(\phi\, n\cdot\partial\psi - \psi n\cdot\partial\phi)&\end{aligned} \tag{5.5}$$

with $\phi(X) = G(X;X'), \psi(X) = K(X;X'')$; with G normalizable:

$$z\partial_z G(X;X') = \Delta_+ G(X;X') \tag{5.6}$$

(in the following it is understood that only the first argument of K has a z-dependence). The left hand side gives

$$\begin{aligned}&\int \mathrm{d}^{d+1}X''\,\sqrt{g}\Big(G\underbrace{(-\Delta+m^2)K}_{=0} - (-\Delta_{X''}+m^2)G(X'';X)\;K(X'';X')\Big)\\ &\qquad = -\int_{\mathcal{M}} \mathrm{d}^{d+1}X''\;\sqrt{g}\,\delta^{(d+1)}(X-X'')K(X'';X') = K(X;X')\end{aligned}$$

and the right hand side is:

$$\begin{aligned}&-\int \mathrm{d}^d x''\;\sqrt{\gamma}\,n^{z''}\Big(G(X'';X')\partial_{z''}K(X'';X') - K(X'';X')\partial_{z''}G(X'';X')\Big)\Big|_{z''=0}\\ &= -L^{d-1}\int \mathrm{d}^d x''\;(z'')^{-d}\Big(G(X'';X')z''\partial_{z''}K(X'';X') - K(X'';X')z''\partial_{z''}G(X'';X')\Big)\Big|_{z''=0}\\ &= -L^{d-1}\int \mathrm{d}^d x''\;(z'')^{-d}\Big(\Delta_- G(X'';X')K(X'';X') - \Delta_+ K(X'';X')G(X'';X')\Big)\Big|_{z''=0}\\ &= -L^{d-1}\int \mathrm{d}^d x''\;(z'')^{-d}\Big(\Delta_- - \Delta_+\Big)G(X'';X')(z'')^{\Delta_-}\delta^{(d)}(x''-x')\Big|_{z''=0},\end{aligned}$$

using (5.20). Equalling both sides gives finally

$$K(z,x;x') = \lim_{z'\to 0}\frac{2\nu}{z'^{\Delta_+}}G(z,x;z',x'). \tag{5.7}$$

It is evident that generally one will use a cut-off ε in the limit.

Question 5.1 Why G is normalizable? Cf for example [28].

The last formula allows us to find directly the expression [27]

$$K(z, x; x') = C_{\Delta_+} \left(\frac{z}{z^2 + (x - x')^2} \right)^{\Delta_+} \tag{5.8}$$

from (4.10) using the limit of the hypergeometric function when z' goes to zero [30, p. 24]

$$F\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \nu + 1; 0 \right) = 1. \tag{5.9}$$

5.2 General expression

We can interpret the solution for $\phi(z, x)$ (2.38) as given by a point source located at the position x' on the boundary, so we can infer the propagator [1, sec. 4.4]

$$K(z, x; x') = \varepsilon^{\Delta_-} \left(\frac{z}{\varepsilon} \right)^{d/2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \frac{K_\nu(kz)}{K_\nu(k\varepsilon)} \, \mathrm{e}^{ik(x - x')}. \tag{5.10}$$

In the same way as the field, we have to rescale it as we approach the boundary, where the propagator becomes a delta function:

$$\varepsilon^{-\Delta_-} K(\varepsilon, x; x') = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \mathrm{e}^{ik(x - x')} = \delta^{(d)}(x - x').$$

To get the propagator in momentum space one has just to remove the field $\phi_0(k)$ from the solution (2.34) in k-space ($\phi_0(k) = 1$ for point source):

$$K_k(z) = \varepsilon^{\Delta_-} \left(\frac{z}{\varepsilon} \right)^{d/2} \frac{K_\nu(kz)}{K_\nu(k\varepsilon)}. \tag{5.11}$$

We note that for $z = \varepsilon$ it is equal to one, corresponding to a delta function.

It is possible to find the boundary behavior of K_k using the equation (5.3) and the asymptotic form (2.16) of $f_k(z)$:

$$K_k(z) = \frac{f_k(z)}{\phi_O(k)} = z^{\Delta_-} + \frac{\phi_1(k)}{\phi_0(k)} \, z^{\Delta_+}. \tag{5.12}$$

5.3 Witten's method

Another simpler expression can be discovered thanks to a Witten's trick [17] [33, sec. 2.4 and 2.5] and [19, lec. 14]: let's pick the point at $z = \infty$. It is possible if we consider the space as compactified, which means that it is just another point (as compactifying $\mathbb{R}^d$ to S^d): then the x part of the metric simplify since

$$\frac{L^2}{z^2} \, \eta_{\mu\nu} \xrightarrow[z \to \infty]{} 0, \tag{5.13}$$

[17] We can summarized this method: 1. solve Laplace equation, 2. show it is singular at some points, 3. singular means sources, 4. check what kind of sources it is and if they are physical. In the case of Witten's solution, we will see it is a point source.

and the space shrinks to a point. For this reason the Green equation (5.1) becomes

$$\big[-z^{d+1}\partial_z(z^{-d+1}\partial_z)+m^2L^2\big]K(z)=0. \tag{5.14}$$

We recognizes the equation (2.3b) for the free scalar field but without the term k^2z^2. Then we have seen that solutions are power-law.

Question 5.2 Do we take a neighbourhood of $z=\infty$ and not strictly $z=\infty$? Because it is weird to have derivatives if we take the function to be valued at one point.

We do the same ansatz as the one for f_k and find that

$$K(z)=C_{\Delta_+}z^{\Delta_+}, \quad \Delta_+=\frac{d}{2}+\sqrt{\frac{d^2}{4}+m^2L^2}, \tag{5.15}$$

keeping only the largest root Δ_+ from (2.18).

Question 5.3 Why are we keeping only the largest root? Further we find that $z^{-\Delta_-}K\to\delta$, which is the same condition as for fields.

