Notes on $2d$ quantum gravity and Liouville theory

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November 2, 2015

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1 Introduction

When studying $2d$ quantum gravity several approaches can be used in complementary ways [46]. For example the matrix models (discretization) is powerful and gives a non-perturbative definition, while Liouville theory (continuous approach) offer a more transparent physical interpretation (states are easier to identify, for example from BRST cohomology) [75].

1.1 Acknowledgements

These notes grew up from lectures by Atish Dabholkar [12], and from discussions with him and its students Tresa Bautista and Matěj Kudrna.

I'm also very grateful to Costas Bachas, Lætitia Leduc, Blagoje Oblak, Sylvain Ribault and Raoul Santachiara for interesting discussions.

This review is still a work in progress, and as such it may contain errors and incomplete material.

Part I
Conformal field theory

2 Conformal field theory

The classic references for $2d$ conformal field theory is [22, chap. 5].

2.1 Coordinates

We consider first an euclidean 2-dimensional manifold with coordinates (τ, σ).

All two dimensional manifolds are complex manifolds (and even Kähler) so that we can use a complex coordinates z. It is often simpler to consider its complex conjugate $\bar{z}$ as independent, and then at the end to restrict oneself to the section $\bar{z} = z^*$.

The metric for a $2d$ Kähler manifold is just diagonal

$$\mathrm{d}s^2 = g_{z\bar{z}} \mathrm{d}z \mathrm{d}\bar{z}. \tag{2.1}$$

We can see that any change of coordinates

$$w = f(z), \qquad \bar{w} = \bar{f}(\bar{z}) \tag{2.2}$$

where f and $\bar{f}$ are any holomorphic and antiholomorphic (independent) functions preserve the form of the line element, and the metric component transforms as

$$g_{w\bar{w}} = |f'(z)|^2 g_{z\bar{z}}. \tag{2.3}$$

We define also the (anti-)holomorphic part of the stress–energy tensor

$$T \equiv T_{zz}, \qquad \bar{T} \equiv T_{\bar{z}\bar{z}}. \tag{2.4}$$

It transforms with an anomalous term

$$T'(w) = \frac{1}{f'^2} \left(T(z) - \frac{c}{12} S(w, z) \right) \tag{2.5}$$

where $S(w,z)$ is the Schwarzian derivative

$$S(w,z) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f}\right)^2. \tag{2.6}$$

Example 2.1 (Flat metric) We may choose complex coordinates

$$z = \tau + i\sigma. \tag{2.7}$$

Then derivatives are given by

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_\tau - i\partial_\sigma) \tag{2.8}$$

and similarly $\bar\partial \equiv \partial_{\bar z}$.

Then the flat metric is simply

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar z. \tag{2.9}$$

Example 2.2 (Cylinder and complex planes) In order to avoid IR problem, it is useful to compactify the spatial direction σ in a circle [48, sec. 4.3], such that

$$\tau \in \mathbb{R}, \qquad \sigma \in [0, 2\pi[. \tag{2.10}$$

These two coordinates parametrize a cylinder.

We can introduce two different coordinates

$$w = \tau + i\sigma, \qquad z = \mathrm{e}^w = \mathrm{e}^{\tau + i\sigma}. \tag{2.11}$$

The transformation $z = e^w$ defines a mapping from the cylinder to the complex plane. On this plane, $\tau = \text{cst}$ corresponds to circles of radius e^τ. Time reversal corresponds to $1/z^*$, parity to z^*. Past infinity is the origin, while future infinity is the complex infinity.

Action invariant under conformal transformations will have the same form in the three coordinates systems

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2.2 Quasi-primary fields

A field $\phi(z,\bar z)$ of scaling dimension Δ and spin s is said to be quasi-primary if it transforms as

$$\phi'(w,\bar w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h}\left(\frac{\mathrm{d}\bar w}{\mathrm{d}\bar z}\right)^{-\bar h}\phi(z,\bar z) \tag{2.12}$$

where h and $\bar h$ are the (anti-)holomorphic conformal dimensions defined as

$$h = \Delta + s, \qquad \bar h = \Delta - s. \tag{2.13}$$

This implies that for a spinless field we have

$$\Delta = h = \bar h \tag{2.14}$$

and

$$\phi'(w,\bar w) = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^{-2\Delta}\phi(z,\bar z) \tag{2.15}$$

This normalization is very convenient for Liouville theory since we will be dealing with a spinless field most of the time [1].

The associated state (on the plane)

$$\left|h, \bar{h}\right\rangle = \phi(0,0) \left|0\right\rangle \tag{2.17}$$

is an eigenvector of L_0 and $\bar{L}_0$

$$L_0 \left|h, \bar{h}\right\rangle = h \left|h, \bar{h}\right\rangle, \qquad \bar{L}_0 \left|h, \bar{h}\right\rangle = \bar{h} \left|h, \bar{h}\right\rangle. \tag{2.18}$$

Using the equality (2.14) for a scalar field we get

$$(L_0 + \bar{L}_0) \left|h, \bar{h}\right\rangle = 2\Delta \left|h, \bar{h}\right\rangle \tag{2.19}$$

instead of just Δ ofr the usual conventions. Recall that L_n are the modes of the Hamiltonian on the plane.

Note that the $T(z)\left|0\right\rangle$ is well behaved only if

$$L_n \left|0\right\rangle = 0, \qquad \forall n \geq -1. \tag{2.20}$$

2.3 Operator product expansion

The stress–energy tensor is a quasi-primary operator, and we can read the central charge from the order 4 term

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{2.21}$$

2.4 Free scalar field

The free scalar field is defined by the action

$$S = \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, (\partial\phi)^2. \tag{2.22}$$

Its stress–energy tensor is

$$T_{\mu\nu} = -\left(\partial_\mu \phi \partial_\nu - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2\right). \tag{2.23}$$

Going to complex coordinates gives

$$T = -(\partial\phi)^2. \tag{2.24}$$

The propagator reads

$$\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\frac{1}{2}\Big(\ln(z-w) + \ln(\bar{z}-\bar{w})\Big). \tag{2.25}$$

We can also take the derivatives with respect to z and w to get

$$\langle \partial_z\phi(z,\bar{z})\partial_w\phi(w,\bar{w})\rangle = -\frac{1}{2}\frac{1}{(z-w)^2} \tag{2.26}$$

[1] The usual definition is [22]

$$h = \frac{1}{2}(\Delta + s), \qquad \bar{h} = \frac{1}{2}(\Delta - s). \tag{2.16}$$

Our new definition amounts to replacing Δ by 2Δ. It means that our scaling dimension is twice the weight of a scalar field [40, p. 12].

and there is no more antiholomorphic dependence. For this reason we now study the holomorphic field $\partial\phi$. It has the OPE with itself

$$\partial\phi(z)\partial\phi(w) \sim -\frac{1}{2}\frac{1}{(z-w)^2}. \tag{2.27}$$

The OPE of T and $\partial\phi$ reads

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{z-w} \tag{2.28}$$

showing that $\partial\phi$ is primary with $h = 1$.

The stress–energy tensor has the OPE with itself

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{2.29}$$

We read the central charge $c = 1$.

2.4.1 Vertex operators

We define vertex operators $V_p = \mathrm{e}^{ipX}$ for the scalar field X. These are primary operators.

2.5 Rational and non-rational CFT

In a rational CFT there is a finite number of primary fields. They often come in families, the most important example of such being the minimal models.

On the other hand non-rational theories have an infinite number of primary fields and they are harder to study. Theories which have a continuous spectrum are non-rational. A well-known example is Liouville theory, which is the main topic of this review.

Finally one defines also quasi-rational theories for which the OPE of two fields involve only a finite number of fields. This is typically the case for theories which possess some conserved current that restrict the quantum numbers. The free boson is such a theory.

Finding non-rational interacting theories can be a difficult task, and understanding their properties can be even harder. For this reason it is interesting to build such theories as the limit of a family of rational models. The limit may be taken in different manners by using the freedom to rescale the fields, and in particular different limits give different theories. Moreover several of these theories received an interpretation from a continuous orbifold.

This technique has been applied to several cases (for a general introduction see the thesis [70]):

- $c = 1$: limit from minimal models [73, 74] and from Liouville theory with $b \to i$ [76], from orbifold [31].
- $c = 1 - 6\frac{(p-p')}{pp'}$: limit from Liouville theory with $b \to i\sqrt{p'/p}$ [57, 76].
- $c = 3/2$: limit from $N = 1$ minimal models and as the limit $b \to i$ of $N = 1$ Liouville theory [28].
- $c = 3$: limit from $N = 2$ minimal models [26], from orbifold [24].
- $c = n - 1$: limit from W_n minimal models and $\mathfrak{sl}(n)$ Toda [23].
- $c = 3n$: limit from $N = (2, 2)$ minimal models W_n (Kazama–Suzuki models) [25].

2.6 Minimal models

The structure constants of the (supersymmetric) minimal models can be written in terms of the Υ function [28, sec. A].

3 Correlation functions in CFT

3.1 Conformal blocks

Conformal blocks appear in the decomposition of the 4-point functions. These are universal quantities determined only by representation theory.

Analytic expressions are known only for some particular values of the parameters. Starting with some assumptions about the analyticity of the conformal blocks, Zamolodchikov has derived reccurence formulas that are very efficient in the context of numerical computations (see also [72]). These formulas can be given in terms of position or elliptic variables, the latter being more efficient. The derivation relies on two data (see also [39]):

- the analytic formula for $c = 1$;
- the first two terms of the (quantum) conformal block $1/\Delta$-expansion can be found from the $1/\delta$-expansion of the classical conformal blocks.[2]

An expression for the conformal blocks can be obtained from Nekrasov partition functions through AGT conjecture [1].

Part II
Liouville theory

4 Two-dimensional gravity

In this section we will review some general aspects of (classical) $2d$ gravity coupled to some matter. We consider a 2-dimensional (euclidean) space $\mathcal{M}$ with metric $g_{\mu\nu}$ and whose coordinates are denoted by σ^μ.

The total action of the theory is written

$$S[g, \psi] = S_{\text{grav}}[g] + S_m[g, \psi] \tag{4.1}$$

where S_{grav} is the action for pure gravity – constructed in section 4.1 –, and S_m is some matter action (not necessarily conformally invariant) for a set of fields denoted collectively as ψ. The action $S_m[g, \psi]$ comes from making the action $S_m[\delta, \psi]$ covariant by minimal coupling ($\delta_{\mu\nu}$ is just the flat metric).

We ask for the action of our theory to:

- be renormalizable;
- be invariant under diffeomorphisms;
- have at most second order derivatives.

General references are [16, 34, 68, 86].

[2] The latter can be obtained from semi-classical computations in Liouville theory.

4.1 Pure gravity

4.1.1 Action and symmetries

We wish to construct the most general action for pure gravity.

Under diffeomorphisms, the coordinates and the metric transform as [3]

$$\sigma'^{\mu} = f^{\mu}(\sigma^{\nu}), \qquad g_{\mu\nu} = \frac{\partial \sigma^{\mu}}{\partial \sigma'^{\rho}} \frac{\partial \sigma^{\nu}}{\partial \sigma'^{\sigma}} \, g'_{\rho\sigma}. \tag{4.2}$$

We will also need to consider (local) Weyl transformations which act only on the metric

$$g_{\mu\nu} = \mathrm{e}^{2\omega(\sigma^{\mu})} g'_{\mu\nu}, \tag{4.3}$$

but we will not require it to be a symmetry of the full action.

Dynamics of gravity is described by the Einstein–Hilbert action, but it is a topological invariant in $2d$

$$S_{\mathrm{EH}}[g] = \int \mathrm{d}^2\sigma \, \sqrt{g}\, R_g = 4\pi\chi, \qquad \chi = 2(1-g), \tag{4.4}$$

g being the genus of the surface. So it can be ignored as soon as we are not interested in topologies (which will be mostly our case). In addition to diffeomorphism this action is invariant under Weyl symmetry (4.3).

Finally diffeomorphisms allow a last piece which is the cosmological constant term

$$S_{\mu}[g] = \mu \int \mathrm{d}^2\sigma \, \sqrt{g} = \mu A \tag{4.5}$$

where A is the area of $\mathcal{M}$. In this case Weyl invariance (4.2) is explicitly broken.

The total action is given by the sum of (4.4) and (4.5)

$$S_{\mathrm{grav}} = S_{\mathrm{EH}} + S_{\mu}. \tag{4.6}$$

4.1.2 Equations of motion

Pure gravity in two dimensions is quite boring: it is purely topological and the equation of motion is trivial.

First let's consider the properties of the Riemann tensor $R_{\mu\nu\rho\sigma}$. In d dimensions it has

$$C_d = \frac{1}{12}\, d^2(d^2-1) \tag{4.7}$$

independent components.

For a maximally symmetric spacetime, the Riemann tensor has only one independent component corresponding to the curvature scalar [7], thus

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}\, (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{4.8}$$

Contracting with $g^{\mu\rho}$ gives the Ricci tensor

$$R_{\mu\nu} = \frac{R}{d}\, g_{\mu\nu}. \tag{4.9}$$

The same holds in two dimensions for $R \neq \mathrm{cst}$ because $C_2 = 1$, implying that there is only one independent component in the Riemann tensor, and its symmetries dictate its form.

[3] For most of the symmetries we will give the old field in term of the new one.

As a consequence proving $R = \text{cst}$ is sufficient for proving that the space it maximally symmetric in two dimensions [7, p. 141].

The Ricci tensor is given by

$$R_{\mu\nu} = \frac{R}{2}\, g_{\mu\nu}. \tag{4.10}$$

Because of this last expression the Einstein tensor vanishes identically

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 0. \tag{4.11}$$

Then the equation of motion for $g_{\mu\nu}$ reduces to

$$\mu = 0. \tag{4.12}$$

This equation has no solution [4] except for the specific case $\mu = 0$. This implies that $2d$ pure gravity exists only for vanishing cosmological constant and thus it always possesses Weyl invariance.

4.2 Partition functions

We continue with a generic matter action $S_m[g, \psi]$. The partition function for $2d$ gravity is then [29, sec. 1, 41, p. 671]

$$Z = \frac{1}{\Omega} \int \mathrm{d}g_{\mu\nu}\, \mathrm{d}_g\psi \;\; \mathrm{e}^{-S[g,\psi]} \tag{4.13a}$$

$$= \frac{1}{\Omega} \int \mathrm{d}g_{\mu\nu} \;\; \mathrm{e}^{-S_\mu[g]}\; Z_m[g], \tag{4.13b}$$

where Ω is the volume of the diffeomorphism group [5] and we have indicated that the ψ measure depends on $g_{\mu\nu}$ (see section 4.5). The total action is given by the sum of S_m and S_μ (4.5)

$$S = S_m + S_\mu, \tag{4.14}$$

and the matter partition function is

$$Z_m[g] = \int \mathrm{d}_g\psi \;\; \mathrm{e}^{-S_m[g,\psi]}. \tag{4.15}$$

In (4.13) the integral $\mathrm{d}g$ is over all metrics: because of the diffeomorphism symmetry we are integrating over a huge number of identical metrics, and the integral will be infinite (as it is usual when we try to integrate in the presence of a gauge symmetry) [29, sec. 2]. We have divided the integral by Ω to indicate that we remove this factor; we will see later how explicitly it cancels the redundant integral. In practice this overall factor will cancel when computing correlation functions. We can remove it by restricting the integral over a gauge slice, instead of integrating over the whole space: this amounts to fix a gauge.

We note that μ should be positive because it contributes as a factor $\mathrm{e}^{-\mu A}$ in the path integral, which would diverge for $\mu < 0$.

[4] Recall that equations of motion are equations for the fields, not for the parameters which are fixed by the definition of the model.

[5] Note that we can not include the Weyl group volume, writing $\Omega = \text{Weyl} \ltimes \text{diffeomorphisms}$ since we can not expect to extract it outside the critical dimension [41, p. 671].

4.3 Equations of motion and stress–energy tensor

We define the (tree order) stress–energy tensor of S and S_m by

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}, \qquad T^{(m)}_{\mu\nu} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{4.16}$$

For the action (4.1), we get

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + 2\pi\mu\, g_{\mu\nu}. \tag{4.17}$$

The classical equation of motions are given by variation of the full action (4.14) with respect to ψ and $g_{\mu\nu}$

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \qquad \frac{\delta S}{\delta \psi} = 0. \tag{4.18}$$

Without specifying the action for the matter we can not go further with the second equation. Nonetheless we can already say a lot about the equation of motion for $g_{\mu\nu}$ which implies that the total stress–energy tensor (4.16) vanishes

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + 2\pi\mu\, g_{\mu\nu} = 0. \tag{4.19}$$

This is three independent equations which appear to be constraints on the metric and matter fields (since it does not contain derivatives of the metric): see section **??** for details on the counting of degrees of freedom.

The trace of (4.19) gives

$$T^{(m)} = -4\pi\mu. \tag{4.20}$$

Then we can rewrite the equation as

$$T^{(m)}_{\mu\nu} - \frac{1}{2}\, T^{(m)}\, g_{\mu\nu} = 0 \tag{4.21}$$

where the left-hand side is just the traceless part of the matter stress–energy tensor.

These last two equations have an important meaning: (4.21) is the stress energy tensor corresponding to the action S_m where all the parameters breaking Weyl invariance have been removed, and this allows us to conclude that it is traceless and thus Weyl invariant. On the other hand, (4.20) will provide

- one equation for ϕ if it is not Weyl invariant;
- a third constraint on the matter fields.

Since (4.20) comes from a trace I think that it will always be Weyl invariant. Then even if this symmetry is not a symmetry of the action, it is a symmetry of the equation of motion for $g_{\mu\nu}$ (similarly to the electric–magnetic duality).

The equation of motion for the matter are not Weyl invariant and one of them will allow us to fix the last metric component.

4.3.1 Conformal matter

If $S_m[\delta, \psi]$ is conformally invariant, then the action $S_m[g, \psi]$ is Weyl invariant [83, chap. 4]

$$S_m[g, \psi] = S_m[\, \mathrm{e}^{2\omega} g, \psi]. \tag{4.22}$$

In this case its stress–energy tensor (4.16) is traceless

$$T^{(m)} = 0. \tag{4.23}$$

Then the trace of $T_{\mu\nu}$ reduces to the cosmological constant, confirming that it is the only term which breaks Weyl symmetry

$$T = 4\pi\mu. \tag{4.24}$$

At the classical level, and similarly to pure gravity, we can not include the cosmological constant because the equation (4.20) implies [3, ex. 2.8]

$$T = 0 \Longrightarrow \mu = 0 \tag{4.25}$$

which has no solution.

The equation of motion reduces to

$$T^{(m)}_{\mu\nu} = 0 \tag{4.26}$$

which consists in two independent equations. The last equation (4.20) being trivially satisfied we have only two constraints. This is quite surprising: in general adding a new gauge symmetry adds a constraint and reduce the number of independent components, whereas here we observe the opposite: adding Weyl invariance to the action removes the trace equation, which was a constraint, so we go from three to two constraints on the matter fields.

4.4 Conformal gauge

4.4.1 Gauge fixing

As we saw when we defined the partition function (4.13), we need to fix a gauge for the diffeomorphism. We will choose the conformal gauge

$$g_{\mu\nu} \longrightarrow g'_{\mu\nu} = \mathrm{e}^{2\phi} h_{\mu\nu} \tag{4.27}$$

where $h_{\mu\nu}$ is fixed (in the other sections we will omit the prime on $g_{\mu\nu}$, but it is crucial to keep track of it here); ϕ is the only remaining degree of freedom [6]. $h_{\mu\nu}$ is sometimes called the fiducial (or non-physical) metric. It can be proved that such a reparametrization always exists, and that it is not possible to make any further reparametrization which preserve this gauge choice [29, sec. 2]. This means that the gauge slice $[h]$ made of all metrics conformal to $h_{\mu\nu}$ is a good one – we will note these conformal classes as

$$[h] = \left\{ g_{\mu\nu} \mid g_{\mu\nu} = \mathrm{e}^{2\phi} h_{\mu\nu} \right\}. \tag{4.28}$$

Even if all these conformal classes are locally equivalent under reparametrization, they can differ by (global) topology, and the global gauge slice is not well-defined [29, sec. 2]. These classes will depend on a finite number of parameters, called moduli (or Teichmüller parameters), denoted collectively as τ. We will mainly ignore them.

We have the following relations between quantities for g and h metrics

$$R_g = (R_h - 2\Delta_h \phi)\, \mathrm{e}^{-2\phi}, \qquad \sqrt{g} = \mathrm{e}^{2\phi} \sqrt{h}. \tag{4.29}$$

When doing this transformation the g measure in (4.13) transforms [56, 67, sec. 2]

$$\mathrm{d}g_{\mu\nu} = \mathrm{d}\tau\, \mathrm{d}_g\phi\, \Delta_{\mathrm{FP}}[g] \tag{4.30}$$

where $\Delta_{\mathrm{FP}}[g]$ is the Jacobian of the transformation (4.27), also called the Faddeev–Popov determinant. $\mathrm{d}\tau$ is the integration over the Teichmüller space.

[6] In $2d$ metric has 3 dof, and 2 are removed by diffeomorphism.

The integral on the right hand side will be on the different conformal classes. $\Delta_{\text{FP}}[g]$ takes into account the variable volume of the diffeomorphism group orbits [29, sec. 2]. The determinant can either be evaluated directly, or it can be represented using ghosts.

Although we should write $\mathrm{d}_{e^{2\phi}h}X$ for the measure on X when the gauge if fixed, we will continue to use $\mathrm{d}_g X$ as a shortcut.

The partition function (4.13) becomes [41, p. 671]

$$\boxed{Z = \int \mathrm{d}_g\phi\ \mathrm{e}^{-S_\mu[g]}\ Z_m[g]\ \Delta_{\text{FP}}[g].} \tag{4.31}$$

We need to express all the objects in terms of $h_{\mu\nu}$ (and possibly ϕ) instead of $g_{\mu\nu}$: this would be trivial if they were invariant under Weyl symmetry, but this is not the case due to the conformal anomaly. We will discuss this point in section 5.

In the previous discussion we have ignored non-trivial Teichmüller parameters, but they may add new terms in the action [32, chap. 1].

4.4.2 Ghost action

As it is common we can use anticommuting fields, called ghosts, to represent determinants. Let's apply this to the Faddeev–Popov determinant $\Delta_{\text{FP}}[g]$ [37, p. 122–124, 68, p. 86–89, 86, p. 87–89] (very sketchy): introduce the notation g'^{ζ} for a transformation of the metric associated to vector ξ^μ, then

$$\delta g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)} \Longrightarrow \frac{\delta g_{\mu\nu}(\sigma)}{\delta\xi^\rho(\sigma')} = \delta_{(\mu}{}^{\rho}\nabla_{\nu)}\delta(\sigma-\sigma'). \tag{4.32}$$

Plugging the relation

$$1 = \Delta_{\text{FP}}(g)\int \mathrm{d}\zeta\ \delta(g - g'^{\zeta}), \tag{4.33}$$

where $\Delta_{\text{FP}}(g)$ is the Faddeev–Popov determinant

$$\Delta_{\text{FP}}(g) = \det\left(\frac{\delta g_{\mu\nu}}{\delta\zeta}\right), \tag{4.34}$$

into the partition function (4.13) one has

$$\begin{aligned} Z[g'_{\mu\nu}] &= \frac{1}{\Omega}\int \mathrm{d}\zeta\, \mathrm{d}_g\psi \mathrm{d}g_{\mu\nu}\ \mathrm{e}^{-S[g,\psi]}\delta(g-g'^{\zeta})\Delta_{\text{FP}}(g) \\ &= \frac{1}{\Omega}\int \mathrm{d}\zeta\, \mathrm{d}_{g'^{\zeta}}\psi\ \mathrm{e}^{-S[g'^{\zeta},\psi]}\Delta_{\text{FP}}(g'^{\zeta}) \end{aligned}$$

where we have integrated over $g_{\mu\nu}$. The determinant can be represented with ghost fields [86, sec. 14]

$$\Delta_{\text{FP}} = \int \mathrm{d}b\, \mathrm{d}c\ \mathrm{e}^{-S_g} \tag{4.35}$$

where the ghost action is

$$S_g = \frac{1}{2\pi}\int \mathrm{d}^2\sigma\ \sqrt{g'}\ b_{\mu\nu}\nabla'^{\mu}c^{\nu}. \tag{4.36}$$

These ghost field are anticommuting and $b_{\mu\nu} = b_{\nu\mu}$.

Nothing depend anymore on ζ since every object is gauge invariant, and the integration gives Ω, cancelling the factor in front of the integral. We are thus left with

$$Z[g'_{\mu\nu}] = \mathrm{e}^{-S_\mu[g']}\int \mathrm{d}_{g'}\psi\, \mathrm{d}_{g'}b\, \mathrm{d}_{g'}c\ \mathrm{e}^{-S_m[g',\psi]-S_g[b,c,g']}.$$

where the only degree of freedom in $g'_{\mu\nu}$ is ϕ. Now we will always write $g'_{\mu\nu} = g_{\mu\nu} = \mathrm{e}^{2\phi} h_{\mu\nu}$:

$$Z[g_{\mu\nu}] = \mathrm{e}^{-S_\mu[g]} \int \mathrm{d}_g\psi\, \mathrm{d}_g b\, \mathrm{d}_g c\ \mathrm{e}^{-S_m[g,\psi]-S_g[b,c,g]}. \tag{4.37}$$

Remark 4.1 *This section needs several improvements. For example Liouville measure is missing in the last formula.*

4.4.3 Emerging Weyl symmetry

If we count naively the number of field components in this gauge, it seems that we have introduced one more: ϕ is one component, and $h_{\mu\nu}$ has three. Diffeomorphisms allow us to fix two of the last three components, but there is still one off-shell degree of freedom. The explanation is that we have adopted a redundant description of our system, and this last component is not physical. As usual redundant components come with gauge symmetries which allow to remove them.

What is the symmetry here? We note that the physical metric

$$g_{\mu\nu} = \mathrm{e}^{2\phi} h_{\mu\nu} \tag{4.38}$$

is left invariant under the transformation

$$h_{\mu\nu} = \mathrm{e}^{2\omega} h'_{\mu\nu}, \qquad \phi = \phi' - \omega. \tag{4.39}$$

This emerging Weyl symmetry ensures that the counting of off-shell degrees of freedom is still valid, since it allows us to remove the last component of $h_{\mu\nu}$. This linear shift of ϕ under a Weyl transformation also shows that ϕ can be interpreted as a Goldstone boson for the broken Weyl invariance (by the choice of h) [34, p. 23].

Note that this emerging Weyl symmetry is not fundamental since it is very specific to the conformal gauge, at the opposite of the Weyl symmetry (4.3): this last symmetry (when it exists) can be used in any gauge and truly reduce the total number of off-shell degrees of freedom, whereas the emerging Weyl is here only not to spoil the counting due to the redundant notation [56]. To stress the distinction, the original Weyl symmetry acts in the conformal gauge as

$$h_{\mu\nu} = h'_{\mu\nu}, \qquad \phi = \phi' + \omega. \tag{4.40}$$

Another reason is that the decompositions

$$g_{\mu\nu} = \mathrm{e}^{2\phi} h_{\mu\nu} = \mathrm{e}^{2\phi'} h'_{\mu\nu} \tag{4.41}$$

are both valid and should lead to the same action [41, p. 671].

Remark 4.2 (Comparison with a vector field) *A similar description can be made for a vector field $\mathcal{A}_\mu$. In general its action will be*

$$S = \int \mathrm{d}^d x \left(\frac{1}{4} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{m^2}{2} \mathcal{A}_\mu \mathcal{A}^\mu \right). \tag{4.42}$$

If $m^2 = 0$ then it enjoys a $\mathrm{U}(1)$ symmetry

$$\mathcal{A}'_\mu = \mathcal{A}_\mu + \partial_\mu \alpha, \tag{4.43}$$

which reduces the d components to $d-1$ off-shell dofs, and furthermore to $d-2$ on-shell dofs. We can decide to separate the spin 0 component (removed by the gauge symmetry if it is presents) from the spin 1 by writing

$$\mathcal{A}_\mu = A_\mu + \partial_\mu a, \tag{4.44}$$

where A_μ and a play respectively the roles of $h_{\mu\nu}$ and ϕ (from the point of view of Lorentz representations, the trace of $g_{\mu\nu}$ is similar to the divergence of $\mathcal{A}_\mu$). The main difference with the 2d gravity is that this is not really a gauge since we have a priori no other symmetries to fix the components of A_μ. If the original system has a $\mathrm{U}(1)$ *symmetry, then it acts as*

$$A'\mu = A_\mu, \qquad a' = a + \alpha \tag{4.45}$$

We could fear that we enlarged our system to $d+1$ components, but we see that we have an emerging $\mathrm{U}(1)$ *symmetry*

$$A'_\mu = A_\mu + \partial_\mu \alpha, \qquad a' = a - \alpha. \tag{4.46}$$

This last symmetry can be used to reduce the number of components of A_μ to $d-1$ (we keep a since a field $\mathcal{A}_\mu$ describes a priori a mixing of spin 0 and 1, so here A_μ can be seen as a pure spin 1).

The field a is the action and can have dynamics in the presence of chiral anomaly, exactly in the same way as the field ϕ: this will be the topic of the later section 5.

4.5 Measures

4.5.1 General properties

We want to construct the measure associated to the different fields Φ appearing in our action [62]: in order to define them we first define the variation $\delta\Phi$ on tangent field space and an inner product

$$(\delta\Phi_1, \delta\Phi_2)_g = \int \mathrm{d}^2\sigma \, \sqrt{g}\, \gamma(\delta\Phi_1, \delta\Phi_2), \tag{4.47}$$

where γ is a metric on the $\delta\Phi$ space (see below to clarify). This product defines a norm

$$|\delta\Phi|_g^2 = (\delta\Phi, \delta\Phi)_g = \int \mathrm{d}^2\sigma \, \sqrt{g}\, \gamma(\delta\Phi, \delta\Phi), \tag{4.48}$$

Then we define the functional measure implicitly through a Gaussian integral such that

$$\int \mathrm{d}_g\delta\Phi \; \mathrm{e}^{-\frac{1}{2}|\delta\Phi|_g^2} = 1. \tag{4.49}$$

The inner products have to respect reparametrization invariance and locality [56]. As usual we desire that the measures be invariant under field translation by some function $\varepsilon(\sigma)$ (but this does not implies that the action be invariant).

All these norms will depend on the metric in a non-trivial way (at least in the volume element), so we may face some difficulties when the metric over which we integrate.

We want to show that Gaussian measures are invariant under field translation

$$\Phi(\sigma) \longrightarrow \Phi'(\sigma) = \Phi(\sigma) + \varepsilon(\sigma). \tag{4.50}$$

This property is very useful because it allows to complete squares and shift integration variables (for example to generate a perturbative expansion and to derive the propagator).

In fact you can see this as follow: shift Φ by ε in the integrand, and then change variables to $\Phi' = \Phi + \varepsilon$

$$\int \mathrm{d}_g\Phi \; \mathrm{e}^{-\frac{1}{2}|\Phi+\varepsilon|_g^2} = \int \mathrm{d}_g\Phi' \, \det \frac{\delta\Phi}{\delta\Phi'} \; \mathrm{e}^{-\frac{1}{2}|\Phi'|_g^2} = \int \mathrm{d}_g\Phi' \; \mathrm{e}^{-\frac{1}{2}|\Phi'|_g^2} = 1 \tag{4.51}$$

The determinant is the Jacobian of the transformation and its value is simply 1, while the last equality comes from the relabelling $\Phi' \to \Phi$.

Below we give the measure for the fields of spin $0, 1, 2$ [16, 59].

4.5.2 Scalar and vector field measures

The measures for the scalar and vector fields which respect locality, reparametrization and field-translation invariance are unique [59]:

- scalar field X:
$$|\delta X|_g^2 = \int \mathrm{d}^2\sigma\, \sqrt{g}\, \delta X^2, \tag{4.52}$$

- vector field V^μ:
$$|\delta V^\mu|_g^2 = \int \mathrm{d}^2\sigma\, \sqrt{g}\, g_{\mu\nu}\, \delta V^\mu \delta V^\nu. \tag{4.53}$$

4.5.3 Gravitational metric measure

In the case of the metric many choices come to us for the measure. The most general form we have is
$$|\delta g_{\mu\nu}|^2 = \int \mathrm{d}^2\sigma\, \sqrt{g}\, G^{\mu\nu\rho\sigma}\, \delta g_{\mu\nu} \delta g_{\rho\sigma} \tag{4.54}$$
where $G^{\mu\nu\rho\sigma}$ is a metric on the symmetric rank 2 tensor tangent space.

Ultralocality [7] restricts its expression to be [86, p. 87]
$$G^{\mu\nu\rho\sigma} = g^{\mu\rho} g^{\nu\sigma} + c\, g^{\mu\nu} g^{\rho\sigma}, \tag{4.55}$$
where c is some constant.

See [56] for some comments.

We want to show that the trace and the traceless part of the metric are orthogonal with this inner product. Define the trace of $\delta g_{\mu\nu}$ as
$$g^{\mu\nu} \delta g_{\mu\nu} = 2\delta\tau, \tag{4.56}$$
and write $\delta g^\perp_{\mu\nu}$ as the traceless part
$$\delta g_{\mu\nu} = g_{\mu\nu}\, \delta\tau + \delta g^\perp_{\mu\nu}. \tag{4.57}$$
Now compute the product
$$\begin{aligned}
G^{\mu\nu\rho\sigma}\, \delta g_{\mu\nu} \delta g_{\rho\sigma} &= \big(g^{\mu\rho} g^{\nu\sigma} + c\, g^{\mu\nu} g^{\rho\sigma}\big)\big(g_{\mu\nu}\, \delta\tau + \delta g^\perp_{\mu\nu}\big)\big(g_{\rho\sigma}\, \delta\tau + \delta g^\perp_{\rho\sigma}\big) \\
&= \big((1+2c) g^{\rho\sigma} \delta\tau + g^{\mu\rho} g^{\nu\sigma} \delta g^\perp_{\mu\nu}\big)\big(g_{\rho\sigma}\, \delta\tau + \delta g^\perp_{\rho\sigma}\big) \\
&= 2(1+2c)(\delta\tau)^2 + g^{\mu\nu} \delta\tau \delta g^\perp_{\mu\nu} + (1+2c) g^{\rho\sigma} \delta\tau \delta g^\perp_{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} \delta g^\perp_{\mu\nu} \delta g^\perp_{\rho\sigma}
\end{aligned}$$
and after simplification
$$G^{\mu\nu\rho\sigma}\, \delta g_{\mu\nu} \delta g_{\rho\sigma} = 2(1+2c) \delta\tau^2 + g^{\mu\rho} g^{\nu\sigma} \delta g^\perp_{\mu\nu} \delta g^\perp_{\rho\sigma}, \tag{4.58}$$
or in term of the original inner product:
$$|\delta g_{\mu\nu}|^2 = 2(1+2c) |\delta\tau|^2 + \left|\delta g^\perp_{\mu\nu}\right|^2, \tag{4.59}$$
where the norm of $\delta\tau$ is the one defined for a scalar field (4.52), while the one for $\delta g^\perp_{\mu\nu}$ is the same as $\delta g_{\mu\nu}$ but with $c = 0$. We see here that $|\delta g_{\mu\nu}|^2$ is positive-definite only if [10]
$$c > -\frac{1}{2}. \tag{4.60}$$

[7] Authors of [5, sec. 2.3] argue that locality is not the most important condition for a theory of quantum gravity, while background independence is a more natural requirement.

We can absorb the coefficient with c in $\delta\tau$ and it will just contribute as an overall factor, so its precise value has no physical meaning and we can choose $c = 0$ for convenience.

Since the two variation are orthogonal, the measure $\mathrm{d}_g g_{\mu\nu}$ factorizes as

$$\mathrm{d}_g g_{\mu\nu} = \mathrm{d}_g \tau \, \mathrm{d}_g g^{\perp}_{\mu\nu}. \tag{4.61}$$

4.6 Computing path integrals

Let's compute the path integral for a scalar field X

$$Z[g] = \int \mathrm{d}_g X \; \mathrm{e}^{-S[g,X]} \tag{4.62}$$

with action

$$S = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \sqrt{g} \, (\partial X)^2 = -\frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \sqrt{g} \, X \Delta X. \tag{4.63}$$

The equation of motion for X is

$$\Delta X = 0. \tag{4.64}$$

The computation can be done by splitting the field into its (constant) zero-mode a fluctuation around it [14]

$$X(\sigma^\mu) = X_0 + \tilde{X}(\sigma^\mu), \qquad X_0 = \mathrm{cst}, \qquad \Delta \tilde{X} = 0. \tag{4.65}$$

The zero-mode and the fluctuation can be taken to be orthogonal with respect to the scalar product defined by (4.52)

$$(X_0, \tilde{X})_g = 0 \Longrightarrow \int \mathrm{d}^2\sigma \, \sqrt{g} \, \tilde{X} = 0. \tag{4.66}$$

The latter fact implies that the measure on the scalar field can be separated

$$\mathrm{d}_g X = \mathrm{d}X_0 \, \mathrm{d}_g \tilde{X} \tag{4.67}$$

where the integration on X_0 is a normal integration (not a path integral). Note that these measures are not canonically normalised. To find the normalisation, the norm of the zero-mode can be computed

$$|X_0|^2 = \int \mathrm{d}^2\sigma \, \sqrt{g} X_0^2 = X_0^2 \, A \tag{4.68}$$

as they are constant.

Since we have normalised the measure $\mathrm{d}_g X$ using (4.49), we can write

$$1 = \int \mathrm{d}_g X \; \mathrm{e}^{-\frac{1}{2}|X|^2} = \int \mathrm{d}X_0 \, \mathrm{d}_g \tilde{X} \; \mathrm{e}^{-\frac{1}{2}|X_0|^2} \, \mathrm{e}^{-\frac{1}{2}|\tilde{X}|^2} = \sqrt{\frac{\pi}{A}} \int \mathrm{d}_g \tilde{X} \; \mathrm{e}^{-\frac{1}{2}|\tilde{X}|^2} \tag{4.69}$$

after performing the Gaussian integral on X_0.

We are now able to compute the above path integral

$$Z[g] = (\det{}' \Delta_g)^{-1/2} \int \mathrm{d}X_0 \int \mathrm{d}_g \tilde{X} \; \mathrm{e}^{-\frac{1}{2}|\tilde{X}|^2} = \Omega \left(\frac{\pi}{A} \, \det{}' \Delta_g \right)^{-1/2} \tag{4.70}$$

where we used the formula (D.2), and $\Omega = \int \mathrm{d}X_0$ is the volume of spacetime and the prime indicates that we omit the zero-mode of the Laplacian.

5 Liouville effective action for quantum gravity with conformal matter

In the previous section we studied the Polyakov action at the classical level: the classical cosmological constant vanishes and the resulting Weyl symmetry allows us to fix also the last metric degree of freedom ϕ, which becomes non dynamical. Things are greatly different at the quantum level: conformal anomaly forces a non-zero cosmological constant and a dynamical Liouville mode.