Now we can do an inversion followed by a x-translation (isometries of adS spacetime)

$$z\longrightarrow\frac{z}{z^2+x^2}, \quad x\longrightarrow x-x', \tag{5.16}$$

where $x^2=\eta_{\mu\nu}x^\mu x^\nu$, to find the boundary propagator

$$K(z,x;x')=C_{\Delta_+}\left(\frac{z}{z^2+(x-x')^2}\right)^{\Delta_+}. \tag{5.17}$$

Now we want to prove that K is singular only at the point $x=x'$ on the boundary [18] [1, sec. 4.4], and thus approaches a delta function, so we computes the integral over x (and setting $x'=0$ due to translation invariance):

$$\begin{aligned}\int \mathrm{d}^dx\; K(z,x)&=C_{\Delta_+}z^{\Delta_+}\int \mathrm{d}^dx\;\frac{1}{(z^2+x^2)^{\Delta_+}}\\ &=C_{\Delta_+}z^{\Delta_+}\Omega_{d-1}\int_0^\infty \mathrm{d}r\;\frac{r^{d-1}}{(z^2+r^2)^{\Delta_+}}\\ &=C_{\Delta_+}z^{d-\Delta_+}\Omega_{d-1}\int_0^\infty \mathrm{d}t\;\frac{t^{d-1}}{(1+t^2)^{\Delta_+}}\end{aligned}$$

with the change of variables $t=r/z$,

$$=\frac{1}{2}C_{\Delta_+}z^{d-\Delta_+}\Omega_{d-1}\int_0^\infty \mathrm{d}x\;\frac{x^{\frac{d}{2}-1}}{(1+x)^{\Delta_+}}$$

[18] This behavior is expected at first glance since K vanishes if $z\to0$, but it diverges if also $(x-x')\to0$.

substituing $x = t^2$,

$$= \frac{1}{2} C_{\Delta_+} z^{d-\Delta_+} \Omega_{d-1} B(d/2, \Delta - d/2)$$
$$= \frac{1}{2} C_{\Delta_+} z^{d-\Delta_+} \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{\Gamma(d/2)\Gamma(\Delta - d/2)}{\Gamma(\Delta)}.$$

So let's choose the normalization constant C_{Δ_+} to be

$$C_{\Delta_+} = \frac{\Gamma(\Delta)}{\pi^{d/2}\,\Gamma(\nu)}. \tag{5.18}$$

K is not exactly a delta function as $z \to 0$ since

$$\int \mathrm{d}^d x \; K(z, x) = z^{\Delta_-}, \tag{5.19}$$

but $z^{-\Delta_-} K$ is:

$$\lim_{z\to 0} z^{-\Delta_-} K(z, x; x') = \delta^{(d)}(x - x'). \tag{5.20}$$

So near the boundary K has the behavior [19, lec. 14]

$$K(\varepsilon, x; x') = \varepsilon^{\Delta_-} \delta^{(d)}(x - x') + C_{\Delta_+} \frac{\varepsilon^{\Delta_+}}{(x - x')^{2\Delta_+}}. \tag{5.21}$$

The first term should be regarded as letting possible to connect the bulk to the boundary. Note that for a massless field, $\Delta_- = 0$ and K becomes really a delta function [33]. If we Fourier transform the general expression (5.12) then we identify the same structure, and we will see that coefficients agree exactly.

We can see that this propagator gives again the relation (2.29) between boundary and bulk fields:

$$\phi(z, x) = C_{\Delta_+} \int \mathrm{d}^d x' \; K(z, x; x') \phi_0(x') \Longrightarrow \phi(\varepsilon, x) = \varepsilon^{\Delta_-} \phi_0(x).$$

Now we have to derived the behavior of K under rescaling. Using the formula [11, sec. 23.5]

$$\frac{1}{(z - w)^2} = \frac{z'^2 w'^2}{(z' - w')^2}, \tag{5.22}$$

where w, z are adS coordinates and w', z' their transformations, we find

$$K(z, x_1; x_2) \longrightarrow (x_1')^{2\Delta_+} K(z, x_1'; x_2'). \tag{5.23}$$

6 Boundary-to-boundary propagator

It is possible to define a boundary-to-boundary propagator $\beta(x; x')$ which coincides with the tree-level two-point function [28]. It is defined similarly as the bulk-to-boundary through a double limit:

$$\beta(x; x') = (2\nu)^2 \lim_{z, z' \to 0} (z z')^{-\Delta_+} G(z, x; z', x'). \tag{6.1}$$

7 Full solution from Witten's Green function

We have seen that near the boundary, the free field solution behaves as (2.27)

$$\phi_{\text{free}}(z, x) \sim_0 z^{\Delta_-} \phi_0(x), \tag{7.1}$$

and we will do as if we were starting from this fact. Since this is equivalent to have a source on the boundary, we have to use the propagator to find a particular solution $\bar{\phi}(z, x)$, using here the Witten's Green function: (5.17):

$$\bar{\phi}(z, x) = \int \mathrm{d}^d x' \, K(z, x; x')\phi_0(x') = \int \mathrm{d}^d x' \, \left(\frac{z}{z^2 + (x - x')^2}\right)^2 \phi_0(x') \tag{7.2}$$

which near the boundary behaves as

$$\bar{\phi}(z, x) \sim_0 C_{\Delta_+} z^{\Delta_+} \int \mathrm{d}^d x' \, \frac{\phi_0(x')}{(x - x')^{2\Delta_+}}. \tag{7.3}$$

The complete solution of the Klein–Gordon equation with source is then

$$\phi(z, x) = \phi_{\text{free}}(z, x) + \bar{\phi}(z, x) \sim_0 z^{\Delta_-} \phi_0(x) + z^{\Delta_+} \phi_1(x) \tag{7.4}$$

with

$$\phi_1(x) = C_{\Delta_+} \int \mathrm{d}^d x' \, \frac{\phi_0(x')}{(x - x')^{2\Delta_+}}. \tag{7.5}$$

Note that this last term, and then the complete solution, is not a local function of $\phi_0(x)$ because of the integration. This asymptotic form is similar to the one discover computing directly the solution (2.16).

Note that this relation can be rewritten

$$\phi(z, x) \sim_0 \int \mathrm{d}^d x' \, \left(z^{\Delta_-} \delta^{(d)}(x - x') + C_{\Delta_+} \frac{z^{\Delta_+}}{(x - x')^{2\Delta_+}}\right) \phi_0(x') \tag{7.6}$$

and in the parenthesis we identify the form (5.21) of the propagator K near the boundary.

8 Witten's diagrams

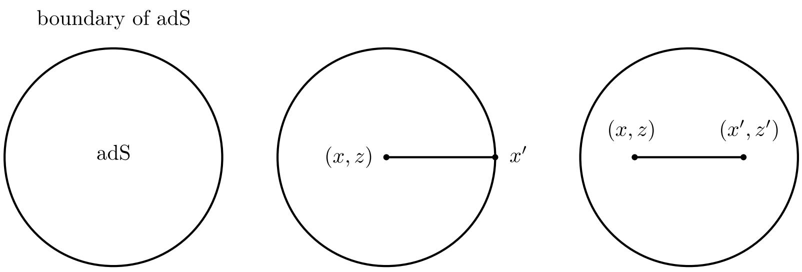
The computation of actions can be summarized by using Witten diagrams: as we have seen the euclidean adS space is equivalent to a ball. With this last representation the boundary of adS space maps to the boundary of the ball, and the bulk corresponds to its interior. Lines between points give propagators (figure 1) [8, 11].

9 Evaluation of the action

Before evaluating the action using different methods, let's pause a second to focus on a potential problem: we have introduced everywhere a cut-off ε and at the end we want to take the limit $\varepsilon \to 0$. But this limit should be take in a consistent manner in all formula since this limit and other expansions (e.g. of Bessel functions) do not commute [35]. Especially we will see that one has to add counter-term to regulate divergences and if the limit is not taken correctly then we get a wrong coefficient for the correlation function [12, 4, app. A], [18, sec. 2.2], [19, lec. 17], [30, sec. 4] [19].

Basically there are two ways to take the limit:

[19] When using the adS/CFT conjecture one can relate 2- and 3-point functions, which determine the "correct" coefficient [12, end of sec. 3.2], which is otherwise not possible in "normal" adS.



(a) Empty adS. (b) Bulk-to-boundary propagator. (c) Bulk-to-bulk propagator.

Figure 1: Witten's diagram for empty adS and propagators.

- first expand all ϕ as $\phi \sim \phi_0 + \phi_1$ using (2.16), and then after taking the limit after simplication (Witten's and alternative momentum methods);
- or do simplifications and then take the limit ("Freedman's" method).

9.1 Position space: Witten's Green function

We will evaluate the scalar field action (1.1) using the second form (1.2): the first term vanishes due to the equation of motion (1.4) and the second, as written in (1.3), reads

$$
\begin{aligned}
S_\varepsilon &= \frac{1}{2} \int_{\partial\mathcal{M}} \mathrm{d}^d y \, \sqrt{\gamma} \, \phi n^\mu \partial_\mu \phi \\
&= \frac{L^{d-1}}{2} \int \mathrm{d}^d x \; z^{-d+1} \phi(z, x) \partial_z \phi(z, x) \Big|_{z=\varepsilon}
\end{aligned}
$$

and using the expression (5.2) for then normal vector, and one introduces the boundary propagator (2.27):

$$
S = \frac{L^{d-1}}{2} \int \mathrm{d}^d x_1 \, \mathrm{d}^d x_2 \; \phi_0(x_1) \mathcal{F}_\varepsilon(x_1, x_2) \phi_0(x_2) \tag{9.1}
$$

where

$$
\mathcal{F}_\varepsilon(x_1, x_2) = z^{-d} \int \mathrm{d}^d x \; K(z, x; x_1) \, z \partial_z K(z, x; x_2) \Big|_{z=\varepsilon}. \tag{9.2}
$$

Using the asymptotic expression (5.21) one can compute this operator to subleading order [19, lec. 14]:

$$
\begin{aligned}
\mathcal{F}_\varepsilon(x_1, x_2) \approx z^{-d} \int \mathrm{d}^d x \; &\left(z^{\Delta_-} \delta^{(d)}(x - x_1) + C_{\Delta_+} \frac{z^{\Delta_+}}{(x - x_1)^{2\Delta_+}} \right) \\
&\times \left(\Delta_- z^{\Delta_-} \delta^{(d)}(x - x_2) + \Delta_+ C_{\Delta_+} \frac{z^{\Delta_+}}{(x - x_2)^{2\Delta_+}} \right) \Bigg|_{z=\varepsilon},
\end{aligned}
$$

and after simplication (the fourth term vanishes in the limit $\varepsilon \to 0$) [20]

$$
\mathcal{F}_\varepsilon(x_1, x_2) \approx \Delta_- \, \varepsilon^{-\nu} \delta^{(d)}(x_1 - x_2) + \frac{d \, C_{\Delta_+}}{(x_1 - x_2)^{2\Delta_+}}, \tag{9.3}
$$

[20] Note that McGreevy [19] has a factor Δ_+ instead of d because he is taking the limit $\varepsilon \to 0$ too soon, approximating $K(x_1)$ by a delta function.

remembering that $\Delta_+ + \Delta_- = d$ and $d - 2\Delta_- = 2\nu$. The first term is a contact term which can be removed by renormalization, by adding a counter-term proportional to ϕ_0^2 on the boundary [20, sec. 4.3], [19, lec. 14], [34, p. 19], [27]. Then the correct expression is [21]

$$S_{\text{ct}} = -\frac{L^{d-1}\Delta_-}{2} \int_{z=\varepsilon} \mathrm{d}^d x \, \sqrt{\gamma}\, \phi(z,x)^2, \tag{9.4}$$

because we have to use 5-dimensional fields to respect covariance [30, sec. 5.3] [22].