To be clear: the last section was also quantum because we used path integral, ghosts, and so on, but we did not compute any effective action. These will arise from the 1-loop conformal anomaly.

The aim of this section is to determine the effective action Γ that we get after integrating out the matter and ghost fields [8] in (4.31)

$$Z[g] \equiv Z[h,\phi] = \mathrm{e}^{-\Gamma[g]} = \mathrm{e}^{-S_\mu[g]}\ \mathrm{e}^{-S_{\text{eff}}[g]} = \mathrm{e}^{-S_\mu[g]}\ Z_m[g]\ \Delta_{\text{FP}}[g], \tag{5.1}$$

where g stands as a shortcut for $g = \mathrm{e}^{2\phi}h$. Including the integration over the metric gives

$$Z[h] = \int \mathrm{d}_g\phi\ Z[h,\phi]. \tag{5.2}$$

Our formalism is very general and we can use any matter for $Z_m[g]$. In this section we consider generic fields ψ with action $S_m[\psi, g]$ invariant under conformal symmetry. We denote by c the central charge, and the partition function is [9]

$$Z_m[g] = \mathrm{e}^{-S_{\text{eff}}[g]} = \int \mathrm{d}_g\psi\ \mathrm{e}^{-S_m}, \tag{5.3}$$

$S_{\text{eff}}[g]$ being the effective action. The stress–energy tensor $T^{(m)}_{\mu\nu}$ is traceless due to conformal invariance.

From the effective action S_{eff} we can compute the stress–energy tensor with quantum corrections

$$\left\langle T^{(m)}_{\mu\nu} \right\rangle = -\frac{4\pi}{\sqrt{g}}\frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}} = \frac{4\pi}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\ln Z_m. \tag{5.4}$$

Reversing the argument, we can compute the effective action if we know the quantum stress–energy tensor, and we will follow this approach.

5.1 Conformal anomaly

5.1.1 General expression

At the quantum level, the stress–energy tensor is no more traceless because of the conformal anomaly [48, p. 65, 68, p. 95, 83, p. 86–87]

$$\left\langle T^{(m)} \right\rangle = -\frac{c}{12} R_g. \tag{5.5}$$

For a computation from Feynman diagrams and current algebra see [37, sec. 3.2.2].

Since it is an anomaly this result is 1-loop exact and can receive only non-perturbative corrections.

[8] In fact we will see that this is not exactly correct: the measure $\mathrm{d}_g\phi$ will also contribute to a term in Γ.

[9] The subscript m is for "matter", but all what we say in this section works if the ψ are ghosts.

5.1.2 Adding the cosmological constant term

The anomaly computation is still valid when one includes S_μ since it does not depend on ψ (which are the matter and ghost fields), so we can take it outside the integral $\mathrm{d}\psi$. This is more clearly seen with the total partition function

$$Z[g] = \mathrm{e}^{-\Gamma[g]} = \mathrm{e}^{-S_\mu[g]}\, Z_m[g] = \mathrm{e}^{-S_\mu[g]} \int \mathrm{d}_g\psi\, \mathrm{e}^{-S_m} \tag{5.6}$$

where Z_m is (5.3), and each object contributes independently to the (quantum) stress–energy tensor. Γ is the new effective action, basically

$$\Gamma = S_{\text{eff}} + S_\mu. \tag{5.7}$$

We can compute the effective action by integrating scalar fields without taking S_μ into account. The result is then

$$\langle T \rangle = -\frac{c}{12}\, R_g + 4\pi\mu. \tag{5.8}$$

More specifically this anomalous part is given because the action is not invariant at quantum level, starting with (5.4):

$$\begin{aligned}
\langle T_{\mu\nu} \rangle &= \frac{4\pi}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \ln Z = \frac{4\pi}{\sqrt{g}} \frac{1}{Z} \frac{\delta Z}{\delta g^{\mu\nu}} \\
&= -\frac{4\pi}{\sqrt{g}} \left(\frac{\delta S_\mu}{\delta g^{\mu\nu}} + \int \mathrm{d}_g\psi\, \frac{\delta S_m}{\delta g^{\mu\nu}}\, \mathrm{e}^{-S_m} \right)
\end{aligned}$$

which gives

$$\langle T_{\mu\nu} \rangle = -\frac{4\pi}{\sqrt{g}} \left(\frac{\delta S_\mu}{\delta g^{\mu\nu}} + \left\langle \frac{\delta S_m}{\delta g^{\mu\nu}} \right\rangle \right), \tag{5.9}$$

and Ward identities tell that (quantum) non-invariance of S_m implies a non-zero expectation value.

5.2 Derivation of the classical Liouville action

The derivation can be found in various places [69, 86, sec. 13.3].

5.2.1 Integrating the conformal anomaly

From the change of variable

$$g_{\mu\nu} = h_{\mu\nu}\, \mathrm{e}^{2\phi} \tag{5.10}$$

we get the variations (see also app. B.3)

$$\delta g^{\mu\nu} = -2 g^{\mu\nu} \delta\phi, \tag{5.11}$$

or

$$g^{\mu\nu} \frac{\delta\phi}{\delta g^{\mu\nu}} = -\frac{1}{2},$$

we can obtain the effective action by integrating $\left\langle T^{(m)} \right\rangle$ from $\phi = 0$ to ϕ, starting from

$$\left\langle T^{(m)} \right\rangle = -\frac{4\pi}{\sqrt{g}}\, g^{\mu\nu} \frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}} = -\frac{4\pi}{\sqrt{g}}\, g^{\mu\nu} \frac{\delta\phi}{\delta g^{\mu\nu}} \frac{\delta S_{\text{eff}}}{\delta\phi} = \frac{1}{2} \frac{4\pi}{\sqrt{g}} \frac{\delta S_{\text{eff}}}{\delta\phi}$$

we deduce

$$\boxed{\frac{\delta S_{\text{eff}}}{\delta\phi} = -\frac{\delta}{\delta\phi} \ln Z_m = \frac{1}{2\pi} \sqrt{g} \left\langle T^{(m)} \right\rangle.} \tag{5.12}$$

Replacing $\langle T^{(m)} \rangle$ with (5.5) we get

$$\frac{\delta S_{\text{eff}}}{\delta \phi} = \frac{1}{2\pi}\sqrt{g}\left(-\frac{c}{12}\,R_g\right) = -\frac{c}{24\pi}\sqrt{h}\,\mathrm{e}^{2\phi}\big(R_h - 2\Delta_h\phi\big)\,\mathrm{e}^{-2\phi}$$

from which we get that [10] [10, 68, p. 326, 83, p. 120]

$$S_{\text{eff}}[\,\mathrm{e}^{2\phi}h] - S_{\text{eff}}[h] = -\frac{c}{24\pi}\int \mathrm{d}^2\sigma\ \sqrt{h}(-\phi\Delta_h\phi + R_h\phi) \tag{5.13}$$

or after integrating by part

$$S_{\text{eff}}[\,\mathrm{e}^{2\phi}h] - S_{\text{eff}}[h] = -\frac{c}{24\pi}\int \mathrm{d}^2\sigma\ \sqrt{h}(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + R_h\phi). \tag{5.14}$$

We define the right hand side to be the (bare) Liouville action without cosmological constant

$$\boxed{s_L = -\frac{6}{c}\,\Big(S_{\text{eff}}[\,\mathrm{e}^{2\phi}h] - S_{\text{eff}}[h]\Big) = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\ \sqrt{h}\Big(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + R_h\phi\Big),} \tag{5.15}$$

and ϕ is called the Liouville mode. We see that the Weyl anomaly generates a kinetic term for it so it becomes dynamical. Said another way the conformal anomaly avoid the decoupling of the trace part which corresponds to a spin 0 field.

5.2.2 Partition function and transformation properties

At the end the old effective action is rewritten

$$\boxed{S_{\text{eff}}[\,\mathrm{e}^{2\phi}h] = S_{\text{eff}}[h] - \frac{c}{6}\,s_L.} \tag{5.16}$$

Taking the exponential gives the fundamental relation

$$\boxed{Z_m[\,\mathrm{e}^{2\phi}h] = \mathrm{e}^{\frac{c}{6}s_L}\,Z_m[h]} \tag{5.17}$$

since we recall that

$$Z_m[h] = \mathrm{e}^{-S_{\text{eff}}[h]}. \tag{5.18}$$

This can also be seen by integrating directly (5.15) [83, p. 119, 86, p. 83].

Then using the path integral representation (5.3) for $S_{\text{eff}}[h]$ we can deduce the transformation of the partition function under a Weyl transformation [46]

$$\boxed{\mathrm{d}_{\,\mathrm{e}^{2\phi}h}\psi = \mathrm{e}^{\frac{c}{6}s_L}\mathrm{d}_h\psi.} \tag{5.19}$$

Since the action (5.3) for ψ is invariant at the classical level, the anomaly must come from the measure $\mathrm{d}_g\psi$.

From now on we omit the index h on the curvature, except when it is needed to avoid confusion.

[10] Factor 2 disappears because the primitive of 2ϕ is ϕ^2. Idem the primitive of $2\,\mathrm{e}^{2\phi}$ is $\mathrm{e}^{2\phi}$ (if we integrate the cosmological constant term).

5.3 Application to $2d$ gravity

We now apply the results of the previous section to the partition function of $2d$ gravity (5.1)

$$Z[g] = \mathrm{e}^{-\Gamma[g]} = \mathrm{e}^{-S_\mu[g]}\ \mathrm{e}^{-S_{\text{eff}}[g]} = \mathrm{e}^{-S_\mu[g]}\ Z_m[g]\ \Delta_{\text{FP}}[g]. \tag{5.20}$$

There the ghosts behave exactly as normal matter, such that we can use the same formula for all the derivation. The peculiar point will come from the negative sign which will be really important.

Let's denote by c_m and c_g the central charges for matter and ghosts. The central charge for all the ghosts is -26:

$$c_g = -26. \tag{5.21}$$

We define the central charge

$$c = -c_g - c_m = 26 - c_m. \tag{5.22}$$

With these notations the relation (5.8) reads

$$\langle T \rangle = \frac{c}{12}\, R_g + 4\pi\mu. \tag{5.23}$$

Because of our definition of c the first term in this formula differs by a minus sign with respect to (5.8) (and thus also with the other formula).

Using the results of the previous section for (5.1) with

$$\mathrm{e}^{-S_{\text{eff}}[g]} = Z_m[g]\ \Delta_{\text{FP}}[g] \tag{5.24}$$

and c given above, we find

$$S_{\text{eff}}[\,\mathrm{e}^{2\phi}h] = S_{\text{eff}}[h] + \frac{c}{6}\, s_L, \tag{5.25}$$

or written in term of the partition function

$$Z_m[g]\ \Delta_{\text{FP}}[g] = \mathrm{e}^{-\frac{c}{6} s_L}\ Z_m[h]\ \Delta_{\text{FP}}[h] \tag{5.26}$$

Then the total effective action is [10, sec. 2]

$$\Gamma[\,\mathrm{e}^{2\phi}h] = S_{\text{eff}}[h] + \frac{c}{6}\, s_L + S_\mu \tag{5.27}$$

where

$$S_\mu = \mu \int \mathrm{d}^2\sigma\ \sqrt{h}\ \mathrm{e}^{2\phi}. \tag{5.28}$$

We can redefine the cosmological constant $\mu \to c\mu/6$ [86, p. 121] and get

$$\boxed{\Gamma[\,\mathrm{e}^{2\phi}h] = S_{\text{eff}}[h] + \frac{c}{6}\, S_L} \tag{5.29}$$

where the complete (bare) Liouville action is now [11]

$$\boxed{S_L[h,\phi] = \frac{1}{4\pi} \int \mathrm{d}^2\sigma\ \sqrt{h}\Big(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + R_h\phi + 4\pi\mu\ \mathrm{e}^{2\phi}\Big).} \tag{5.30}$$

[11] If we had integrated (5.8), then the cosmological constant term would have been $4\pi\mu\,(\,\mathrm{e}^{2\phi} - 1)$ [29, sec. 9]; but the additional term is independent of ϕ. The relation would have been

$$\Gamma[\,\mathrm{e}^{2\phi}h] = \Gamma[h] + \frac{c}{6}\, S_L.$$

The path integral (5.2) for $2d$ gravity thus reduces to

$$Z[h] = \int \mathrm{d}_g\phi \ \mathrm{e}^{-\frac{c}{6}S_L} \ Z_m[h] \ \Delta_{\mathrm{FP}}[h]. \tag{5.31}$$

We note that the Liouville measure, coming from (4.31), is still given by the metric $g = \mathrm{e}^{2\phi} h$; we will discuss this problem in sec. 5.5. Moreover the path integral seems to be well defined only if $c \geq 0$, that is $c_m \leq 26$, otherwise the exponential factor would come with a positive sign; we will come back later to this problem.

In the last expression $Z_m[h]$ and $\Delta_{\mathrm{FP}}[h]$ do not depend on ϕ and they can be taken outside the ϕ integration; we get three decoupled sectors: the matter, the ghosts and the Liouville mode [67]

$$\boxed{Z[h] = Z_m[h] \ \Delta_{\mathrm{FP}}[h] \int \mathrm{d}_g\phi \ \mathrm{e}^{-\mathcal{S}_L}} \tag{5.32}$$

where we have defined

$$\mathcal{S}_L = \frac{c}{6}\, S_L. \tag{5.33}$$

5.4 Few properties of the classical Liouville action

As we said this theory is not yet well-defined because the measure over ϕ is $\mathrm{d}_g\phi$: to get the final expression we need to transform this measure to $\mathrm{d}_h\phi$. General arguments indicates that contributions of this transformation can be reabsorbed into the coefficients of the terms in the action, so it is equivalent to a renormalization of these parameters. For this reason it will be interesting to use different names for the parameters in front of the different terms since they may receive different corrections. This explains why we speak of classical Liouville gravity even it it arises as an effective action: to really get the full quantum effective action we need to take into account all contributions. Moreover we will see later that the semi-classical limit is given by the classical Liouville action (sec. 6.8), hence the name is well-chosen.

5.4.1 Central charge

In the previous section we derived the classical Liouville action (5.33). We introduce the parameters

$$\boxed{Q = \frac{1}{b} = \sqrt{\frac{c}{6}}} \tag{5.34}$$

and we rescale the Liouville field and the cosmological constant

$$\phi \longrightarrow b\phi \tag{5.35}$$

in order to get [12]

$$S_L = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \ \sqrt{h} \Big(h^{\mu\nu} \partial_\mu\phi \partial_\nu\phi + QR\phi + 4\pi\mu \, \mathrm{e}^{2b\phi} \Big). \tag{5.36}$$

With these parameters, the physical metric g reads [34, p. 19]

$$g_{\mu\nu} = \mathrm{e}^{2b\phi} h_{\mu\nu}. \tag{5.37}$$

The emerging Weyl symmetry (4.39) is then modified and can be written in two equivalent way

$$h_{\mu\nu} = \mathrm{e}^{2b\omega} h'_{\mu\nu}, \qquad \phi = \phi' - \omega. \tag{5.38a}$$

[12] We write S_L for $\mathcal{S}_L$.

or [34, p. 23]

$$h_{\mu\nu} = \mathrm{e}^{2\omega} h'_{\mu\nu}, \qquad \phi = \phi' - \frac{1}{b}\,\omega = \phi' - Q\,\omega. \tag{5.38b}$$

In section 6.4 we will prove that the trace of the stress–energy tensor (6.29) is given by

$$T = -\frac{Q^2}{2}\,R. \tag{5.39}$$

Comparing with (5.5) and writing

$$T = -\frac{c_L}{12}\,R, \tag{5.40}$$

we can thus interpret the Liouville action as a conformal field theory with central charge

$$\boxed{c_L \equiv c = 6Q^2 = 26 - c_m.} \tag{5.41}$$

This is a very striking fact since the action presented here is purely classical but already shows some quantum features [12].

If we interpret the Liouville sector as independent, then the total central charge of the theory vanishes

$$c_{\text{tot}} = c_L + c_m - 26 = 0. \tag{5.42}$$

So even if we replace the Polyakov action by some other kind of matter, the Liouville central charge will take the necessary value to cancel the total central charge. Then in any case the full theory is conformally invariant.

The result presented here can also be found by using canonical quantization and studying the Virasoro algebra [54].

5.4.2 Polyakov action and critical string theory

The matter is made of d scalar fields X^μ. The central charge for one scalar field is 1, and we have d of them, so

$$c_m = d. \tag{5.43}$$

The formula (5.17) gives for $Z_m[g]$ and Δ_{FP} the following transformation

$$Z_m[\,\mathrm{e}^{2\phi}h] = Z_m[h]\,\mathrm{e}^{\frac{c_m}{6}\,S_L}, \qquad \Delta_{\text{FP}}[\,\mathrm{e}^{2\phi}h] = \Delta_{\text{FP}}[h]\,\mathrm{e}^{-\frac{26}{6}\,S_L}. \tag{5.44}$$

Translating this to the measures we have

$$\mathrm{d}_g X\,\mathrm{d}_g b\,\mathrm{d}_g c = \mathrm{e}^{-\frac{(26-c_m)}{6}\,S_L}\,\mathrm{d}_h X\,\mathrm{d}_h b\,\mathrm{d}_h c. \tag{5.45}$$

Both terms cancel if $c_m = 26$ – or $c = 0$ – (which justify the string theory procedure). If $c_m \neq 26$ then quantizing quantum gravity reduces to the quantization of Liouville theory. Another way to reach $c = 0$ is to add conformal matter which will contribute to Weyl anomaly [86].

5.5 Changing the Liouville mode measure

We may think that the measure over the Liouville mode $\mathrm{d}_g\phi$ is the same as any scalar field and that the Jacobian will be the same when changing the variable from g to h, but this would be a mistake [86, sec. 14, 21]. In fact the measure is defined from the norm [16]

$$|\delta\phi|^2 = \int \mathrm{d}^2\sigma\,\sqrt{g}\,\delta\phi^2 = \int \mathrm{d}^2\sigma\,\sqrt{h}\,\mathrm{e}^{2\phi}\delta\phi^2, \tag{5.46}$$

which depends on the field ϕ and not only on its variation. Thus this expression is Weyl invariant, but it is not invariant by translation, and we do not know how to make sense of this measure for the quantum theory. For this reason we want to find a new measure $\mathrm{d}_h\phi$ defined with respect to the norm [13]

$$|\delta\phi|^2 = \int \mathrm{d}^2\sigma \, \sqrt{h} \, \delta\phi^2. \tag{5.47}$$

The change of variables reads [56]

$$\mathrm{d}_g\phi = \mathrm{d}_h\phi \;\operatorname{Det}\mathcal{J} \tag{5.48}$$

where the operator inside the Jacobian $\operatorname{Det}\mathcal{J}$ is

$$\mathcal{J} = \frac{\delta g_{\mu\nu}(\sigma)}{\delta h_{\mu\nu}(\sigma')} = \mathrm{e}^{2\phi(\sigma)}\delta^{(2)}(\sigma - \sigma'). \tag{5.49}$$

Several approaches are possible to find the expression of the Jacobian:

1. make some assumptions on its form and show that it is given by the Liouville action with different parameters, as was done by David and Distler and Kawai (DDK) [13, 16];

2. make an ansatz for the stress–energy tensor, and ask for the closure of Virasoro algebra [78];

3. compute it explicitly as did Mavromatos and Miramontes [10, 11, 56].

Finally note that [20] propose to define rigorously the Liouville measure by parametrizing the Liouville field in terms of the area and a Kähler potential: this allows them to use Bergmann matrices and tools from matrix models.