Here the use of the Witten's Green function is equivalent to first expanding field and then taking the limit, so we have to do the same in S_{ct}. Using the asymptotic expression (2.16) (in position space) one gets

$$S_{\text{ct}} \approx -\frac{L^{d-1}\Delta_-}{2} \int_{z=\varepsilon} \mathrm{d}^d x \, \Big(z^{-2\nu}\phi_0(x)^2 + 2\phi_0(x)\phi_1(x) \Big). \tag{9.5}$$

Using the expression (7.5) of ϕ_1 in term of K we rewrite the second term as

$$-\frac{L^{d-1}\Delta_-}{2} \int_{z=\varepsilon} \mathrm{d}^d x_1 \mathrm{d}^d x_2 \; 2\phi_0(x_1)\phi_0(x_2) \frac{C_{\Delta_+}}{(x_1 - x_2)^{2\Delta_+}},$$

which results in the total action [34, p. 19]

$$S = \frac{L^{d-1}}{2} \int \mathrm{d}^d x_1 \, \mathrm{d}^d x_2 \; \phi_0(x_1) \widetilde{\mathcal{F}}_\varepsilon(x_1, x_2) \phi_0(x_2) \tag{9.6}$$

where

$$\widetilde{\mathcal{F}}(x_1, x_2) = \mathcal{F}_\varepsilon(x_1, x_2) - \Delta_- z^{-2\nu} \delta^{(d)}(x_1 - x_2) - 2\Delta_- \frac{C_{\Delta_+}}{(x_1 - x_2)^{2\Delta_+}} \tag{9.7a}$$

$$= \frac{2\nu\, C_{\Delta_+}}{(x_1 - x_2)^{2\Delta_+}} \tag{9.7b}$$

is the renormalized kernel: it is independent of ε.

The final renormalized action is then [26, sec. 4.2]

$$S[\phi_0] = \nu\, L^{d-1} C_{\Delta_+} \int \mathrm{d}^d x_1 \, \mathrm{d}^d x_2 \; \frac{\phi_0(x_1)\phi_0(x_2)}{(x_1 - x_2)^{2\Delta_+}}. \tag{9.8}$$

In section 3 we said that ϕ_1 is the conjugate variable to ϕ_0. We are now able to prove this assertion by computing the derivative of $S[\phi_0]$ [22]

$$\frac{\delta S[\phi_0]}{\delta \phi_0(x)} = \nu\, L^{d-1} C_{\Delta_+} \int \mathrm{d}^d x' \; \frac{\phi_0(x')}{(x - x')^{2\Delta_+}} = \nu\, L^{d-1}\, \phi_1(x) \tag{9.9}$$

using the relation (7.5).

9.2 Momentum space: Green function ("Freedman's" method)

The previous method is not always available, for example when the interior is not adS anymore. So we have to work in momentum space, using the expression (2.34) (as we note earlier it is almost the propagator, so we will save some pain by using this expression instead

[21] Some people are using directly the boundary field, giving a different factor.
[22] Note also that this boundary term ensures stability[27].

of the propagator). The insertion of the Fourier transform (2.7) in the action (1.3) gives [11, sec. 23.10]

$$\begin{aligned} S[\phi_0] &= \frac{L^{d-1}}{2} \int \mathrm{d}^d x \; z^{-d+1} \phi(z,x) \partial_z \phi(z,x) \Big|_{z=\varepsilon} \\ &= \frac{L^{d-1}}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d k'}{(2\pi)^d} \, \mathrm{d}^d x \; \mathrm{e}^{ix(k+k')} \, z^{-d+1} f_k(z) \partial_z f_{k'}(z) \Big|_{z=\varepsilon} \\ &= \frac{L^{d-1}}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d k'}{(2\pi)^d} \, \mathrm{d}^d x \; \mathrm{e}^{ix(k+k')} \, \phi_0(k) \phi_0(k') \, z^{-d+1} K_k(z) \partial_z K_{k'}(z) \Big|_{z=\varepsilon} \end{aligned}$$

and using (5.3)

$$S = \frac{L^{d-1}}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \phi_0(k) \, \mathcal{F}_\varepsilon(k) \, \phi_0(-k) \tag{9.10}$$

where

$$\mathcal{F}_\varepsilon(k) = z^{-d} K_{-k}(z) \, z\partial_z K_k(z) \Big|_{z=\varepsilon}. \tag{9.11}$$

By plugging (5.11) this can be rewritten [23]

$$\mathcal{F}_\varepsilon(k) = \varepsilon^{2\Delta_-} \, \varepsilon^{-d+1} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \ln \Big((k\varepsilon)^{d/2} K_\nu(k\varepsilon) \Big). \tag{9.12}$$

In the parenthesis we recognize $f_k(\varepsilon)/a_k$ from (2.15) so we can use the asymptotic (2.16) to compute the derivative [24]:

$$\begin{aligned} \varepsilon \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \ln \Big(\varepsilon^{d/2} K_\nu(k\varepsilon) \Big) &\approx \varepsilon \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \ln \left(\frac{1}{a_k} \Big(\phi_0 \varepsilon^{\Delta_-} + \phi_1 \varepsilon^{\Delta_+} \Big) \right) \\ &= \varepsilon \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left[-\ln a_k + \ln \phi_0 + \Delta_- \ln \varepsilon + \ln \left(1 + \frac{\phi_1}{\phi_0} \varepsilon^{2\nu} \right) \right] \end{aligned}$$

so expand the logarithm and take to zero the derivative of the two constants,

$$= \Delta_- + 2\nu \, \frac{\phi_1}{\phi_0} \varepsilon^{2\nu}.$$

The first term is analytic in k (if we were going further in the development, there would be only integer power of k), so it corresponds to contact term and it is removed by renormalization: the corresponding counter-term is the one written previously (9.4). Note that we have to take the limit $\varepsilon \to 0$ directly, using the limit (2.27):

$$S_{\mathrm{ct}} \approx -\frac{L^{d-1} \Delta_-}{2} \int_{z=\varepsilon} \mathrm{d}^d x \, \sqrt{\gamma} \, \varepsilon^{2\Delta_-} \phi_0(x)^2. \tag{9.13}$$

The second term is non-analytic and it corresponds to the absorptive part of the two-point function:

$$\mathcal{F}_\varepsilon(k) = 2\nu \, \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(\frac{k}{2} \right)^{2\nu} \tag{9.14}$$

where we used (2.20), and the $\varepsilon^{2\Delta_-}$ factor cancels the $\varepsilon^{-2\Delta_-}$ appearing in the product [25]. So finally the action is

$$S[\phi_0] = \nu \, L^{d-1} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \phi_0(-k) \phi_0(k) \left(\frac{k}{2} \right)^{2\nu}. \tag{9.15}$$

[23] We add a factor $(k/k)^{d/2}$ for convenience.