5.5.1 Ansatz for the jacobian

One can argue that the Jacobian $\operatorname{Det}\mathcal{J}$ should take the same form as Liouville action since there is no other possible term allowed by the symmetries, thus it leads to a renormalization of the coefficient appearing in the Liouville action [13, 16, 47, app. E, 48, p. 148–150, 86, p. 7], that we defined to be

$$S_L = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \sqrt{h} \Big(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + QR\phi + 4\pi\mu \, \mathrm{e}^{2b\phi} \Big) \tag{5.50}$$

and we will fix the relation between them by asking that the physical metric g does not depend on the specific splitting we choose, as discussed when we fixed the gauge (sec. 4.4.3); we will mainly use [34, p. 17–19, 86, sec. 21.2] for the derivation. This has the same form as (5.36) but Q and b are now independent. Remember that we called the Liouville action without cosmological constant s_L.

However the computations is more difficult here: for the classical action the three transformations (5.38) were equivalent (recall that we used these ones to keep the exponential term invariant). But the parameters b and Q receive different corrections; naively we may try to keep the transformation in terms of b since it is the most obvious candidate to keep $\mathrm{e}^{2b\phi}$ invariant. But then we would discover that the kinetic and curvature terms are not invariant… Moreover exponentials of quantum fields can get anomalous dimensions from

[13] In [86, p. 120] it is said that one believes that we can replace a non-linear measure by a linear one by adding local terms to the Lagrangian.

quantum corrections, and it would spoil the transformation with b. Thus we use the last transformation

$$h_{\mu\nu} = \mathrm{e}^{2\omega} h'_{\mu\nu}, \qquad \phi = \phi' - Q\,\omega, \tag{5.51}$$

and it will appear to be correct. Another approach would be to transform h normally and then shift ϕ by some function, which is determined by the invariance of the kinetic plus curvature terms [86, sec. 21.2]. At the end we need to show that the cosmological constant term is invariant independently (see section 5.6).

Transforming the kinetic and curvature terms The inverse metric transforms as

$$h^{\mu\nu} = \mathrm{e}^{-2\omega} h'^{\mu\nu} \tag{5.52}$$

and the kinetic term reads

$$\begin{aligned}\sqrt{h}\, h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi &= \sqrt{h'}\, h'^{\mu\nu}\partial_\mu\left(\phi' - Q\,\omega\right)\partial_\nu\left(\phi' - Q\,\omega\right)\\ &= \sqrt{h'}\, h'^{\mu\nu}\left(\partial_\mu\phi'\partial_\nu\phi' + Q^2\,\partial_\mu\omega\partial_\nu\omega - 2Q\,\partial_\mu\phi'\partial_\nu\omega\right).\end{aligned}$$

We recall the formula

$$R = \mathrm{e}^{-2\omega}(R' - 2\Delta'\omega). \tag{5.53}$$

before computing the transformation of the curvature term:

$$\begin{aligned}\sqrt{h}\, R_h\phi &= \sqrt{h'}\ \mathrm{e}^{2\omega} R\,(\phi' - Q\,\omega) = \sqrt{h'}\,(R' - 2\Delta'\omega)\,(\phi' - Q\,\omega)\\ &= \sqrt{h'}\ (R'\phi' - Q\,R'\omega - 2\phi'\Delta'\omega + 2Q\,\omega\Delta'\omega)\\ &= \sqrt{h'}\,(R'\phi' - Q\,R'\omega) + \sqrt{h'}\, h'^{\mu\nu}(2\,\partial_\mu\phi'\partial_\nu\omega - 2Q\partial_\mu\omega\partial_\nu\omega)\end{aligned}$$

where the last step is an integration by part (valid under integral). Adding these two pieces, the cross term $\partial_\mu\phi'\partial_\nu\omega$ cancels and we get

$$s'_L = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\ \sqrt{h'}\Big(h'^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' + QR'\phi'\Big) - \frac{Q^2}{4\pi}\int \mathrm{d}^2\sigma\ \sqrt{h'}\Big(h'^{\mu\nu}\partial_\mu\omega\partial_\nu\omega + R'\omega\Big), \tag{5.54}$$

that is

$$s'_L[\phi', \omega] = s_L[\phi'] - Q^2\, s_L[\omega]. \tag{5.55}$$

We also need to take into account the transformation of the path integral measures of the other sectors. From the expression (5.26) we get

$$Z_m[h]\ \Delta_{\mathrm{FP}}[h] = \mathrm{e}^{-\frac{c}{6}s_L[\omega]}\ Z_m[h']\ \Delta_{\mathrm{FP}}[h'], \qquad c = 26 - c_m. \tag{5.56}$$

The measure for ϕ is invariant under its translation since we made it this way. Since ω is a Weyl rescaling independent of ϕ, the measure for ϕ transforms as the one for a normal scalar field [41, p. 673]

$$\mathrm{d}_{\mathrm{e}^{2\omega}h}\phi = \mathrm{e}^{\frac{1}{6}s_L}\mathrm{d}_h\phi. \tag{5.57}$$

Adding all the pieces together, we see that the variation is given by

$$\delta S = \left(\frac{c-1}{6} - Q^2\right) s_L, \tag{5.58}$$

and it vanishes if

$$c = 1 + 6Q^2, \tag{5.59}$$

or written differently

$$6Q^2 = 25 - c_m, \qquad Q = \sqrt{\frac{25 - c_m}{6}}. \tag{5.60}$$

Again this means that the total central charge vanishes

$$c_{\text{tot}} = c + c_m - 26 = 0, \tag{5.61}$$

and the quantum Liouville theory (plus the ghosts and matter) thus possesses exact conformal symmetry at the quantum level, with respect to the h metric. We can interpret the 1 in c as the quantum correction (due to one scalar field) to the classical central charge $6Q^2$ [12].

An infinitesimal derivation exists [34, p. 18–19, 67, sec. 3]. There is also a mixed derivation in [41, sec.25.1].

Transforming the cosmological constant term In a conformal theory, exponentials $\mathrm{e}^{2a\phi}$ receive quantum corrections to their dimensions

$$\Delta(\mathrm{e}^{2a\phi}) = a(Q - a). \tag{5.62}$$

Since every term in an action should have conformal dimension $(1, 1)$, the coefficient b from the cosmological constant term should satisfies

$$b(Q - b) = 1 \tag{5.63}$$

or

$$Q = \frac{1}{b} + b. \tag{5.64}$$

Again we recognize a quantum correction to the classical value b^{-1}, and we recover this value in the semi-classical limit $b \to 0$.

5.5.2 Explicit computation

We will compute the determinant of (5.49) using standard techniques [56].

In order to find the ϕ dependence of $\operatorname{Det}\mathcal{J}$ for

$$\mathcal{J} = \frac{\delta g_{\mu\nu}(\sigma)}{\delta h_{\mu\nu}(\sigma')} = \mathrm{e}^{2\phi(\sigma)}\delta^{(2)}(\sigma - \sigma') \tag{5.65}$$

we look at the variation $\sigma \to \sigma + \delta\sigma$ of its logarithm [6]. Then integrating will give the effective action S_{eff}.

Its logarithm gives the effective action, and for a small variation $\delta\phi$ it reads

$$\delta S_{\text{eff}} = \ln \operatorname{Det}\mathcal{J} = \operatorname{Tr}\ln\mathcal{J} = 2\int \mathrm{d}^2\sigma\ \delta\phi(\sigma)\,\delta^{(2)}(0). \tag{5.66}$$

The $\delta^{(2)}(0)$ is infinite and does not make sense, so we need to regularize it: we replace it by $G(\varepsilon; \sigma, \sigma')$ where G is the heat kernel of the Laplacian (C.40). Its small time expansion is

$$G(\varepsilon; \sigma, \sigma') = \sqrt{g}\left(\frac{1}{4\pi\varepsilon} + \frac{1}{12\pi}\,R_g\right) = \sqrt{h}\left(\frac{\mathrm{e}^{2\phi}}{4\pi\varepsilon} + \frac{1}{12\pi}\,(R_h + 2\Delta\phi)\right) \tag{5.67}$$

We can now replace

$$\delta S_{\text{eff}} = 2\int \mathrm{d}^2\sigma\ \sqrt{h}\,\delta\phi\left(\frac{\mathrm{e}^{2\phi}}{4\pi\varepsilon} + \frac{1}{12\pi}\,(R_h + 2\Delta\phi)\right) \tag{5.68}$$

We can integrate and get the effective action (after an integration by part)

$$S_{\text{eff}} = \frac{1}{4\pi\varepsilon} \int \mathrm{d}^2\sigma \, \sqrt{h} \; \mathrm{e}^{2\phi} + \frac{1}{12\pi} \int \mathrm{d}^2\sigma \, \sqrt{h} \, (\partial_\mu \phi \partial^\mu \phi + R_h \phi). \tag{5.69}$$

The cosmological constant piece is divergent and we need to introduce counter-terms

$$S_{\text{ct}} = \frac{\mu_0}{\varepsilon} \int \mathrm{d}^2\sigma \, \sqrt{h} \; \mathrm{e}^{2\phi} + S_{\text{fin}}. \tag{5.70}$$

The first piece removes the divergence, whereas S_{fin} contains all the finite (and arbitrary) counter-terms that are allowed by the (quantum) symmetries; especially it needs to be invariant under the (emerging) Weyl symmetry:

$$S_{\text{fin}}[h, \phi] = S_{\text{fin}}[\,\mathrm{e}^{2\omega} h, \phi - \omega]. \tag{5.71}$$

For the moment we consider $S_{\text{fin}} = 0$ and we will come back on this point later (section 5.6), where we will also discuss the cosmological constant term renormalization. Then the effective action is [14]

$$S_{\text{eff}} - S_{\text{ct}} = \frac{1}{12\pi} \int \mathrm{d}^2\sigma \, \sqrt{h} \, (\partial_\mu \phi \partial^\mu \phi + R_h \phi) = \frac{1}{6} \, s_L. \tag{5.72}$$

By looking at formula (5.25) we see that adding this term amounts to renormalize the central charge from $c = 6Q^2$ to

$$c = 1 + 6Q^2 \tag{5.73}$$

in agreement with the previous section.

The previous computation is not totally rigorous: we obtained the renormalization for the central charge c directly from the kinetic and curvature terms, but the computation of b is still a bit ad hoc. It is possible to improve this computation by using other methods (a better heat kernel, Schwinger–Dyson equations, background field expansion and Ward identities for the Weyl symmetry) [10, 11].

In [10], D'Hoker checks that Green functions computed with both measures and actions agree at all order in a perturbative development in b (the zero-mode is treated exactly). But then additional non-perturbative divergences arise for $c < 0$ and lead to shrink the radius of convergence to zero. He concludes that even if we can find agreement for the expansion, it is not sure that Liouville it renormalizable non-perturbatively.

5.5.3 Summary

The quantum Liouville action

$$S_L = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \sqrt{h} \Big(h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + Q R \phi + 4\pi\mu \, \mathrm{e}^{2b\phi} \Big) \tag{5.74}$$

is defined by the parameters

$$c = 1 + 6Q^2, \qquad Q = \sqrt{\frac{25 - c_m}{6}} = \frac{1}{b} + b. \tag{5.75}$$

We end this section by a comment on the previous derivations. The DDK approach does not have any firm foundation, and the direct computation raises also some questions. The main problem is to assume that one can separate the kinetic and curvature terms from the cosmological constant: the first are used to compute the Jacobian, the second the renormalization of the interaction. But the effect of one on the others has been ignored [10, sec. 5]. It is only recently that a non-perturbative approach has been found [5, 19, 20].

[14] We have a factor 1/2 wrong.

5.6 Interactions with matter

In section 5.5.2 we saw that we can add finite counter-terms to our action: if we have a matter theory with (spinless) fields ψ_i of conformal dimensions Δ_i, then [13, 16] (see also [34, p. 21–22, 47, app. E, 53, sec. 1, 56, 78, sec. 4.3])

$$S_{\text{fin}} = \sum_i U_i \int \mathrm{d}^2\sigma \, \sqrt{h} \, \psi_i \, \mathrm{e}^{2a_i\phi}, \tag{5.76}$$

where U_i are coupling constants. Matter fields are gravitationally dressed by exponentials of the Liouville field to ensure that the whole term has conformal dimension $(1, 1)$. The conformal dimension $h_i = a_i(Q - a_i)$ of the exponential is such that

$$\Delta_i + h_i = \Delta_i + a_i(Q - a_i) = 1. \tag{5.77}$$

Solutions for a_i are

$$a_i = \frac{Q}{2} \pm \frac{1}{2}\sqrt{\Delta_i - 1 + Q^2} = \frac{Q}{2}\left(1 \pm \sqrt{1 + \frac{\Delta_i - 1}{Q^2}}\right), \tag{5.78}$$

and replacing the value of Q gives [15]

$$a_i = \sqrt{\frac{25 - c_m}{24}}\left(1 - \sqrt{1 + \frac{24(\Delta_i - 1)}{25 - c_m}}\right). \tag{5.79}$$

In the semiclassical limit $b \to 0$, or $c_m \to -\infty$, we should have

$$a_i = 1 - \Delta_i \tag{5.80}$$

and this selects the minus sign.

The effect of including these terms is to define the theory outside from the critical point, for matter coupled to Liouville [16].

If we have no matter we still have the identity operator with $\Delta_i = 0$, and it will contribute as

$$S_{\text{fin}} = \Lambda \int \mathrm{d}^2\sigma \, \sqrt{h} \;\, \mathrm{e}^{2b\phi}. \tag{5.81}$$

Since this "renormalized" cosmological constant has been generated as a finite counter-term, its coefficient is arbitrary. The exponent is real if $c_m < 1$ or $c_m > 25$.

In the general case, we also have $c_m < 1$ or $c_m > 25$ if we allow imaginary a_i for $i \neq 0$, otherwise we should restrict to $c_m < 1$.

Finally to end the coupling of the conformal matter to gravity, we need to replace derivative by minimal coupling, and to dress all the operators.

5.7 Non-local effective action

The trace of the equation for the effective action (5.4)

$$-\frac{4\pi}{\sqrt{g}}\, g^{\mu\nu}\frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}} = \frac{4\pi}{\sqrt{g}}\, g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} \ln Z_m = \left\langle T^{(m)} \right\rangle = \frac{c}{12}\, R \tag{5.82}$$

[15] Note that [13, 56] got the formula wrong as they forgot the square root for the first factor.
[16] All these terms are marginal operators.

can be directly integrated without having to choose a gauge [69]

$$S_{\text{eff}} = \frac{c}{6} \int \mathrm{d}^2\sigma_1 \mathrm{d}^2\sigma^2 \, \sqrt{g(\sigma_1)}\sqrt{g(\sigma_2)}\, R_g(\sigma_1) G(\sigma_1, \sigma_2) R_g(\sigma_2) \tag{5.83}$$

where G is the Green function for the Laplacian Δ_g and this action is covariant. The price we pay for not introducing a gauge is that the action is non-local. Note that we do not include the cosmological constant for the moment.

A perturbative analysis is done in [42].

It is said that this action is equivalent to the Liouville one and emerges as the quantum effective action of conformal matter, but this is not true at finite area because fixing the gauge involves the Mabuchi action (see section 14) while the latter never appears for conformal matter [19, sec. 3.2.4].

As we see here there are no emerging Weyl symmetry, explaining why this symmetry is less fundamental.

We may introduce an auxiliary field φ such that the action becomes local

$$S = \frac{c}{6} \int \mathrm{d}^2\sigma \, \sqrt{g} \left(\frac{1}{2}\, g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + Q R_g \varphi \right). \tag{5.84}$$

This action looks like Liouville, except that it is formulated in terms of the physical metric g; in fact this action corresponds to $2d$ dilatonic gravity. Then we can go the conformal gauge and get an action for φ and ϕ. One the field should not be dynamical.

5.8 Comment on Liouville action status

Here we got Liouville gravity as the effective $2d$ gravity: it is the exact perturbatively since we derive it from an anomaly and we know that anomalies are 1-loop exact (but nothing prevents non-perturbative corrections). But we should not forget that the Liouville action is really the gravity action, and the Liouville mode should be treated as matter. In this sense the metric h that appears is not physical.

Nonetheless it is possible to study Liouville action by itself without referring to $2d$ gravity, and then the Liouville mode corresponds to the matter, while h is the physical metric. From this point of view it is also possible to couple the Liouville action to (tree-level) gravity, and this system will be really different from pure $2d$ gravity (the idea is similar to dilaton gravity).

This is just a perspective choice, but one has to be clear about one's position.

5.9 Gravitational anomaly

In the rest of this paper we consider the Liouville action, obtained from the conformal anomaly, as the effective action for $2d$ quantum gravity. Starting from the equations of motion

$$\nabla^\mu T_{\mu\nu} = 0, \qquad T = \frac{c}{12}\, R, \tag{5.85}$$

using the change of variable

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{c}{12}\, R\, g_{\mu\nu} \tag{5.86}$$

one can trade the conformally anomaly for a gravitational anomaly

$$\nabla^\mu \tilde{T}_{\mu\nu} = -\frac{c}{12}\, \partial_\nu R, \qquad \tilde{T}. \tag{5.87}$$

The first formula can be made Weyl covariant, and it is the starting point for another effective action. This path has been studied in [44, 45].

6 Properties of Liouville action

In this section we study the Liouville action for itself, without referring anymore to its origin.
In particular we will use two approximations:

- minisuperspace: consider only the time dependence;
- ultralocal: consider only the constant dependence.

6.1 Definitions

The Euclidean Liouville action is

$$S_L = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \sqrt{h} \Big(h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + QR\phi + 4\pi\mu \, \mathrm{e}^{2b\phi} \Big) \tag{6.1}$$

with parameters

$$c = 1 + 6Q^2, \qquad Q = \sqrt{\frac{25 - c_m}{6}} = \frac{1}{b} + b. \tag{6.2}$$

We note that the Liouville action is very similar to that of the Coulomb gas (sec. C.5) with an imaginary background charge $Q = -iq$. We can express b in terms of Q [16, 34, p. 19]

$$b = \frac{Q}{2} - \frac{1}{2}\sqrt{Q^2 - 4} = \frac{1}{\sqrt{24}} \left(\sqrt{25 - c_m} - \sqrt{1 - c_m} \right), \tag{6.3}$$

and we choose the minus sign to get agreement with the semi-classical limit $b \to 0$ [10, sec. 4].

The potential will be denoted by

$$U(\phi) = \mu \, \mathrm{e}^{2b\phi}. \tag{6.4}$$

The cosmological constant needs to be positive in order to have a well-defined path integral (see section 4.2). We will find that $\mu \geq 0$ is also necessary within the Hamiltonian formalism. Since the path integral is invariant under constant shift of $\phi \to \phi + c$ the value of μ can be changed at will since

$$\mu \, \mathrm{e}^{2b\phi} \longrightarrow (\mu \, \mathrm{e}^{2bc}) \, \mathrm{e}^{2b\phi}. \tag{6.5}$$

If $\mu > 0$ then the new cosmological constant is also positive as the exponential is always positive.

Even if we will rarely use it except to gain some intuition, the original (and physical) metric g is [34, p. 19]

$$g_{\mu\nu} = \mathrm{e}^{2b\phi} h_{\mu\nu}. \tag{6.6}$$

Because of this splitting the Liouville action (6.1) is invariant under Weyl transformations

$$h_{\mu\nu} = \mathrm{e}^{2\omega} h'_{\mu\nu}, \qquad \phi = \phi' - Q\,\omega. \tag{6.7}$$

6.1.1 Coordinates

In order to avoid some IR divergences, we compactify the space direction on a circle in order to obtain a cylinder [48, sec. 4.3]

$$\tau \in \mathbb{R}, \qquad \sigma \in [0, 2\pi[. \tag{6.8}$$

In order to use all the tools of radial quantization we will also study the theory on the plane. For this we can use a Weyl transformation (6.7) such that $h' = \delta$. Then the action becomes

$$S_L = \frac{1}{4\pi} \int \mathrm{d}^2\sigma' \left(\partial_\mu \phi \partial^\mu \phi + QR\phi + 4\pi\mu\, \mathrm{e}^{2b\phi} \right) \tag{6.9}$$

where σ' are coordinates on flat space. We kept the curvature term even if $R = 0$ because it will contribute to the equations of motion.

This map for other quantities is easier to establish in complex coordinates and we delay the discussion to section 6.5.

6.1.2 Values of the central charge

For $c_m \leq 1$ both b and Q are real and this is the domain where the theory is the best understood (we speak about spacelike Liouville theory); on the other hand we loose the spacetime interpretation of strings except for $c_m = 1$ (where we can use Polyakov action with $d = 1$).

For $c_m > 25$ both Q and b are pure imaginary [16]: then it makes sense to do a Wick rotation on Q, b and ϕ in order to obtain real parameters; this theory is called timelike Liouville, we will study it in section 13. In this case the kinetic term changes sign since $c < 1$. Especially for $c_m = 25$ the field $X^0 = -i\phi$ provides the timelike coordinate and we get back critical string theory.

Finally for the interval $c_m \in]1, 25[$, Q is imaginary and b is complex and we don't have much control in the continuum approach.

For $c_m \leq 1$ it might me convenient to represent the matter sector as a Coulomb gas with a charge q (see section C.5) [16, sec. 5]. In [10] the authors comments on the various ranges, and he argues that doing an analytic continuation of the parameters may help to regularize the divergences.

6.2 Partition function at fixed area

It is possible to expand the partition function as

$$Z = \sum_g \int \mathrm{d}A\, Z_g[A] \tag{6.10}$$

where g is the genus and A the area.

We define the partition function at fixed area A by [16, 34, p. 20]

$$Z[A] = \int \mathrm{d}\phi\ \mathrm{e}^{-S_L}\, \delta\left(\int \mathrm{d}^2\sigma\, \sqrt{h}\ \mathrm{e}^{2b\phi} - A \right). \tag{6.11}$$

In [16] the authors explain why this delta function may be ill-defined, and that we are saved by the fact that μ is arbitrary. Then the partition function can be recovered through the Laplace transform

$$Z = \int \mathrm{d}A\ \mathrm{e}^{-\mu A} Z[A]. \tag{6.12}$$

The correlation function at fixed area of an operator $\mathcal{O}$ is defined by

$$\langle \mathcal{O} \rangle_A = Z[A]^{-1} \int \mathrm{d}\phi\, \mathcal{O}\ \mathrm{e}^{-S_L}\, \delta\left(\int \mathrm{d}^2\sigma\, \sqrt{h}\ \mathrm{e}^{2b\phi} - A \right). \tag{6.13}$$

Several scaling laws can be obtained using this partition function. For example the critical string exponent γ_{str} (to be determined in the next section) is defined by

$$Z[A] \sim A^{\gamma_{\text{str}} - 3}. \tag{6.14}$$

If one adds some matter interaction (see section 5.6)

$$S_{\text{int}} = \int \mathrm{d}^2\sigma\, \sqrt{h}\, \psi\, \mathrm{e}^{2a\phi}, \tag{6.15}$$

where ψ is a primary field of dimension Δ_0, then the gravitational scaling dimension Δ is defined by

$$\langle S_{\text{int}} \rangle_A \sim A^{1-\Delta}. \tag{6.16}$$

6.3 Critical exponents

From formula (5.17) we can easily get the way the partition function depends on the size of the system for $\mu = 0$ [13, 86, p. 84]. Take $\phi = \ln L$, then the rescaling

$$g_{\mu\nu} \longrightarrow L^2 g_{\mu\nu} \tag{6.17}$$

gives

$$Z[L^2 g] = \mathrm{e}^{-\frac{c}{24\pi}\ln L \int \mathrm{d}^2\sigma\, \sqrt{g} R}\, Z[g] = \mathrm{e}^{-\frac{c}{6}(1-g)\ln L^2}\, Z[g].$$

We thus find that

$$Z[L^2 g] = A^{\kappa} Z[g], \qquad \kappa = \frac{c}{6}(1-g). \tag{6.18}$$

The critical exponent γ is given by [33, p. 9–10]

$$\kappa = (\gamma - 2)(1-g) - 1 \Longrightarrow \gamma = \frac{1}{12}\Big(d - 1 - \sqrt{(d-1)(d-25)}\Big) = -\frac{1}{m} \tag{6.19}$$

where m is defined by

$$c = 1 - \frac{6}{m(m+1)}. \tag{6.20}$$

Note that $d > 25$ does not make sense. For pure gravity one has

$$d = 0, \quad m = 2, \quad \gamma = -\frac{1}{2}. \tag{6.21}$$

6.4 Equations of motion and classical solutions

6.4.1 Equations

Our first interest is in computing the $h_{\mu\nu}$ and ϕ variations of S_L (6.1)

$$\begin{aligned}
\delta_h S_L = \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h}\, \delta h^{\mu\nu} \bigg[&- \frac{1}{2}\, h_{\mu\nu} \Big(h^{\rho\sigma} \partial_\rho\phi \partial_\sigma\phi + QR\phi + 4\pi\mu\, \mathrm{e}^{2b\phi} \Big) \\
&+ \Big(\partial_\mu\phi \partial_\nu\phi + QR_{\mu\nu}\phi + Q\big(h_{\mu\nu}\Delta\phi - \nabla_\mu\nabla_\nu\phi\big)\Big) \bigg],
\end{aligned} \tag{6.22a}$$

$$\delta_\phi S_L = \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h}\, \delta\phi \Big(-2\Delta\phi + QR + 8\pi\mu b\, \mathrm{e}^{2b\phi} \Big). \tag{6.22b}$$

The equation of motion for ϕ

$$\frac{\delta S_L}{\delta \phi} = 0 \tag{6.23}$$

gives [34, sec. 3.1]

$$QR[h] - 2\Delta\phi = -8\pi\mu b\, \mathrm{e}^{2b\phi}. \tag{6.24}$$

For flat fiducial metric $h = \delta$ it reduces to [63, sec. 2.3]

$$\partial_\mu \partial^\mu \phi = 4\pi\mu b\, \mathrm{e}^{2b\phi}. \tag{6.25}$$

We can now compute the stress–energy tensor associated to the $h_{\mu\nu}$ metric

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{h}} \frac{\delta S}{\delta h^{\mu\nu}} \tag{6.26}$$

which reads

$$T_{\mu\nu} = -\left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\, h_{\mu\nu} h^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi\right) + Q\big(\nabla_\mu \nabla_\nu \phi - h_{\mu\nu} \Delta\phi\big) + 2\pi\mu\, \mathrm{e}^{2b\phi} h_{\mu\nu}. \tag{6.27}$$

The trace of this tensor is

$$T = -Q\Delta\phi + 4\pi\mu\, \mathrm{e}^{2b\phi}. \tag{6.28}$$

Using the equation of motion (6.24) we can rewrite it as

$$T = -\frac{Q^2}{2}\, R. \tag{6.29}$$

Taking into account the anomalous contribution of ϕ, we get the quantum expectation

$$\langle T \rangle = -\frac{c}{12}\, R = -\frac{1}{12} - \frac{Q^2}{2}\, R. \tag{6.30}$$

6.4.2 Classical solutions

6.4.3 Backlünd transformation

Liouville theory can be mapped to a free field by a Backlünd transformation [9, 79, app. A.1].

6.5 Complex coordinates

6.5.1 General computations

It is interesting to use complex coordinates to study Liouville theory [34, sec. 3.1, 3.3, 40, sec. 2], since it brings the discussion closer to usual CFT. If $\mu = 0$, then Liouville theory would just be a CFT; but this situation is more difficult (for example product of exponentials are not given by a free field expansion).

If we have some coordinates z, then we recall that the metric reads

$$\mathrm{d}s^2 = h_{\mu\nu} \mathrm{d}\sigma^\mu \mathrm{d}\sigma^\nu = h_{z\bar z} \mathrm{d}z \mathrm{d}\bar z. \tag{6.31}$$

Under a conformal transformation, the metric changes as

$$h_{w\bar w} = \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|^2 h_{z\bar z}, \tag{6.32}$$

and from this we read the conformal factor

$$\omega = \ln \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|. \tag{6.33}$$

As a consequence the Liouville mode transforms as (6.7)

$$\phi' = \phi - Q \ln \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|. \tag{6.34}$$

Note that this reduces to

$$\phi' = \phi - \frac{1}{b} \ln \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right| \tag{6.35}$$

in the semiclassical limit. Then its derivative follows

$$\partial_z \phi \longrightarrow \frac{\mathrm{d}w}{\mathrm{d}z}\, \partial_w \phi + Q\, \frac{\mathrm{d}}{\mathrm{d}z} \ln \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|. \tag{6.36}$$

The field $\partial\phi$ transforms inhomogeneously, thus it is not a primary field.

We can guess that fields $\mathrm{e}^{2a\phi}$ will be primary fields since they transform as [40, sec. 2.2, 78]

$$\mathrm{e}^{2a\phi(z)} \longrightarrow \mathrm{e}^{2a\phi(w)} = \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|^{2aQ} \mathrm{e}^{2a\phi(z)} \tag{6.37}$$

(we consider only the holomorphic part).

Let's compute the change of T under a transformation $w = w(z)$. We plug (6.36) into the expression (6.49) (omitting absolute values)

$$T'_{zz} = -\frac{1}{2} \left(w'\, \partial_w \phi + Q\, \frac{\mathrm{d}}{\mathrm{d}z} \ln w' \right)^2 + Q\, \frac{\mathrm{d}}{\mathrm{d}z} \left(w'\, \partial_w \phi + Q\, \frac{\mathrm{d}}{\mathrm{d}z} \ln w' \right).$$

and after simplification we get

$$T'_{zz} = (w')^2\, T_{w\bar{w}} + Q^2\, S(w, z) \tag{6.38}$$

with

$$S(w, z) = \frac{\mathrm{d}^2}{\mathrm{d}z^2} \ln w' - \frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}z} \ln w' \right)^2. \tag{6.39}$$

We used the fact that

$$\frac{\mathrm{d}}{\mathrm{d}z} \ln w' = \frac{w''}{w'} \tag{6.40}$$

in order to cancel the term proportional to $Q\partial_w\phi$.

The first term is the usual term that we get from transforming a rank 2 tensor. The other term needs a bit more thought for its interpretation. First we note that

$$S(w, z) = T_{z\bar{z}}(\phi = \ln w'). \tag{6.41}$$

Operating with all the derivatives, we obtain

$$S(w, z) = \frac{w'''}{w'} - \frac{3}{2} \left(\frac{w''}{w'} \right)^2 \tag{6.42}$$

which is the usual expression of the Schwartzian derivative. This formula is valid also for generic coordinates, just that the proof is harder.

This is a great fact that one can obtain this expression by a simple computation, whereas proving this formula just from CFT computation is much longer. It is also very nice to see that we have a classical field theory (in the semiclassical limit) which possesses nevertheless many features of quantum CFT.

6.5.2 From the cylinder to the plane

Let's consider S^2. Two sets of complex coordinates will interest us

$$w = \tau + i\sigma, \qquad z = \mathrm{e}^{w} = \mathrm{e}^{\tau + i\sigma}. \tag{6.43}$$

The first coordinates describe a cylinder, while the second describes the complex plane. For this reason the metric in z-coordinates is flat

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z}. \tag{6.44}$$

We will mostly work in z-coordinates and use the previous rules to translate the result in w-coordinates (and thus in terms of τ and σ).

In this case the curvature term of the action disappears and one needs to impose the boundary condition [40, sec. 2.1]

$$\phi(z, \bar{z}) \xrightarrow[|z|\to\infty]{} -Q \ln |z|^2 + O(1). \tag{6.45}$$

This ensures that the physical metric is smooth on S^2, see (6.34). This is equivalent to concentrating the curvature at infinity through the insertion of an operator there (cf also the discussion with the Coulomb gas).

Putting the theory on the sphere directly may introduce divergences, and for regulating them one needs to set a cutoff for the integration and to introduce boundary terms. Hence the action is equivalent to the large R of

$$S_L = \frac{1}{4\pi} \int_D \mathrm{d}^2\sigma \, (\partial_\mu \phi \partial^\mu \phi + 4\pi\mu \, \mathrm{e}^{2b\phi}) + \frac{Q}{\pi} \int_{\partial D} \mathrm{d}\theta \, \phi + 2Q^2 \ln R. \tag{6.46}$$

The equation of motion (6.24) reads

$$\partial\bar{\partial}\phi = 4\pi\mu b \, \mathrm{e}^{2b\phi}, \tag{6.47}$$

while the stress–energy tensor (6.27) is

$$T = -(\partial\phi)^2 + Q\partial^2\phi + 2\pi\mu \, \mathrm{e}^{2b\phi}, \tag{6.48}$$

or after using the equation of motion

$$T = -\frac{1}{2}(\partial\phi)^2 + Q\partial^2\phi. \tag{6.49}$$

It is straightforward to check that $T_{z\bar{z}} = 0$. The last term is is called an improvement term: if we had set directly $R = 0$ into the action, then we would miss it. Again we note the similarity with the Coulomb gas expression (section C.5).

The Schwarzian derivative (2.6) for the change of variables

$$w = \ln z \tag{6.50}$$

reads

$$S(w, z) = \frac{1}{2z^2}, \tag{6.51}$$

and we get the expression for $T_{w\bar{w}}$

$$T_{w\bar{w}} = -\frac{1}{2}(\partial\phi)^2 + Q\partial^2\phi + \frac{Q^2}{2}. \tag{6.52}$$

The difference between the two tensors is just a constant shift of the vacuum energy.

The vertex operator $e^{2a\phi}$ on the cylinder creates a state with momentum

$$ip = a - \frac{Q}{2} \tag{6.53}$$

on the cylinder because of the anomalous term in the transformation (6.36) of $\partial_z \phi$

$$\partial_z \phi \longrightarrow \frac{1}{z} \left(\partial_w \phi - Q \right). \tag{6.54}$$

This last property may seem to be incompatible with the action (6.1) since it contains the operator

$$V_b = e^{2b\phi} \tag{6.55}$$

which, on the cylinder, is transformed to

$$V_b = e^{(2b-Q)\phi}. \tag{6.56}$$

But the point is that the term $\sqrt{h}\, V_b$ has been designed to be Weyl invariant, which implies that we still have V_b for any choice of h, and the term looks the same on both the plane and the cylinder – this is a consequence of the Weyl invariance (6.7).

6.6 Lorentzian theory

According to the discussion on Wick rotation in appendix A.4, the Liouville action gets an additional minus sign for Lorentzian signature [67, pp. 125–126]

$$S_L = -\frac{1}{4\pi} \int d^2\sigma \, \sqrt{h} \Big(h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + QR\phi + 4\pi\mu \, e^{2b\phi} \Big). \tag{6.57}$$

6.6.1 Hamiltonian formalism

The canonical momentum conjugate to ϕ is

$$p = \frac{\delta S}{\delta \dot{\phi}} = \frac{\dot{\phi}}{2\pi}. \tag{6.58}$$

We see that μ should be positive otherwise the Hamiltonian is unbound from below. Note also that for Euclidean signature the Hamiltonian is negative.

6.6.2 Minisuperspace

The minisuperspace approximation (also called reduced particle dynamics) consists in treating the Liouville mode as constant over space [32, sec. 2.4, 57]

$$\phi(t, \sigma) = \phi_0(t), \tag{6.59}$$

and by considering only the flat metric. A lot of subtleties of Liouville theory already appear in the treatment of this zero mode.

The action becomes [17] [34, sec. 3.4, 86, sec. 22.3]

$$S = \int L \, dt = \int dt \left(\frac{\dot{\phi}_0^2}{2} - 2\pi\mu \, e^{2b\phi_0} \right) \tag{6.60}$$

[17] Note that many authors [57] write the Euclidean Liouville action before giving the minisuperspace equation, and the switch to Lorentzian signature is implicit.

taking into account the integration over $\sigma \in [0, 2\pi]$, and the dot denotes the derivative with respect to the time.

The conjugate momentum is [18]

$$p_0 = \dot{\phi}_0 \tag{6.61}$$

and we can find the Hamiltonian [78, sec. 4.2]

$$H_0 = p_0 \dot{\phi}_0 - L = \frac{p_0^2}{2} + 2\pi\mu\, \mathrm{e}^{2b\phi_0}. \tag{6.62}$$

Note that if we had taken Euclidean signature the Hamiltonian would have been negative definite.

6.7 States

In this section we begin by describing the states, and we will show later how to obtain them. Normalisable states can be found from minisuperspace analysis, while the non-normalisable ones cannot.

Primary states are vertex operators

$$V_a = \mathrm{e}^{2a\phi}. \tag{6.63}$$

Recall that the associated weight is

$$h_a = a(Q - a). \tag{6.64}$$

The momentum a can generically be written as

$$a = \frac{Q}{2} + ip, \tag{6.65}$$

with associated weight

$$h = \frac{Q^2}{4} + p^2 = \frac{c-1}{24} + p^2. \tag{6.66}$$

As we are asking for unitarity, the conformal dimension needs to be positive

$$h \geq 0. \tag{6.67}$$

Solving this inequality implies that the spectrum is made of two categories of states:

- normalisable

$$p \in \mathbb{R}, \qquad h \geq \frac{Q^2}{4} = \frac{c-1}{24}; \tag{6.68}$$

- non-normalisable

$$a \in [0, Q], \qquad p \in i\left[-\frac{Q}{2}, \frac{Q}{2}\right], \qquad 0 \leq h \leq \frac{Q^2}{4} = \frac{c-1}{24}. \tag{6.69}$$

Note that due to the reflection property of vertex operators, one can restrict states to

$$p \in \mathbb{R}_+ \tag{6.70}$$

and to

$$a \leq \frac{Q}{2}, \tag{6.71}$$

the latter being known as the *Seiberg bound*.

When studying the semiclassical approximation, two types of states will be singled out:

[18] Note that the minisuperspace momentum is related to the full momentum by $p = p_0/2\pi$ which means that p is a spatial density, as it is usual when discretizing space in QFT.

- Light states: their momentum is proportional to b, i.e.

$$a = b\,\sigma \tag{6.72}$$

where σ is finite.

Their particularity is to have a finite dimension as $b \to 0$

$$h_{b\sigma} \longrightarrow \sigma. \tag{6.73}$$

The vertex operator becomes independent of b in the semiclassical limit

$$V_{b\sigma} = \mathrm{e}^{2\sigma\phi_{\mathrm{cl}}}. \tag{6.74}$$

and insertions of these operators into the path integral will not contribute to the equations of motion (i.e. they do not change the saddle point). Then in correlation functions they just need to be evaluated at the solution for ϕ_{cl}.

- Heavy states: their momentum is proportional to $1/b$, i.e.

$$a = \frac{\eta}{b} \tag{6.75}$$

where η is finite.

Their conformal dimension diverges as $b \to 0$. The associated vertex operators will contain a power $1/b^2$

$$V_{\eta/b} = \mathrm{e}^{\frac{2\eta\phi_{\mathrm{cl}}}{b^2}}, \tag{6.76}$$

which is the same than the one which appears in the semiclassical action (6.81), and for this reason they will contribute as delta functions in the equations of motions (they modify the saddle point). Indeed one can write

$$V_{\eta/b}(\sigma) = \exp \frac{2\eta}{b^2} \int \mathrm{d}^2\sigma'\, \delta^{(2)}(\sigma - \sigma')\, \phi_{\mathrm{cl}}(\sigma'). \tag{6.77}$$

6.8 Semi-classical limit

The semi-classical analysis of Liouville theory is described in [40], where the Liouville field is continued to complex values.