[24] Freedman [11, p. 548] directly expands the Bessel function, the result is the same but our way avoid expanding again.

[25] As we said in section 2.5 most of people use the boundary condition at $z = 0$ instead of $z = \varepsilon$, leading to a factor $\varepsilon^{-2\Delta_-}$, which they throw away saying that one has to rescale the field.

Transforming this back to position space gives

$$\begin{aligned}
S[\phi_0] &= \nu\, L^{d-1}\, \frac{\Gamma(-\nu)}{2^{2\nu}\Gamma(\nu)} \int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\; \phi_0(x_1)\phi_0(x_2) \int \frac{\mathrm{d}^d k}{(2\pi)^d} k^{2\nu}\, \mathrm{e}^{ik(x_1-x_2)} \\
&= \nu\, L^{d-1}\, \frac{\Gamma(-\nu)}{2^{2\nu}\Gamma(\nu)} \int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\; \phi_0(x_1)\phi_0(x_2)\, \frac{2^{2\nu}}{\pi^{d/2}} \frac{\Gamma(\Delta_+)}{\Gamma(-\nu)} \frac{1}{(x_1-x_2)^{2\Delta_+}},
\end{aligned}$$

and finally

$$S[\phi_0] = \nu\, L^{d-1}\, C_{\Delta_+} \int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \frac{\phi_0(x_1)\phi_0(x_2)}{(x_1-x_2)^{2\Delta_+}}. \tag{9.16}$$

We get the same result as in the previous section.

The computation for ν integer will be multiplied by a logarithm.

9.3 Momentum space: Green function (alternative method)

If we compute the kernel $\mathcal{F}_\varepsilon(k)$ (9.11) without writing it as a log and use (5.12), then we need to use the action (9.5) where ϕ is expanded [34, p. 19]:

$$\begin{aligned}
\mathcal{F}_\varepsilon(k) &= z^{-d} K_{-k}(z)\, z\partial_z K_k(z)\Big|_{z=\varepsilon} \\
&= z^{-d} \left(z^{\Delta_-} + \frac{\phi_1}{\phi_0} z^{\Delta_+} \right) z\partial_z \left(z^{\Delta_-} + \frac{\phi_1}{\phi_0} z^{\Delta_+} \right) \Big|_{z=\varepsilon} \\
&= \Delta_- \varepsilon^{2\nu} + d\, \frac{\phi_1}{\phi_0}.
\end{aligned}$$

Here the factor of the second term is d, so subtracting $2\Delta_-$ gives the expected 2ν.

10 Cubic interactions

To the free action (1.1) for three fields ϕ_i

$$S_0 = -\frac{1}{2} \int \mathrm{d}^{d+1} X\, \sqrt{g}\, (g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i + m^2_{ij}\phi_i\phi_j), \qquad m_{ij} = m_i\delta_{ij}, \tag{10.1}$$

we add a cubic interaction

$$S_{\text{int}} = -\int \mathrm{d}^{d+1} X\, \sqrt{g}\, \lambda_{ijk}\phi_i\phi_j\phi_k. \tag{10.2}$$

This gives the equation of motion

$$(-\,\Delta + m_i^2)\phi_i = \lambda_{ijk}\phi_j\phi_k. \tag{10.3}$$

The solution for ϕ_i is obtained perturbatively by introducing more and more source on the boundary and interactions in the bulk:

$$\begin{aligned}
\phi_i(z,x) = &\int \mathrm{d}^d x_1\; K_i(z,x;x_1)\phi_{i0}(x_1) \\
&+ \sum_{j,k} \lambda_{ijk} \int \mathrm{d}^{d+1} X'\; G_i(z,x;z',x') \times \\
&\quad \times \int \mathrm{d}^d x_1 \mathrm{d}^d x_2\; K_j(z',x';x_1) K_k(z',x';x_2) \phi_{0j}(x_1)\phi_{0k}(x_2) \\
&+ \cdots
\end{aligned} \tag{10.4}$$

Each term in this expansion can be represented as a Witten diagram.

Now we can compute the action, keeping only terms with three sources; the free action part will not contribute and so only the first term in the ϕ_i expansion is needed:

$$S_{\text{int}}^{(3)} = \int \mathrm{d}^d x_1 \mathrm{d}^d x_2 \mathrm{d}^d x_3 \; \mathcal{F}_{ijk}(x_1, x_2, x_3) \phi_{0i}(x_1) \phi_{0j}(x_2) \phi_{0k}(x_3) \tag{10.5}$$

where

$$\mathcal{F}_{ijk}(x_1, x_2, x_3) = \lambda_{ijk} \int \mathrm{d}^{d+1} X \; K_i(z, x; x_1) K_j(z, x; x_2) K_k(z, x; x_3) \tag{10.6}$$

(no summation). Note that the interaction part can not give any contribution to the term quadratic in the sources, so the result we found in the free theory for this quadratic part is still valid. Moreover there are no ε left so we do not need renormalization [23]. The corresponding Witten's diagram is given in figure 2 (bulk-to-bulk propagator appears with four sources).

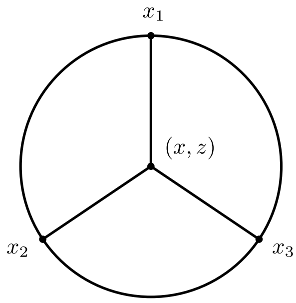


Figure 2: Witten's diagram for cubic interaction.

Question 10.1 Check that the free part does not contribute. Add references. Cf Kiritsis [15, p. 429, 551].

11 Comments

Generalization to the Lifschitz space corresponding to the metric

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{z^{2z}} + \frac{\mathrm{d}z^2}{z^2} + \frac{1}{z^2} \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{11.1}$$

where the symmetries on the boundary lie in the Schrödinger group (non-relativistic scale invariant theory) can be found in [2, 31, 13]; z is the dynamical exponent which gives the anisotropy between time and space under scale transformation

$$t \longrightarrow \lambda^z, \qquad x^i \longrightarrow \lambda x^i. \tag{11.2}$$

$z = 1$ corresponds to full conformal symmetry and $z = 2$ to the Schrödinger equation.