We saw that the semi-classical limit is given by the equivalent limit

$$b \to 0, \qquad c_m \to -\infty, \qquad c \to \infty \tag{6.78}$$

Then Q reduces to

$$Q = \frac{1}{b} = \sqrt{\frac{c}{6}}. \tag{6.79}$$

b^2 can be thought as controlling the semi-classical limit of the path integral since it gives the saddle points for $b \to 0$. For this reason we rename (ϕ, μ) as $(\phi_{\mathrm{cl}}, \mu_{\mathrm{cl}})$ [19].

[19] We will not write the indices when no doubt is possible. In fact these variables are the ones that were used in (5.33).

6.8.1 Semiclassical action

After rescaling the Liouville mode and the cosmological constant

$$\phi_{\rm cl} = b\phi, \qquad \mu_{\rm cl} = \mu b^2, \tag{6.80}$$

the Liouville action (6.1) becomes

$$S_L = \frac{1}{4\pi b^2} \int \mathrm{d}^2\sigma \, \sqrt{h} \Big(h^{\mu\nu} \partial_\mu \phi_{\rm cl} \partial_\nu \phi_{\rm cl} + R\phi_{\rm cl} + 4\pi \mu_{\rm cl} \, \mathrm{e}^{2\phi_{\rm cl}} \Big). \tag{6.81}$$

The equation of motion (6.24) becomes [40, sec. 1]

$$\boxed{R[\,\mathrm{e}^{2\phi_{\rm cl}} h] = (R[h] - 2\Delta\phi_{\rm cl})\, \mathrm{e}^{-2\phi_{\rm cl}} = -8\pi\mu_{\rm cl}.} \tag{6.82}$$

This is the well-known Liouville equation [20] whose solutions are metrics g of constant negative curvature (since $\mu \geq 0$). Moreover this last equation is explicitly invariant under (5.38). A solution of this equation corresponds to a saddle point of the path integral.

6.8.2 Correlation functions on the sphere

In this section we restrict ourselves to $\mathcal{M} = S^2$, and we will follow [40, sec. 2]. The goal is to obtain semi-classical expressions for the correlation functions in order to compare them to exact expressions obtained after the quantization.[21]

In complex coordinates the action (6.46) simply becomes

$$b^2 \, S_L = \frac{1}{4\pi} \int_D \mathrm{d}^2\sigma \, (\partial_\mu \phi_c \partial^\mu \phi_c + 4\pi\mu_c \, \mathrm{e}^{2\phi_c}) + \frac{1}{\pi} \int_{\partial D} \mathrm{d}\theta \, \phi_c + 2\ln R. \tag{6.83}$$

We consider semiclassical computations of correlation functions

$$\left\langle \prod_i \mathrm{e}^{2a_i \phi/b} \right\rangle = \int \mathrm{d}\phi \, \prod_i \mathrm{e}^{2a_i\phi/b} \; \mathrm{e}^{-S[\phi]}. \tag{6.84}$$

As said above, light states $a_i = b\sigma_i$ do not modify the saddle point, while a heavy operator $a_i = \eta_i/b$ will contribute [40, sec. 1].

For three heavy states, the saddle point is real only if the η_i satisfy the following conditions (called the physical region)

$$\eta_i \in \mathbb{R}, \qquad \eta_i < 1/2, \qquad \sum_i \eta_i > 1, \tag{6.85}$$

and the result is

$$\left\langle \prod_{i=1}^3 \mathrm{e}^{2\eta_i\phi} \right\rangle = \mathrm{e}^{-S[\phi_{\rm cl}]}, \qquad S[\phi_{\rm cl}] = \frac{1}{b^2} \, G(\eta_1, \eta_2, \eta_3) + O(1). \tag{6.86}$$

If the η_i are not in the physical region then $G(\eta_i)$ becomes multivalued and there are monodromies [40, sec. 1.1]. In particular the normalisable states are not in the physical region since they correspond to

$$\operatorname{Re} \eta_i = \frac{1}{2}, \qquad \operatorname{Im} \eta_i > 0. \tag{6.87}$$

[20] It was used in the context of Riemann uniformization problem: “Can any metric be related to one with negative constant curvature?” Liouville theory can be viewed as the quantum version of this problem.

[21] In particular this is useful for deciding if theories which are not defined through the Liouville action are equivalent to it (for example in the case of matrix model or conformal bootstrap approaches).

6.9 Ultralocal approximation

In the ultralocal approximation only the constant zero-mode is kept

$$\phi(t,\sigma)=\phi_0=\mathrm{cst}. \tag{6.88}$$

This approximation is sometimes also called minisuperspace but we reserve the latter for the time-dependent zero-mode.

The exponential interaction (6.4) can be approximated by a wall located at

$$\phi_w=-\frac{1}{2b}\ln\mu \tag{6.89}$$

which corresponds to the point where the interaction becomes strong

$$U(\phi_w)=\mu\,\mathrm{e}^{2b\phi_w}\sim 1. \tag{6.90}$$

7 Quantization

Several approaches can be used in order to quantize the Liouville theory (canonical quantization, path integral, BRST quantization, "conformal bootstrap"...). They are all complementary as some answers are easier to get in one of them.

7.1 Canonical quantization

The first approach is to quantize canonically by promoting fields to operators [34, sec. 3.3].

We can expand ϕ and p in Fourier modes [78, sec. 4.2]

$$\phi(t,\sigma)=\phi_0(t)+\sum_{n\neq 0}\frac{i}{n}\big(a_n(t)\,\mathrm{e}^{-in\sigma}+b_n(t)\,\mathrm{e}^{-in\sigma}\big), \tag{7.1a}$$

$$p(t,\sigma)=p_0(t)+\frac{1}{4\pi}\sum_{n\neq 0}\big(a_n(t)\,\mathrm{e}^{-in\sigma}+b_n(t)\,\mathrm{e}^{-in\sigma}\big). \tag{7.1b}$$

We have

$$a_n^\dagger=a_{-n},\qquad b_n^\dagger=b_{-n}. \tag{7.2}$$

Imposing the equal time commutation relation

$$[\phi(t,\sigma),p(t,\sigma')]=\delta(\sigma-\sigma') \tag{7.3}$$

gives the following relations for the modes

$$[\phi_0,p_0]=i,\qquad [a_n(t),b_m(t)]=n\,\delta_{n,m}. \tag{7.4}$$

7.2 Operator formalism

The authors of [9] proved that Liouville does not have a translation invariant vacuum. We can guess this result by taking the expectation value of the quantum equation of motion

$$-\Delta\Phi+8\pi\mu b\,\mathrm{e}^{2b\Phi}=0, \tag{7.5}$$

then assuming that $|0\rangle$ is translation invariant we have

$$\langle 0|\,\Delta\Phi\,|0\rangle=0 \tag{7.6}$$

which implies

$$\langle 0|\,\mathrm{e}^{2b\Phi}\,|0\rangle=0. \tag{7.7}$$

This is a contradiction with the fact that the exponential is strictly positive.

7.3 Minisuperspace quantization

We continue the study of minisuperspace (called also the quantum mechanical model) [57, sec. 2, 77, sec. 3.1, 86, sec. 22.3] (see also [34, sec. 3.4, 4.3]) [22]

$$S = \int \mathrm{d}t \left(\frac{\dot\phi_0^2}{2} - 2\pi\mu\, \mathrm{e}^{2b\phi_0} \right), \tag{7.8a}$$

$$H_0 = \frac{p_0^2}{2} + 2\pi\mu\, \mathrm{e}^{2b\phi_0} \tag{7.8b}$$

that we begun in section 6.6.2. It can be used to obtain semi-classical approximations to n-point functions.

The discussion can be extended to include matter [53, sec. 1].

7.3.1 Canonical quantization

Canonical quantization goes with the replacement

$$p_0 = -i\, \frac{\mathrm{d}}{\mathrm{d}\phi_0}. \tag{7.9}$$

The zero-mode Hamiltonian (6.62) reads [78, sec. 4.2]

$$H_0 + N = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\phi_0^2} + 2\pi\mu\, \mathrm{e}^{2b\phi_0} + N \tag{7.10}$$

where we added a zero-point energy N [63, sec. 2.3].

In order to find interpret the eigenvalues of this operator, let's go back to the plane. There the Hamiltonian is given by the dilatation operator $L_0 + \bar L_0$ (these are the Virasoro modes defined on the plane)

$$(L_0 + \bar L_0)\psi_\Delta = 2\Delta\, \psi_\Delta \tag{7.11}$$

because of the relation (2.14). The Hamiltonian on the cylinder is given [23]by [63, sec. 2.3]

$$H_0 + N = L_0 + \bar L_0 - \frac{c}{12} = L_0 + \bar L_0 - \frac{1}{12} - \frac{Q^2}{2}. \tag{7.12}$$

and this allow us to identify the zero-point energy

$$N = -\frac{1}{12}. \tag{7.13}$$

Finally applying H_0 on ψ_Δ with (7.11) we get

$$H_0\psi_\Delta(\phi_0) = \left(2\Delta - \frac{Q^2}{2}\right)\psi_\Delta(\phi_0). \tag{7.14}$$

Defining

$$\Delta = \frac{Q^2}{4} + p^2, \qquad p \in \mathbb{C} \tag{7.15}$$

and renaming ψ_Δ to ψ_p the Wheeler–deWitt equation [24] for a wave functions [25] $\psi_p(\phi_0)$

$$H_0\psi_p = 2p^2\psi_p, \tag{7.16}$$

[22]McElgin is defining $a = (Q + ip)/2$.

[23]Both operators L_0 and $\bar L_0$ are shifted by $-c/24$ and $-\bar c/24$ on the cylinder, but we have $\bar c = c$.

[24]The WdW equation is a constraint on wave functions since the latter should be invariant under time diffeomorphisms.

[25]Do not confound the momentum p in the eigenvalue with the canonical momentum.

or after using the explicit expression (7.10) we get the

$$\left(-\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}\phi_0^2} + 2\pi\mu \, \mathrm{e}^{2b\phi_0} \right) \psi_p = 2p^2 \, \psi_p. \tag{7.17}$$

In order to simplify this equation we can do the change of variable

$$\ell = \mathrm{e}^{b\phi_0}. \tag{7.18}$$

This new variable ℓ can be interpreted as the circumference of the $1d$ universe in the physical metric.

We compute the change of variables for the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}\phi_0} = \frac{\mathrm{d}\ell}{\mathrm{d}\phi_0} \frac{\mathrm{d}}{\mathrm{d}\ell} = b \, \ell \frac{\mathrm{d}}{\mathrm{d}\ell}, \tag{7.19a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi_0^2} = \left(\frac{\mathrm{d}\ell}{\mathrm{d}\phi_0} \frac{\mathrm{d}}{\mathrm{d}\ell} \right)^2 = b^2 \left(\ell \frac{\mathrm{d}}{\mathrm{d}\ell} \right)^2 = b^2 \left(\ell^2 \frac{\mathrm{d}}{\mathrm{d}\ell^2} + \ell \frac{\mathrm{d}}{\mathrm{d}\ell} \right). \tag{7.19b}$$

We obtain the equation

$$\left(\ell^2 \frac{\mathrm{d}}{\mathrm{d}\ell^2} + \ell \frac{\mathrm{d}}{\mathrm{d}\ell} - 4(\hat{\mu} \, \ell^2 - \hat{p}^2) \right) \psi_p = 0 \tag{7.20}$$

where we defined

$$\hat{\mu} = \frac{\pi\mu}{b^2}, \qquad \hat{p} = \frac{p}{b}. \tag{7.21}$$

Note that $\hat{\mu}$ is proportional to the semi-classical cosmological constant, and $\hat{p} = bp$ is a light momentum (since it is proportional to b which goes to zero in this limit).

7.3.2 Wave functions

In the limit $\phi_0 \to -\infty$ (short physical distance, i.e. UV), the potential energy tends to zero and we get the equation [78, sec. 4.2]

$$\frac{\mathrm{d}^2\psi_p}{\mathrm{d}\phi_0^2} + 4p^2 \, \psi_p = 0, \tag{7.22}$$

which is just the equation for normalizable states which behave as plane waves

$$\psi_p \sim \sin(2p \, \phi_0). \tag{7.23}$$

Plane waves with energy $\pm p$ behaves identically. For $\phi_0 \to \infty$, the wall grows exponentially and reflects waves, such that the two waves $\pm p$ are related and we should select only one. Then we see that the effect of the wall is to truncate half of the states.

In (7.20) we recognize the modified Bessel equation (E.30) with imaginary parameter, and whose solution is

$$\psi_p(\ell) = \alpha_p \, K_{2i\hat{p}}(2\sqrt{\hat{\mu}} \, \ell) + \beta_p \, I_{2i\hat{p}}(2\sqrt{\hat{\mu}} \, \ell). \tag{7.24}$$

Looking at the asymptotic form (E.31) for the modified Bessel equations in the limit $\phi_0 \to \infty$ corresponding to $\ell \to \infty$, we need to set

$$\beta_p = 0 \tag{7.25}$$

to discard the growing exponential.

In order to find α_p we need a normalization condition. We choose to take incoming plane waves $\mathrm{e}^{2ip\phi_0}$ with unit coefficient when $\phi_0 \to -\infty$ or $\ell \to 0$. Using again the limit of the modified Bessel functions (E.31), we get

$$\psi_p(\phi_0) \sim \alpha_p \left[\frac{\Gamma(2i\hat{p})}{2} \left(\frac{1}{\sqrt{\hat{\mu}}\ell} \right)^{2i\hat{p}} + \frac{\Gamma(-2i\hat{p})}{2} \left(\sqrt{\hat{\mu}}\ell \right)^{2i\hat{p}} \right] \tag{7.26a}$$

$$= \alpha_p \frac{\Gamma(-2i\hat{p})}{2\hat{\mu}^{-i\hat{p}}} \left[\mathrm{e}^{2ip\phi_0} + \frac{\Gamma(2i\hat{p})}{\Gamma(-2i\hat{p})} \hat{\mu}^{-2i\hat{p}} \, \mathrm{e}^{-2ip\phi_0} \right] \tag{7.26b}$$

(the p in the exponential in order to remove the b in $\ell = \mathrm{e}^{b\phi_0}$). Choosing

$$\alpha_p = \frac{2\hat{\mu}^{-i\hat{p}}}{\Gamma(-2i\hat{p})}, \tag{7.27}$$

we finally obtain

$$\psi_p(\ell) = \frac{2\hat{\mu}^{-i\hat{p}}}{\Gamma(-2i\hat{p})} \, K_{2i\hat{p}}(2\sqrt{\hat{\mu}}\,\ell) \tag{7.28a}$$

or substituting $\hat{p}$ and $\hat{\mu}$

$$\psi_p(\ell) = \frac{2(\pi\mu/b^2)^{-ip/b}}{\Gamma(-2ip/b)} \, K_{2ip/b}(2\sqrt{\pi\mu}\,\ell/b). \tag{7.28b}$$

As we have written above for $\phi_0 \to -\infty$ one gets

$$\psi_p(\phi_0) \sim \mathrm{e}^{2i\hat{p}\phi_0} + R_0(p)\, \mathrm{e}^{-2i\hat{p}\phi_0}. \tag{7.29}$$

The wave function decomposes in two plane waves, one ingoing, and one outgoing with a reflection coefficient

$$R_0(p) = \frac{\Gamma(2i\hat{p})}{\Gamma(-2i\hat{p})} \, \hat{\mu}^{-2i\hat{p}}. \tag{7.30}$$

This may be seem as the effect of the wall, which implies that both waves are not independent. Using the recursion (E.3) for the Gamma function we obtain the usual expression for the coefficient

$$R_0(p) = -\frac{\Gamma(1+2i\hat{p})}{\Gamma(1-2i\hat{p})} \, \hat{\mu}^{-2i\hat{p}}. \tag{7.31}$$

For $p \in \mathbb{R}$, the reflection coefficient is a pure phase [57, sec. 2]

$$|R_0(p)| = \left| \frac{\Gamma(2i\hat{p})}{\Gamma(-2i\hat{p})} \right| = 1, \tag{7.32}$$

using the formula (E.6), and this is expected for a completely reflecting potential.

It is easy to check that

$$R_0(-p) = \frac{\alpha_p}{\alpha_{-p}} = \frac{1}{R_0(p)}. \tag{7.33}$$

In particular this implies

$$\psi_{-p} = R_0(-p)\, \psi_p. \tag{7.34}$$

One may see that it corresponds to the limit $b \to 0$ of the full reflection coefficient (9.26).

7.3.3 States

Wave functions such that p is real

$$p \in \mathbb{R} \tag{7.35}$$

corresponds to normalizable states, i.e. they correspond to square integrable functions. Moreover functions with $p \in \mathbb{R}$ form an orthonormal set [26]

$$\int_{-\infty}^{\infty} \mathrm{d}\phi_0\, \psi_p^*(\phi_0)\psi_{p'}(\phi_0) = \int_0^{\infty} \frac{\mathrm{d}\ell}{\ell}\, \psi_p^*(\ell)\psi_{p'}(\ell) = \pi\, \delta(p - p'), \tag{7.36}$$

using the orthonormalization relations for the modified Bessel functions. In view of this formula, the normalization (7.27) for α corresponds to a canonical normalization of the wave functions. For this set of states, the conformal dimensions (7.15) are bounded from below [78, sec. 4.2]

$$\Delta \geq \frac{Q^2}{4}. \tag{7.37}$$

When p is a pure imaginary number [78, sec. 4.2]

$$p = i\omega, \qquad \omega \in \mathbb{R}, \tag{7.38}$$

the wave function (7.28a) diverges as $\phi_0 \to -\infty$

$$\psi_p(\phi_0) \sim \mathrm{e}^{-2\hat{\omega}\phi_0}. \tag{7.39}$$

Moreover conformal dimensions (7.15)

$$\Delta = \frac{Q^2}{4} - \omega^2 \tag{7.40}$$

are bounded from above

$$\Delta \leq \frac{Q^2}{4}. \tag{7.41}$$

and positive values (for unitarity) are achieved only for

$$|\omega| \leq \frac{Q}{2}. \tag{7.42}$$

As a consequence values of the parameter a are

$$a \in [0, Q] \tag{7.43}$$

if one requires unitarity (but note that wave functions corresponding to $a \notin [0, Q]$ are solution anyway).

As we have seen wave functions $\phi_{\pm p}$ are equivalent, which is due to the reflection of the wall, and they should be identified. Hence the independent parameters are limited to

$$\text{normalisable:} \quad p \in \mathbb{R}_+, \qquad \text{non-normalisable:} \quad a \in [0, Q/2]. \tag{7.44}$$

The second is called the *Seiberg bound* [78, sec. 4.2].

For more details see also [32, sec. 2.5].

[26] In principle we normalize wave functions to $2\pi\delta(p - p')$, but here the wave functions is $\mathrm{e}^{2ip\phi_0}$ and we have $\delta(2p) = \delta(p)/2$ [63, p. 13].

7.3.4 Correlation functions

The wave functions can be used to compute a semi-classical approximation to the 3-point structure constant $C(a_1, a_2, a_3)$ (see for example [57, 76]). In particular the limit $b \to 0$ of $C(a_1, a_2, a_3)$ evaluated with the following weights[27]

$$a_1 = \frac{Q}{2} + ibp_1, \qquad a_2 = b\sigma, \qquad a_3 = \frac{Q}{2} + ibp_3 \tag{7.45}$$

matches the integral (assuming $\sigma \equiv ip_2 > 0$)

$$C_0(a_1, a_2, a_3) = \int_{-\infty}^{\infty} \mathrm{d}\phi_0 \; \psi_{bp_1}(\phi_0)\, \mathrm{e}^{2b\sigma\phi} \psi_{bp_3}(\phi_0) \tag{7.46a}$$

$$= \left(\frac{\pi\mu}{b^2}\right)^{-2\tilde{p}} \Gamma(2\tilde{p}) \prod_i \frac{\Gamma\big((-1)^i 2\tilde{p}_i\big)}{\Gamma(2p_i)} \tag{7.46b}$$

where we defined

$$2\tilde{p} = \sum_i p_i, \qquad \tilde{p}_i = \tilde{p} - p_i, \qquad i = 1, 2, 3. \tag{7.47}$$

8 Liouville duality

Quantum Liouville theory exhibits a duality under [57, sec. 2]

$$b \longleftrightarrow \frac{1}{b}. \tag{8.1}$$

The presence of this duality is responsible for the two-dimensional lattice of poles of the three-point function [65], which can be understood as coming from the presence of a second exponential interaction (called the dual cosmological constant) in the action. Hence the cosmological constant and its dual are

$$U_b = \mu\, \mathrm{e}^{2b\phi}, \qquad U_{b^{-1}} = \tilde{\mu}\, \mathrm{e}^{2\phi/b}. \tag{8.2}$$

The Liouville action can be obtained by the reduction of a WZW model. Then conformal invariance at the classical level requires its potential to be the sum of the two exponentials (whereas classically the reduction leads to a single exponential) [65]. This is obtained from a differential equation on the potential, which is needed to get it invariant under conformal symmetry; then this equation is second-order in the quantum theory, and only first order in the classical theory.

8.1 Dual cosmological constant

Vertex operators are defined by exponentials of the Liouville field

$$V_a = \mathrm{e}^{2a\phi}. \tag{8.3}$$

Operators that have a conformal weight $(1, 1)$ are denoted by

$$V_b = \mathrm{e}^{2b\phi} \tag{8.4}$$

where b is a solution of the equation

$$Q = \frac{1}{b} + b. \tag{8.5}$$

[27]For simplifying the notations we do not write the that on p_i and σ,but these quantities really correspond to the hatted ones of the previous formulas.

This equation admits two solutions for b

$$b_\pm = \frac{Q}{2} \pm \frac{1}{2}\sqrt{Q^2 - 4}. \tag{8.6}$$

We can see that we have the relations

$$b_+ b_- = 1, \qquad b_+ + b_- = Q. \tag{8.7}$$

We define

$$b_- \equiv b \tag{8.8a}$$

from which it follows that

$$b_+ = \frac{1}{b}. \tag{8.8b}$$

We thus obtain two vertex operators of conformal dimensions $(1, 1)$

$$V_b \equiv V_- = \mathrm{e}^{2b_-\phi} \equiv \mathrm{e}^{2b\phi}, \qquad V_{b^{-1}} \equiv V_+ = \mathrm{e}^{2b_+\phi} \equiv \mathrm{e}^{2\phi/b}. \tag{8.9}$$

These operators are very special for several reasons. First of all they can be added to the Lagrangian as marginal deformations. In general only V_- is added since it appears from the classical cosmological constant term, but the others could be present as a non-perturbative effect. It would them seem very natural to include both in the action. Another reason is that this second term is required by crossing symmetry [57, sec. 2]. This has been studied in the context of the Coulomb gas [18, chap. 7].

We define the interaction term

$$U = U_- + U_+ = \mu_- V_- + \mu_+ V_+ = \mu\,\mathrm{e}^{2b\phi} + \tilde{\mu}\,\mathrm{e}^{2\phi/b} \tag{8.10}$$

with the alternative notations for the coupling constants

$$\mu \equiv \mu_-, \qquad \tilde{\mu} \equiv \mu_+, \tag{8.11}$$

μ being the usual cosmological constant and $\tilde{\mu}$ is called the dual cosmological constant. This new term is adding a second growing wall. Then the Lagrangian reads

$$S_L = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}\Big(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + QR\phi + 4\pi\mu\,\mathrm{e}^{2b\phi} + 4\pi\tilde{\mu}\,\mathrm{e}^{2\phi/b}\Big). \tag{8.12}$$

The action we have just written is invariant under the weak-strong transformation

$$b \longrightarrow \frac{1}{b} \tag{8.13a}$$

if we also exchange the cosmological constants

$$\mu \longleftrightarrow \tilde{\mu} \tag{8.13b}$$

since Q is obviously invariant and so is the two interaction terms taken together.

Moreover the two cosmological constants are related by [57, sec. 2]

$$\big(\pi\mu\gamma(b^2)\big)^{1/b} = \big(\pi\tilde{\mu}\gamma(b^{-2})\big)^{b}. \tag{8.14}$$

This relation is used a simplification in the computations of [65] (it seems to be an assumption, despite the fact that they claim in the introduction to derive it). It can be obtained by asking for the invariance of the 3-point function under the duality. One can check that this relation is invariant under the duality (8.13). Note that we can rewrite it as

$$\big(\pi\mu_-\gamma(b_-^2)\big)^{1/b_-} = \big(\pi\mu_+\gamma(b_+^2)\big)^{1/b_+}. \tag{8.15}$$

Finally it can be used to write $\tilde\mu$ in terms of μ

$$\tilde\mu = \frac{1}{\pi\,\gamma(b^{-2})}\big(\pi\mu\gamma(b^2)\big)^{1/b^2}. \tag{8.16}$$

In particular this relation implies that the identity [57, sec. 2]

$$S_L[\phi](b,\mu) = S_L[\phi + \tfrac{1}{2b}\ln\mu](b,1) - \frac{Q\chi}{2b}\ln\mu \tag{8.17}$$

is preserved (χ is the Euler number).

Since an exponential term is growing very fast it can be approximated by a hard wall located at the position where the interaction term is of order 1 [34, sec. 0.2]. In our case we get

$$\mu\,\mathrm{e}^{2b\phi_-}\sim 1 \Longrightarrow \phi_- \sim \frac{1}{2b}\ln\frac{1}{\mu}, \tag{8.18a}$$

$$\tilde\mu\,\mathrm{e}^{2\phi_+/b}\sim 1 \Longrightarrow \phi_+ \sim \frac{b}{2}\ln\frac{1}{\tilde\mu} \tag{8.18b}$$

for the positions of the two walls $\phi_\pm$. Using the expression (8.16) we find that

$$\phi_+ = \phi_- + \frac{b}{2}\ln\pi\,\gamma(b^{-2}) - \frac{1}{2b}\ln\pi\,\gamma(b^2) \tag{8.19}$$

from

$$\begin{aligned}\phi_+ &\sim \frac{b}{2}\ln\frac{1}{\tilde\mu} = \frac{b}{2}\ln\pi\,\gamma(b^{-2})\,\big(\pi\mu\gamma(b^2)\big)^{-1/b^2}\\ &= \frac{1}{2b}\ln\frac{1}{\mu} + \frac{b}{2}\ln\pi\,\gamma(b^{-2}) - \frac{1}{2b}\ln\pi\gamma(b^2).\end{aligned}$$

Another expression for this relation is

$$\phi_+ + \frac{b_+}{2}\ln\pi\,\gamma(1/b_+^2) = \phi_- + \frac{b_-}{2}\ln\pi\,\gamma(b_-^{-2}). \tag{8.20}$$

We can also note that since $b<1$ the dual exponential is growing much faster, but depending on the value of $\tilde\mu$ it can start to grow later.

An interesting relation is [57, sec. 2]

$$\frac{\partial\tilde\mu}{\partial\mu} = \frac{1}{b^2}\frac{\tilde\mu}{\mu} = R(b) \tag{8.21}$$

where $R(b)$ is the reflection coefficient evaluated at b. From this last point we obtain that

$$\frac{\partial U_-}{\partial\mu} = \frac{\partial U_+}{\partial\mu} = V_b \tag{8.22}$$

since

$$\frac{\partial U_+}{\partial\mu} = \frac{\partial\tilde\mu}{\partial\mu}V_{b^{-1}} = R(b)V_{b^{-1}} = V_b \tag{8.23}$$

because of $1/b = Q/2 - b$ and using the reflection property (9.3).

At the self-dual point

$$b = 1 \tag{8.24}$$

we obtain

$$V_- = V_+ = \mathrm{e}^{2\phi}, \qquad \tilde\mu = \mu, \tag{8.25}$$

the second relation following from (8.16).

8.2 Quantum theory

By looking at the Ward identities, it is shown in [64] that the relations

$$Q = b_- b_+, \qquad b_- b_+ = 1 \tag{8.26}$$

still holds in the quantum regime.

9 Correlation functions

In this section we consider correlation functions of vertex operators

$$V_a(z, \bar{z}) = \mathrm{e}^{2a\phi(z,\bar{z})}, \qquad \Delta_a = a(Q - a). \tag{9.1}$$

We will often omit the z dependence.

We identify the identity operator as

$$\mathrm{id} = \lim_{a \to 0} V_a. \tag{9.2}$$

We recall that V_a and V_{Q-a} are related by reflection

$$V_{Q-a} = R(a) V_a \tag{9.3}$$

due to the wall [28]. The reflection coefficient is such that

$$R(a) R(Q - a) = 1. \tag{9.4}$$

9.1 2-point function

It is explained in [64] how to interpret this reflection if the two-point function is given. Defining the inner product

$$\langle a_1, a_2 \rangle = \lim_{x \to 0} x^{2\Delta_1} A_2(Q - a_1, a_2; x) \tag{9.5}$$

between primary states, one has

$$\begin{pmatrix} \langle a, a \rangle & \langle a, Q - a \rangle \\ \langle Q - a, a \rangle & \langle Q - a, Q - a \rangle \end{pmatrix} = \begin{pmatrix} 1 & R(Q - 2a)^{-1} \\ R(Q - 2a) & 1 \end{pmatrix} \delta(0). \tag{9.6}$$

The determinant of this matrix is zero which implies that one linear combination of states has zero norm and decouples

$$|a\rangle - R(Q - 2a)\, |Q - a\rangle = 0. \tag{9.7}$$

9.2 3-point function and DOZZ formula

The three-point function of Liouville theory was found by Dorn and Otto and by Zamolodchikov and Zamolodchikov [17, 84] by generalizing the result of computations done with the path integral at specific values of the momenta (see also [2] and [15]).

On the other hand, this 3-point function can be obtained by solving some equations of the conformal bootstrap, as was shown by Teschner (and refined by others) [82]. This will be the topic of the section 10, where more references will be given.

[28] Some authors use $R(a)^{-1}$.

It was latter explained by O'Raifeartaigh et al. how to get this formula by generalizing the path integral to include the dual cosmological constant interaction [64, 65] (see also section 8). In this case it is possible to obtain the normalization coefficient of the 3-point function, which cannot be derived by the two other methods (instead an ansatz).

Various references on the DOZZ formula and the recursion relations include [40, 57, sec. 5, 63, 85].

Given a set of three variables $\{x_1, x_2, x_3\}$, one defines [76, p. 5]

$$2\tilde{x} = \sum_i x_i = x_1 + x_2 + x_3, \qquad \tilde{x}_i = \tilde{x} - x_i. \tag{9.8}$$

For example one has

$$\tilde{x}_1 = x_2 + x_3 - x_1. \tag{9.9}$$

Conformal invariance dictates the form of the 3-point function to be [40, sec. 2.3]

$$\langle V_{a_1} V_{a_2} V_{a_3} \rangle = \frac{C(a_1, a_2, a_3)}{|z_{12}|^{2\Delta_{12}} |z_{13}|^{2\Delta_{13}} |z_{23}|^{2\Delta_{23}}} \tag{9.10}$$

where we defined

$$z_{ij} = z_i - z_j, \quad \Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3. \tag{9.11}$$

The structure constant $C(a_1, a_2, a_3)$ are one of the fundamental element to define a CFT. Its expression is given by the DOZZ formula

$$C(a_1, a_2, a_3) = \left[\pi\mu\gamma(b^2) b^{2-2b^2}\right]^{(Q-2\tilde{a})/b} \frac{\Upsilon_b'(0)}{\Upsilon_b(2\tilde{a} - Q)} \prod_i \frac{\Upsilon_b(2a_i)}{\Upsilon_b(2\tilde{a}_i)}. \tag{9.12}$$

We refer to the appendix E.3 for the properties of the various functions. The limit $b \to 0$ agrees with (**??**) [57, p. 7].

We can obtain the reflection amplitude from the identity

$$C(a_1, a_2, a_3) = R(a_1) C(a_1 - Q, a_2, a_3). \tag{9.13}$$

We can also get this coefficient from the limit of C, but there is no ambiguity here: we don't have to know that $V_a \to \text{id}$ when $a \to 0$ in order to compute it, while the limit requires this [76, sec. 3].

9.2.1 2-point function limit

We want to check if the limit of $C(a_1, a_2, a_3)$ gives the correct 2-point function. We consider

$$a_1 = \frac{Q}{2} + ip_1, \qquad a_3 = \frac{Q}{2} + ip_2, \qquad a_2 = \varepsilon. \tag{9.14}$$

The various combination of a_i are

$$\sum a - Q = i(p_1 + p_2) + \varepsilon, \tag{9.15a}$$
$$a_{12} = i(p_1 - p_2) + \varepsilon, \tag{9.15b}$$
$$a_{23} = -i(p_1 - p_2) + \varepsilon, \tag{9.15c}$$
$$a_{13} = i(p_1 + p_2) - \varepsilon + Q. \tag{9.15d}$$

The DOZZ formula gives then (we already remove the ε which will not contribute)

$$C(Q/2+ip_1,\varepsilon,Q/2+ip_2) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{i\frac{(p_1+p_2)}{b}} \times \\ \times \frac{\Upsilon_b'(0)\Upsilon_b(Q+2ip_1)\Upsilon_b(Q+2ip_2)\Upsilon_b(2\varepsilon)}{\Upsilon_b\big(i(p_1+p_2)\big)\Upsilon_b\big(i(p_1+p_2)+Q\big)\Upsilon_b\big(i(p_1-p_2)+\varepsilon\big)^2}. \tag{9.16}$$

We use the formula (E.42) to remove the Q in the Υ_b functions. The total power of b is given by

$$-i(2-2b^2)\frac{p_1+p_2}{b} - 4 + 4i(p_1+p_2)\left(\frac{1}{b}-b\right) + 2 - 2i(p_1+p_2)\left(\frac{1}{b}-b\right) = -2, \tag{9.17}$$

where the second and third terms comes from $\Upsilon_b(Q+2ip_1)\Upsilon_b(Q+2ip_1)$, and the fourth and fifth from $\Upsilon_b\big(i(p_1+p_2)+Q\big)$.

We can also use (E.44) to expand $\Upsilon_b(2\varepsilon)$ as

$$\Upsilon_b(2\varepsilon) = 2\varepsilon\,\Upsilon_b'(0) \tag{9.18}$$

The formula reduces to

$$C(Q/2+ip_1,\varepsilon,Q/2+ip_2) = \frac{1}{b^2}\left[\pi\mu\gamma(b^2)\right]^{i\frac{(p_1+p_2)}{b}} \frac{\Upsilon_b(2ip_1)\Upsilon_b(2ip_2)}{\Upsilon_b\big(i(p_1+p_2)\big)^2}\times \\ \times \frac{2\varepsilon\,\Upsilon_b'(0)^2}{\Upsilon_b\big(i(p_1-p_2)+\varepsilon\big)^2} \frac{\gamma(2ip_1b+1)\gamma(2ip_1b^{-1})\gamma(2ip_2b+1)\gamma(2ip_2b^{-1})}{\gamma\big(i(p_1+p_2)b+1\big)\gamma\big(i(p_1+p_2)b^{-1}\big)}. \tag{9.19}$$

The denominator has a zero only when $p_1 = p_2$, so that we can deduce directly that the function vanishes when $\varepsilon \to 0$ if $p_1 \neq p_2$. If the two momenta are equal we need to expand also $\Upsilon_b\big(i(p_1-p_2)+\varepsilon\big)$, giving

$$C(Q/2+ip_1,\varepsilon,Q/2+ip_2) = \frac{1}{b^2}\left[\pi\mu\gamma(b^2)\right]^{i\frac{(p_1+p_2)}{b}} \frac{\Upsilon_b(2ip_1)\Upsilon_b(2ip_2)}{\Upsilon_b\big(i(p_1+p_2)\big)^2}\times \\ \times \frac{2\varepsilon\,\Upsilon_b'(0)^2}{\big(i(p_1-p_2)+\varepsilon\big)^2\Upsilon_b'(0)^2} \frac{\gamma(2ip_1b+1)\gamma(2ip_1b^{-1})\gamma(2ip_2b+1)\gamma(2ip_2b^{-1})}{\gamma\big(i(p_1+p_2)b+1\big)\gamma\big(i(p_1+p_2)b^{-1}\big)}. \tag{9.20}$$

We make use of the relation

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{(p_1-p_2)^2+\varepsilon^2} = \pi\,\delta(p_1-p_2) \tag{9.21}$$

and this allows us to set $p_2 = p_1$.

We obtain the 2-point function [40, sec. 2.4]

$$\lim_{\varepsilon\to 0} C(Q/2+ip_1,\varepsilon,Q/2+ip_2) = 2\pi G(a_1)\delta(p_1-p_2). \tag{9.22}$$

where

$$G(a) = \frac{1}{R(a)} = \frac{1}{b^2}\left[\pi\mu\gamma(b^2)\right]^{\frac{Q-2a_1}{b}} \gamma(2ba_1-b^2)\gamma(2a_1/b-1-1/b^2) \tag{9.23}$$

To make contact we the previous formula we note that

$$2ip_1 = Q-2a_1, \qquad 2ip_1b+1 = 2ba_1-b^2, \qquad 2ip_1b^{-1} = \frac{2a_1}{b} - 1 - \frac{1}{b^2}. \tag{9.24}$$

For generic α_1, α_3 we get

$$\lim_{a_2\to 0} C(a_1,a_2,a_3) = 2\pi\delta(a_1+a_3-Q) + G(a_1)\,\delta(a_1-a_3). \tag{9.25}$$

9.2.2 Reflection coefficient

From the DOZZ formula the reflection coefficient is found to be [57, sec. 1, 2]

$$R(a) = \left[\pi\mu\gamma(b^2)\right]^{(Q-2\tilde{a})/b} \frac{\gamma(2ab - b^2)}{b^2\, \gamma(2 - 2a/b + b^{-2})}, \tag{9.26a}$$

$$R(p) = -\left[\pi\mu\gamma(b^2)\right]^{-ip/b} \frac{\Gamma(1 + ip/b)}{\Gamma(1 - ip/b)} \frac{\Gamma(1 + ibp)}{\Gamma(1 - ibp)}. \tag{9.26b}$$

Part III
Extensions

10 Conformal bootstrap

In the context of the conformal bootstrap, the theory is defined only by its symmetries. Its spectrum and its correlation functions are determined from consistency conditions. This allows to remove many assumptions and to get a more general theory that we still call Liouville theory.

The conformal bootstrap approach to Liouville theory was initiated by Tescher [82] who derived recursion relations for the 3-point function. He still relied on the existence of an action, but at the price of few assumptions it is possible to get rid of it [66].

By giving up the formulation in terms of an action one gets more freedom in defining the theory because it can be extended to cases where the action is not well-defined [71].

We will greatly follow [71, 72].