Computations of n-point functions with arbitrary polynomial can be found in [23].

In all computations we used Poincaré coordinates as a vacuum. Global coordinates are also often used, and sometimes more convenient. We know that the vacuum choice can have an impact when quantizing a field theory on curved space; the authors of [3] relate modes of different vacua (using adS/CFT) and show that the only difference is an energy shift of the states.

A Conventions

A.1 Basic

The metric signature is taken to be mostly plus.

We define by g the absolute value of the metric determinant:

$$g = |\det g_{\mu\nu}| = -\det g_{\mu\nu}. \tag{A.1}$$

A.2 Wick rotation

The passage from Minkowski metric

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2 \tag{A.2}$$

to euclidean metric

$$\mathrm{d}s_E^2 = \delta_{\mu\nu}\mathrm{d}x_E^\mu \mathrm{d}x_E^\nu = \mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{x}^2 \tag{A.3}$$

is done through the substitution of the real time t by the euclidean time τ [36, sec. 3.4]

$$t = -i\tau. \tag{A.4}$$

Then the action becomes [32]

$$S = \int \mathrm{d}^d x\, \mathcal{L} = -i \int \mathrm{d}^d x_E\, \mathcal{L} = i \int \mathrm{d}^d x_E\, \mathcal{L}_E = iS_E \tag{A.5}$$

so that

$$S = iS_E, \quad \mathcal{L} = -\mathcal{L}_E, \quad Z = \int \mathrm{d}\phi\ \mathrm{e}^{iS} = \int \mathrm{d}\phi\ \mathrm{e}^{-S_E} \tag{A.6}$$

Since the Euclidean action is now positive definite, the minus sign in the partition function gives exponential damping. In the case of curved spacetime the metric determinant simply becomes

$$\sqrt{-g} = \sqrt{g_E}. \tag{A.7}$$

As an example look at the scalar lagrangian with potential:

$$\mathcal{L} = -\frac{1}{2}\Big((\partial^\mu \phi)^2 + m^2\phi^2\Big) - V(\phi) \tag{A.8}$$

which gives the equation of motion

$$(-\Delta + m^2)\phi = V'(\phi). \tag{A.9}$$

Plugging plane-waves into the free equation ($V = 0$) gives the mass-shell condition

$$p^2 = -m^2 \tag{A.10}$$

and the Green function

$$G(p) = \frac{1}{p^2 + m^2} \tag{A.11}$$

has a singularity.

Applying the Wick rotation gives

$$\mathcal{L}_E = \frac{1}{2}\Big((\partial_E^\mu \phi)^2 + m^2\phi^2\Big) + V(\phi), \tag{A.12}$$

which is positive definite, and the equation of motion

$$(-\Delta_E + m^2)\phi = -V'(\phi). \tag{A.13}$$

with Green function

$$G(p) = \frac{1}{p_E^2 + m^2}. \tag{A.14}$$

This function has no singularity since plane-waves are not anymore solutions of the Klein-Gordon equation (said another way, there is no particle in Euclidean space).

B Anti de Sitter space

This apppendix mainly draw from [11, sec. 22.1], and [5] is another useful review.

In this chapter we will use as indices:

- A for $(d+1)$-dimensional (adS);
- μ for d-dimensional Minkowski;
- i for $(d-1)$-dimensional spatial.

In some cases we will also consider the embedding into $(d+2)$-dimensional space with indices α, and we will note $a = \{i\}, d$.

B.1 Action and equation of motion

Start with the Einstein–Hilbert action in vacuum with cosmological constant Λ in $D = d+1$ dimensions

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^{d+1}x\, \sqrt{-g}(R - \Lambda). \tag{B.1}$$

If we write

$$\Lambda = -\frac{1}{L^2}d(d-1) \tag{B.2}$$

the equation of motion reads

$$R_{\mu\nu} = -\frac{d}{L^2}g_{\mu\nu}, \qquad R = -\frac{1}{L^2}d(d+1), \tag{B.3a}$$

which means that the space is an Einstein manifold, and moreover it is maximally symmetric [7, p. 141]:

$$R_{\mu\nu\rho\sigma} = -\frac{1}{L^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{B.3b}$$

L will be the radius of the space (equivalently size of the throat).

B.2 Coordinates

B.2.1 Global

Let's denote generically the metric by

$$\mathrm{d}s^2 = g_{AB}\,\mathrm{d}X^A\mathrm{d}X^B \tag{B.4}$$

where the A indices are splitten between a radial coordinate and d-dimensional $\mu = 0, \dots, d-1$ (in some cases they will correspond to Minkowski indices).

From the embedding into $\mathbb{R}^{d,2}$ with

$$\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, \dots, 1, -1), \qquad \alpha = 0, \dots, d+1, \tag{B.5}$$

adS$_{d+1}$ is defined as an hyperboloid of radius L

$$\begin{aligned}\eta_{\alpha\beta} Y^\alpha Y^\beta &= \eta_{\mu\nu} Y^\mu Y^\nu + (Y^d)^2 - (Y^{d+1})^2 = -L^2 \\ &= -(Y^0)^2 + (Y^a)^2 - (Y^{d+1})^2 = -L^2,\end{aligned}$$

where we have denoted $a = \{i\}, d$ and as usual $i = 1, \dots, d-1$.

The choice of parametrization

$$Y^0 = \sqrt{L^2 + r^2} \sin \frac{t}{L}, \qquad Y^{d+1} = \sqrt{L^2 + r^2} \cos \frac{t}{L}, \qquad Y^a = r \hat{x}^a \tag{B.7}$$

with

$$r \geq 0, \qquad t \in [0, 2\pi L[, \qquad (\hat{x}^a)^2 = 1 \tag{B.8}$$

gives the metric

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{L^2}\right) \mathrm{d}t^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{d-1}^2. \tag{B.9}$$

The time is periodic and closed time-like curved exists. To get rid of this unwanted feature one can take the covering space where $t \in \mathbb{R}$ – we do not use a new name but implictly we use this covering space. In these coordinates the SO$(d, 2)$ symmetry is manifest [27].

The change of radial variable

$$\mathrm{sh}\, \frac{y}{L} = \frac{r}{L}, \qquad \mathrm{ch}\, \frac{y}{L} = \sqrt{1 + \frac{r^2}{L^2}}, \qquad y \geq 0 \tag{B.10}$$

gives

$$\mathrm{d}s^2 = -\,\mathrm{ch}^2 \frac{y}{L}\, \mathrm{d}t^2 + \mathrm{d}y^2 + L^2\, \mathrm{sh}^2 \frac{y}{L}\, \mathrm{d}\Omega_{d-1}^2. \tag{B.11}$$

From this last coordinate system we can introduce again a new radial coordinates and a rescaled time

$$t = L\tau, \qquad \mathrm{ch}\, \frac{y}{L} = \frac{1}{\cos\rho}, \qquad \rho \in [0, \pi/2[, \tag{B.12}$$

which gives the metric

$$\mathrm{d}s^2 = \frac{L^2}{\cos^2\rho} \left(-\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sin^2\rho\, \mathrm{d}\Omega_{d-1}^2\right). \tag{B.13}$$

This metric is a conformal factor times another metric known as the Einstein static universe, which is $\mathbb{R} \times S^{d-1}$; but since to cover the whole sphere one needs $\rho \in [0, \pi[$, only the upper half plane is covered here.

The Cauchy problem is ill-defined in adS space because information from spatial infinity can reach the origin in finite time: it will explain why in adS/CFT one needs to give boundary conditions.

B.2.2 Patches

Another set of very useful coordinates is the Poincaré patch, given by

$$\mathrm{d}s^2 = \frac{L^2}{r^2}\,\mathrm{d}r^2 + \frac{r^2}{L^2}\,\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu, \qquad r > 0, \tag{B.14}$$

or by rescaling $r = Lu$:

$$\mathrm{d}s^2 = L^2\left(\frac{\mathrm{d}u^2}{u^2} + u^2\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right). \tag{B.15}$$

Note that only a part of the adS space is covered by these coordinates, and the $r \to 0$ limit is an horizon, not a singularity. For the euclidean version $r = 0$ shrinks to a point. In the case og minkowskian signature we will need to consider ingoing or outgoing waves as boundary conditions when solving the wave equations.

The change of variables

$$z = \frac{L^2}{r} = \frac{1}{u} \tag{B.16}$$

brings the metric to the form

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\Big(\mathrm{d}z^2 + \eta_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu\Big). \tag{B.17}$$

The boundary is at $z \to 0$.

Null geodesics are given by [27]

$$z = \pm(t - t_0) \tag{B.18}$$

for which the boundary $z = 0$ is reached in a finite time, while it takes an infinite time to reach $z = \infty$.

The normal vector to a surface $z = \mathrm{cst}$ is

$$n = \frac{1}{\sqrt{g_{zz}}}\,\partial_z = \frac{z}{L}\,\partial_z \tag{B.19}$$

(normalized such that $n^2 = 1$) and the induced metric is

$$\gamma_{\mu\nu} = \frac{L^2}{z^2}\,\eta_{\mu\nu}. \tag{B.20}$$

B.3 Properties

Given two points $X = (z, x)$ and $X' = (x', z')$, the unique conformal invariant that can be constructed is [27]

$$\xi = \frac{2zz'}{z^2 + z'^2 + (x - x')^2}. \tag{B.21}$$

B.4 Boundary

Consider the equation of embedding (B.6)

$$-(Y^0)^2 + (Y^a)^2 - (Y^{d+1})^2 = -L^2 \tag{B.22}$$

and introduce light-cone coordinates [1, sec. 4.1.1]

$$U = Y^{d+1} + Y^d, \qquad V = Y^{d+1} - Y^d. \tag{B.23}$$

The constraint equation becomes

$$-UV - (Y^0)^2 + (Y^i)^2 = -L^2, \qquad i = 1, \dots, d-1. \tag{B.24}$$

If $L = 0$ then one can solve for V if $U \neq 0$ as

$$V = \frac{-(Y^0)^2 + (Y^a)^2}{U}. \tag{B.25}$$

Note that we also have the symmetry where $Y^\alpha \to \lambda Y^\alpha$, and we mod it out by using it to fix one of the coordinates, say $U = 1$ (this is equivalent to identifying two points if they differ by a rescaling). The induced metric is then d-dimensional Minkowski

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{B.26}$$

However we can also include the point $U = 0$ by reversing the roles of U and V; concretely we have added the point at infinity and the space we get has not exactly the topology of Minkowski but the one of $(S^1 \times S^{d-1})/\mathbb{Z}_2$, or $\mathbb{R} \times S^{d-1}$ if we consider the universal covering.

It is not possible to define directly the induced metric on the boundary $z = 0$ because it is divergent, so we define it as the limit [1, sec. 5.2] (omitting the L^2 factor)

$$\gamma_{\mu\nu} = \lim_{z\to 0} \frac{f^2}{z^2}\, \eta_{\mu\nu} \tag{B.27}$$

for some function f such that $f(z) \sim_0 z$. Thus the induced metric depends on f and is defined up to a conformal transformation, which means that we have a conformal structure on the boundary (i.e. we can measure angles but not distances) [26]. Doing a conformal transformation $f \to \mathrm{e}^w f$ implies

$$\gamma'_{\mu\nu} = \mathrm{e}^{2w} \gamma_{\mu\nu}, \tag{B.28}$$

so the metric has conformal weight -2.

C AdS/CFT correspondence

C.1 The dictionnary

The basic correspondence says that to each boundary field there is an associated boundary operator from a CFT. The generating function is given by

$$Z_{\text{CFT}} = \left\langle \exp\left(\int_{\partial\mathcal{M}} \phi_0 \mathcal{O} \right) \right\rangle \tag{C.1}$$

and one postulates that it is given by the extremum of the string theory action, i.e. at tree level it corresponds to the action of supergravity evaluated with classical configuration:

$$Z_{\text{CFT}} = \exp(-S[\phi_0]) \tag{C.2}$$

where the ϕ satisfy the asymptotic condition (2.27).

If ϕ couples to an operator $\mathcal{O}$ on the boundary, we can use the following trick to find the scaling of this operator [19]:

$$S \ni \int_{z=\varepsilon} \mathrm{d}^d x\, \sqrt{\gamma_\varepsilon} \phi(\varepsilon, x) \mathcal{O}(\varepsilon, x) = \int_{z=\varepsilon} \mathrm{d}^d x\, \left(\frac{L}{\varepsilon}\right)^d \varepsilon^{\Delta_-} \phi_0(x) \mathcal{O}(x)$$

[26] This is expected in context of adS/CFT, where the CFT lives on the boundary.

where we used the boundary field (2.29)

$$\phi(\varepsilon, x) = \varepsilon^{\Delta_-} \phi_0(x), \tag{C.3}$$

defined in the previous section, and so

$$\mathcal{O}(\varepsilon, x) = \varepsilon^{\Delta_+} \mathcal{O}(x). \tag{C.4}$$

C.2 Choice of coordinate systems

A question that can be asked concerns the choice between the global coordinate system (B.9)

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{L^2}\right)\mathrm{d}t^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{d-1}^2 \tag{C.5a}$$

and the Poincaré upper-half plane (B.17)

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\Big(\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\Big). \tag{C.5b}$$

The later presents an horizon at $z = \infty$, but we will study only systems with finite time and length, so the horizon will never be crossed.

It is worth to note that the dual CFT is defined on S^{d-1} instead of $\mathbb{R}^{d-1}$ [27].

C.3 Correlation functions

C.3.1 1-point

C.3.2 2-point

Using (9.16) the 2-point function is given by

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = -\frac{\delta}{\delta\phi_0(x_1)}\frac{\delta}{\delta\phi_0(x_2)}S[\phi_0]\bigg|_{\phi_0=0} = -\nu\, L^{d-1} C_{\Delta_+}\,\frac{1}{(x_1 - x_2)^{2\Delta_+}} \tag{C.6}$$

which agrees with the result one would expect from conformal invariance.

C.3.3 3-point

If we turn on a cubic interaction in the bulk for three fields ϕ_i as in (10.2), then the 3-point function for the associated operators $\mathcal{O}_i$ is given by deriving (10.5) with respect to the sources:

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \mathcal{F}_{ijk}(x_1, x_2, x_3) \tag{C.7}$$

with

$$\mathcal{F}_{ijk}(x_1, x_2, x_3) = \lambda_{ijk}\int \mathrm{d}^{d+1}x\; K_i(z, x; x_1)K_j(z, x; x_2)K_k(z, x; x_3) \tag{C.8}$$

Using the Witten's expression for K_i and the scaling relation (5.23)

$$K(z, x_1; x_2) \longrightarrow (x_1')^{2\Delta_+} K(z, x_1'; x_2') \tag{C.9}$$

one finds that

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle \longrightarrow (x_1')^{2\Delta_i}(x_2')^{2\Delta_j}(x_3')^{2\Delta_k}\,\langle \mathcal{O}_i(x_1')\mathcal{O}_j(x_2')\mathcal{O}_k(x_3')\rangle \tag{C.10}$$

which is the correct transformation for a conformal 3-point function.

D Special functions

D.1 Gamma and beta functions

The beta function is

$$B(x,y) = \int_0^1 \mathrm{d}t\, t^{x-1}(1-t)^{y-1} \tag{D.1}$$

We can obtain the equivalent forms

$$B(x,y) = \int_0^\infty \mathrm{d}t\, \frac{t^{x-1}}{(1+t)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{D.2}$$

D.2 Bessel functions

Bessel differential equation is

$$x^2 f'' + x f' - (x^2 - n^2) f = 0 \tag{D.3}$$

admits $J_{\pm n}$ (Bessel functions of first kind) as solutions if $n \notin \mathbb{N}$.

If $n \in \mathbb{N}$ then $J_{\pm n}(x)$ are not linearly independent since

$$J_{-n} = (-1)^n J_n. \tag{D.4}$$

In this case one has to introduce Bessel functions of second kind $Y_n(x)$ (denoted sometime $N_n(x)$)

$$Y_n(x) = \frac{J_n(x)\cos(nx) - J_{-n}(x)}{\sin(nx)}, \tag{D.5}$$

the limit where n is an integer being regular.

Asymptotic forms are pure imaginary exponentials.

D.3 Modified Bessel functions

Modified Bessel functions $I_n(x)$ and $K_n(x)$ are solutions of the differential equation

$$x^2 f'' + x f' - (x^2 + n^2) f = 0. \tag{D.6}$$

The modified Bessel functions have exponential behavior.

One has the asymptotic forms

$$I_n(x) \sim_0 \frac{1}{\Gamma(n+1)} \left(\frac{x}{2}\right)^n, \qquad K_n(x) \sim_0 \frac{\Gamma(n)}{2} \left(\frac{2}{x}\right)^n, \tag{D.7a}$$

$$I_n(x) \sim_\infty \frac{\mathrm{e}^x}{\sqrt{x}}, \qquad K_n(x) \sim_\infty \frac{\mathrm{e}^{-x}}{\sqrt{x}}? \tag{D.7b}$$

The derivative of the Bessel function is

$$\frac{\mathrm{d}K_n(x)}{\mathrm{d}x} = -\frac{1}{2}\Big(K_{n-1}(x) + K_{n+1}(x)\Big), \tag{D.8}$$

and

$$x\, \frac{\mathrm{d}K_n(x)}{\mathrm{d}x} = nK_n(x) - x\, K_{n+1}(x). \tag{D.9}$$

The complete limits for K_n is [8, p. 548]

$$K_n(x) \sim_0 \frac{\Gamma(n)}{2} \left(\frac{2}{x}\right)^n + \frac{\Gamma(-n)}{2} \left(\frac{x}{2}\right)^n. \tag{D.10}$$

If n is integer, then there is an extra factor $\ln x$ in the second term.

In terms of Bessel functions they are given as

$$\begin{aligned} I_n(x) &= i^{-n} J_n(ix), \\ K_n(x) &= \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin(nx)}. \end{aligned} \tag{D.11}$$

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