10.1 Hypothesis and setup

The main hypothesis is that the spectrum is the same as for $c \geq 25$ Liouville theory

$$p \in \mathbb{R}. \tag{10.1}$$

10.2 Teschner's recursion relations

10.2.1 Derivation

We will again use the notations (9.8)

$$2\tilde{x} = \sum_i x_i, \qquad \tilde{x}_i = \tilde{x} - x_i, \qquad i = 1, 2, 3. \tag{10.2}$$

10.2.2 First solution

10.2.3 Second solution

The second solution to the recursion relations, called $\widehat{\text{DOZZ}}$ or timelike DOZZ, was obtained by [85] and independently in [50, sec. 3, 51], but it already appears as part of the analytic continuation derived by Schomerus [76]. Finally it was shown that this function can be computed from the usual path integral with a different cycle of integration [40] and from a Coulomb gas computation in [35]. Good reviews of the properties and derivation include [40, sec. 7.1, 57, sec. 5, 72].

The 3-point function reads

$$\widehat{C}(\alpha_1,\alpha_2,\alpha_3) = \frac{2\pi}{\beta}\Big[-\pi\mu\gamma(-\beta^2)\,\beta^{2+2\beta^2}\Big]^{-(q+2\tilde{\alpha})/\beta}\,\mathrm{e}^{i\pi(q+2\tilde{\alpha})/\beta}\times$$
$$\times\,\frac{\Upsilon_\beta(\beta-q-2\tilde{\alpha})}{\Upsilon_\beta(\beta)}\prod_i\frac{\Upsilon_b(\beta-2\tilde{\alpha}_i)}{\Upsilon_b(\beta-2\alpha_i)}. \tag{10.3}$$

We refer to the appendix E.3 for the properties of the various functions.

Note that

$$\mathrm{e}^{i\pi(q+2\tilde{\alpha})/\beta} = (-1)^{(q+2\tilde{\alpha})/\beta} \tag{10.4}$$

and this cancels the minus sign in the first parenthesis of (10.3).

10.3 Crossing symmetry

10.4 Modular invariance

11 Runkel–Watts–Schomerus theories

The Runkel–Watts–Schomerus (RWS) theories exist for rational values of the central charge

$$\beta^2 = \frac{p'}{p} \in \mathbb{Q}. \tag{11.1}$$

Recall that in this case $c \leq 1$.

The first such theory was obtained by Runkel and Watts by taking carefully the limit $p \to \infty$ of the unitary minimal models M_p and it has $c = 1$ [73, 74]. It was later shown by Schomerus that this theory arises as a specific case from the analytical continuation of the DOZZ formula to the above values of the central charge [76]. As Schomerus was only interested in the rolling tachyon, he did not compute the limit explicitly for the cases $c \neq 1$, and the corresponding formulas have been provided by McElgin [57]. It was proved numerically in [72] that these theories are crossing symmetric.

The original motivation for taking this limit is that it reproduces the 3-point function computed from the $c = 1$ minisuperspace approximation. Moreover by taking the limit $\alpha_2 \to 0$ of the 3-point function the non-analytic factor gives a diagonal 2-point function, which also agrees with the minisuperspace [76]. As shown in McElgin the theories with $p = 1$ also have diagonal two-point functions. But as we explained in other sections one can identify the 2-point function from the limit of the 3-point function only if the identity is the only field with vanishing dimension.

The only unitary theory of this family is the $c = 1$ theory since the weights are such that

$$h \leq \frac{c-1}{24}, \tag{11.2}$$

and the theory contains negative weights for any central charge less than one. As it is special we call this one the Runkel–Watts model. Note that it also arises from the orbifold of the free boson [31].

11.1 Three-point function from analytical continuation

As explained in the appendix E.3.4, Υ_b functions can not be continued to $b \in i\mathbb{R}$. Nonetheless the specific combination that appear in the 3-point function admits a well-defined limit which will take the form

$$\widetilde{C}(a_1,a_2,a_3) = P(a_1,a_2,a_3)\,\widehat{C}(a_1,a_2,a_3) \tag{11.3}$$

where $\widehat{C}$ is the timelike DOZZ formula, and P is a non-analytic function whose expression depends on the domain of the a_i.

We will follow the computations from [57, sec. 6, 76].[29]

We wish to make the analytical continuation

$$b = i\beta, \qquad a_i = i\alpha_i \tag{11.4}$$

of (9.12)

$$C(a_1, a_2, a_3) = \left[\pi\mu\gamma(b^2) b^{2-2b^2}\right]^{(Q-2\tilde{a})/b} \frac{\Upsilon_b'(0)}{\Upsilon_b(2\tilde{a}-Q)} \prod_i \frac{\Upsilon_b(2a_i)}{\Upsilon_b(2\tilde{a}_i)}. \tag{11.5}$$

In the appendix we got the formula

$$\Upsilon_b(a) = \frac{H_b(a)}{\Upsilon_{ib}(-ia+ib)} = \frac{H_{-i\beta}(-i\alpha)}{\Upsilon_\beta(\beta-\alpha)} \tag{11.6}$$

with

$$H_b(x) = \exp\left[\frac{i\pi}{2}\left(x^2 + \frac{x}{b} - xb + \frac{b^2}{4} - \frac{3}{4b^2} - \frac{1}{4}\right)\right] \frac{\vartheta_1(x/b)}{\vartheta_1(1/2+1/2b^2)}, \tag{11.7}$$

and for simplicity we defined

$$\vartheta_1(x) \equiv \vartheta_1(x, \tau), \qquad \tau \equiv \frac{1}{b^2} = -\frac{1}{\beta^2}. \tag{11.8}$$

Since $tau \in \mathbb{R}$ corresponds to the boundary of analyticity of ϑ_1, we assume that τ has a small imaginary part.

We can see easily that the only difference with the timelike DOZZ (10.3) (the prime indicates that we removed few factors from the original formula)

$$\widehat{C}'(\alpha_1, \alpha_2, \alpha_3) = \Big[-\pi\mu\gamma(-\beta^2)\,\beta^{2+2\beta^2}\Big]^{-(q+2\tilde{\alpha})/\beta} \frac{\Upsilon_\beta(\beta-q-2\tilde{\alpha})}{\Upsilon_\beta(\beta)} \prod_i \frac{\Upsilon_b(\beta-2\tilde{\alpha}_i)}{\Upsilon_b(\beta-2\alpha_i)} \tag{11.9}$$

will come from contributions from H_b functions, thus we can write

$$\widetilde{C}(\alpha_1, \alpha_2, \alpha_3) = P(\alpha_1, \alpha_2, \alpha_3)\, \widehat{C}'(\alpha_1, \alpha_2, \alpha_3) \tag{11.10}$$

with $P(\alpha_i)$ gathering contributions from H_b. This function will provide the poles necessary to get a diagonal 2-point function.

The derivative of Υ_b taken at $x = 0$ is

$$\Upsilon_b'(0) = \frac{H_b'(0)}{\Upsilon_\beta(\beta)} = \exp\left[\frac{i\pi}{2}\left(\frac{b^2}{4} - \frac{3}{4b^2} - \frac{1}{4}\right)\right] \frac{\vartheta_1'(0)}{\Upsilon_\beta(\beta)} \tag{11.11}$$

using the fact that $\vartheta_1(0) = 0$ which implies $H_b(0) = 0$ (the – constant – exponential factor will cancel with the one of $\Upsilon_b(Q - \sum a_i)$).

We now try to evaluate

$$P(\alpha_1, \alpha_2, \alpha_3) = \frac{H_b'(0)}{H_b(Q - \sum a_i)} \prod_i \frac{H_b(2a_i)}{H_b(2\tilde{a}_i)}. \tag{11.12}$$

First we note that there are the same number of functions in the numerator and denominator, so factor independent of the function argument will cancel (in particular inside the exponentials). After continuing the parameters, we are left with

$$P(\alpha_1, \alpha_2, \alpha_3) = E(\alpha_1, \alpha_2, \alpha_3)\Theta(\alpha_1, \alpha_2, \alpha_3) \tag{11.13}$$

[29] Similar computations were already done in [38, 81].

with

$$\Theta(\alpha_1, \alpha_2, \alpha_3) = \frac{\vartheta_1'(0)}{\vartheta_1(q/\beta + \tilde{\alpha}/\beta)} \prod_i \frac{\vartheta_1(2\alpha_i/\beta)}{\vartheta_1(2\tilde{\alpha}_i/\beta)}, \tag{11.14a}$$

$$\ln E(\alpha_1, \alpha_2, \alpha_3) = \frac{i\pi}{\beta}(q + 4\tilde{\alpha}). \tag{11.14b}$$

Note that we can remove the minus sign in the parenthesis of (11.9) by multiplying with

$$\mathrm{e}^{\frac{i\pi}{\beta}(q+2\tilde{\alpha})} = (-1)^{(q+2\tilde{\alpha})/\beta}. \tag{11.15}$$

Our results agree completely with McElgin [57, sec. 6]: he does not have the minus sign in (11.9) such that he has

$$\ln E(\alpha_1, \alpha_2, \alpha_3) = \frac{2i\pi\tilde{\alpha}}{\beta}. \tag{11.16}$$

We observe a difference in $E(\alpha_i)$ with Schomerus [76, sec. 4] who gets

$$\ln E(\alpha_1, \alpha_2, \alpha_3) = \frac{i\pi}{\beta}(q + 2\tilde{\alpha}) - 2\pi i q\, \tilde{\alpha} + \ln \mathcal{N}_0(\beta). \tag{11.17}$$

This might come from the definition of $H_b(x)$ that Schomerus gives: the exponential inside is

$$\exp -\frac{i\pi}{2}\left(x^2 + bx - \frac{x}{b}\right). \tag{11.18}$$

According to McElgin [57, p. 23], the 3-point function is non-trivial only when

$$\tau = \frac{1}{b^2} = -\frac{1}{\beta^2} = r + i\varepsilon, \qquad r \in \mathbb{Q}. \tag{11.19}$$

The latter can be obtained by letting

$$\beta \longrightarrow \beta + i\varepsilon, \qquad \beta^2 = \frac{p'}{p} \tag{11.20}$$

since

$$\tau = -\frac{1}{(\beta + i\varepsilon)^2} \sim -\frac{1}{\beta^2 + i\varepsilon} \sim -\frac{1}{\beta^2} + i\varepsilon. \tag{11.21}$$

11.2 Limit from minimal models

11.3 Continuous orbifold

12 Complex Liouville theory

In section 10 we have considered the conformal bootstrap of Liouville theory from an algebraic point of view. Then the reality of the Liouville mode and the condition $c \geq 25$ could be relaxed. We will explore in this section the properties of this theory.

12.1 Lagrangian study

The properties of the complex Liouville Lagrangian were studied in [40]. This paper is agnostic concerning the spectrum and computes correlation functions for any values of the conformal dimensions.

Classical solutions and minisuperspace analysis can be found in [87, sec. 3].

12.2 $c \leq 1$ Liouville theory

We wish to distinguish $c \leq 1$ Liouville from timelike Liouville (which is the topic of section 13) because they do not arise in the same contexts and their spectrum might be different.

We recall that the spectrum of $c \leq 1$ Liouville is $p \in \mathbb{R}$.

This theory received an interpretation in terms of a loop model in [43]: it was shown that the 3-point functions of both models agree.

12.2.1 Correlation functions

The 2-point function can be obtained by an analytical continuation from (9.23) [40, sec. 7.1, 81, sec. 3]

$$\widehat{G}(\alpha) = -\frac{1}{\beta^2}\Big[-\pi\mu\gamma(-\beta^2)\Big]^{-(q+2\alpha)/\beta} \mathrm{e}^{i\pi(q+2\alpha)/\beta}\gamma(\beta^2 - 2\alpha\beta)\gamma(1/\beta^2 + 2\alpha/\beta - 1). \quad (12.1)$$

As explained in section 10 the correct 3-point function for $c \leq 1$ Liouville is given by (10.3).

The limit where one of the dimension vanishes reads [40, sec. 7.1, 81]

$$\widehat{C}(\alpha_1, 0, \alpha_3) = \lim_{\alpha_2 \to 0} \widehat{C}(\alpha_1, \alpha_2, \alpha_3) = \frac{2\pi}{\beta}\Big[-\pi\mu\gamma(-\beta^2)\,\beta^{2+2\beta^2}\Big]^{-(q+\alpha_1+\alpha_3)/\beta} \mathrm{e}^{i\pi(q+\alpha_1+\alpha_3)/\beta} \times$$
$$\times\, \frac{\Upsilon_\beta(\beta - q - \alpha_1 - \alpha_3)\Upsilon_\beta(\beta - \alpha_1 - \alpha_3)\Upsilon_\beta(\beta - \alpha_1 + \alpha_3)\Upsilon_\beta(\beta + \alpha_1 - \alpha_3)}{\Upsilon_\beta(\beta)^2\,\Upsilon_\beta(\beta - 2\alpha_1)\Upsilon_\beta(\beta - 2\alpha_3)} \quad (12.2)$$

because the denominator has no poles, and the numerator has no zeros. This signals that that the field with vanishing dimension $\Delta = 0$ is *not the identity* and that the quantity $\widehat{C}(\alpha_1, 0, \alpha_3)$ is a genuine 3-point function for this field and two other fields. In particular the full 3-point function still depends on the position z_2 of the second operator.

For equal momenta $\alpha_1 = \alpha_3$ the above formula gives correctly the 2-point function (12.1)

$$\widehat{C}(\alpha, 0, \alpha) = \frac{2\pi}{\beta}\,\widehat{G}(\alpha). \quad (12.3)$$

An important quantity is the reflection coefficient $R(\alpha)$: it can be extracted from the 3-point function by computing $C(q - \alpha_1, \alpha_2, \alpha_3)$ such that we do not need the 2-point function to compute it.

12.2.2 The fake-identity operator

In this section we want to comment the status of the field V_0 which is called *fake identity* and which corresponds to zero dimension $\alpha = \Delta = 0$. We stress again that it is different from the identity

$$V_0 \neq \mathrm{id}\,. \quad (12.4)$$

In theories one encouters generally $\widehat{C}(\alpha_1, 0, \alpha_3)$ is diagonal, meaning that $\widehat{C}(\alpha_1, 0, \alpha_3) = 0$ if $\alpha_1 \neq \alpha_3$, and reproduces the 2-point function. This fact is due to the decoupling of null vectors (recall that this decoupling implies differential equations on the correlation functions). Hence one can conclude that the operator V_0 is non-degenerate.

On the other hand the 2-point function is always diagonal because it involves fields with equal conformal dimensions.

In [43] this object was interpreted in the equivalent loop model as a *marking operator*.

A similar problem occurs for the limit $p \to \infty$ of minimal models M_∞ (a non-rational CFT) where the Hilbert space does not contain any state with conformal weight 0 (which means that there are no vacuum state) [73, 74]. There it is possible to define correlation functions for the identity and the stress–energy tensor consistently even if the Hilbert space does not contain the states corresponding to the field; what is important is that the Hilbert space contains a state for every possible physical state of the system, and this completeness may be used to insert the full sum over the states. Note that only a field of conformal weight 0 can have a non-vanishing one-point function, and this makes the OPE a bit subtle; a way out is to introduce smeared field over the parameter value (this also allows to define the the identity state as a limit of states, but which is not in the Hilbert space). In this context the identity can be defined as [57, sec. 6, 73, 74]

$$\mathrm{id} = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} V_{-i\alpha}. \tag{12.5}$$

13 Timelike Liouville theory

13.1 Definition

For $c_m \geq 25$, quantities become purely imaginary: it is possible to make an analytic continuation of the parameters [40, sec. 7, 57, sec. 3]. The theory can still be Lorentzian or Euclidean.

We define the following quantities [30]

$$\phi = i\chi, \qquad Q = iq, \qquad b = -i\beta, \qquad a = -i\alpha, \qquad p = -i\omega. \tag{13.1}$$

The formula for q in terms of β, for the central charge and for the weight become

$$q = \frac{1}{\beta} - \beta, \qquad c = 1 - 6q^2, \qquad \Delta_\alpha = \alpha(q + \alpha). \tag{13.2}$$

The Euclidean action is [57, sec. 3]

$$S_L = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \sqrt{h} \Big(- h^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - qR\chi + 4\pi\mu \, \mathrm{e}^{2\beta\chi} \Big). \tag{13.3}$$

The name "timelike" comes from the negative sign in front of the kinetic term: the action is not positive definite.

It may also be useful to not continue the Liouville mode and to work with ϕ: then the action is complex but we do not have to worry about this negative kinetic term. Similarly sometimes we will continue to use the original parameters even if they are complex.

Looking at the definition of the Coulomb gas (sec. C.5) we see that timelike Liouville is very close: we just need to continue also ϕ.

This provides an interesting model for $4d$ quantum gravity, since the conformal factor comes with the "wrong sign" [34, p. 19]. The second point is that this model is closer to $4d$ gravity in the sense that $c_m \geq 25$ instead of $c_m \leq 1$, the last case being further from our intuition on "normal" matter [67, p. 125, 8].

From the string theory interpretation, for $\beta = 1$ and hence $q = 0$ the potential corresponds to a closed string tachyon growing exponentially with time (note that at $X^0 \equiv \chi \to -\infty$ the tachyon is small and we recover flat space) [38, 81]. Moreover since this is a time dependent background there is closed string pair production, thus timelike Liouville theory provides a toy model to study tachyon condensation.

Timelike Liouville theory was used as the matter CFT in [50, 51].

[30] In Harlow et al. [40, sec. 7] they define $a = i\alpha$, which implies that $\Delta_\alpha = \alpha(\alpha - q)$. McElgin is noting $q \equiv \Lambda$ [57].

Interesting comments on the continuation also appeared in [38].

Since we had $b \in (0,1]$ we also have $\beta \in (0,1]$, implying $c \leq 1$.

The main difference with (spacelike) Liouville theory with $c \leq 1$ defined in section 10 comes from the spectrum which is different (the proposed 3-point function being identical).

13.2 Equations of motion and semi-classical limit

The equation of motion can be obtained from (13.3) or by an analytical continuation of the spacelike result (6.24)

$$qR[h] - 2\Delta\chi = 8\pi\mu\beta\, e^{2\beta\chi}. \tag{13.4}$$

The semi-classical limit corresponds to

$$\beta \to 0, \qquad c_m \to \infty, \qquad c \to -\infty \tag{13.5}$$

and we define

$$\chi_{\text{cl}} = \beta\chi, \qquad \lambda_{\text{cl}} = \mu\beta^2 \tag{13.6}$$

(noting that $\chi_{\text{cl}} = \phi_{\text{cl}}$ since $b\phi = \beta\chi$, but $\lambda_{\text{cl}} = -\mu_{\text{cl}}$). The equation of motion becomes

$$R[\, e^{2\chi_{\text{cl}}} h] = (R[h] - 2\Delta\chi_{\text{cl}})\, e^{-2\chi_{\text{cl}}} = 8\pi\lambda_{\text{cl}}. \tag{13.7}$$

Solutions correspond to space of constant *positive* curvature [57, sec. 3, 40, sec. 7].

13.3 Classical solutions

13.4 States

Again we start by discussing the spectrum of timelike Liouville, and only later we will derive it.

Considering states of the form

$$\alpha = -\frac{q}{2} + i\omega, \qquad h_\alpha = -\frac{q^2}{4} - \omega^2 = \frac{c-1}{24} - \omega^2, \tag{13.8}$$

there exists three categories of states:

- continuous real
$$\omega \in \mathbb{R}, \qquad h \geq \frac{c-1}{24}; \tag{13.9}$$

- discrete imaginary
$$\omega \in \mathbb{D} = i\left(\frac{\mathbb{N}\beta}{2} + \frac{\mathbb{N}}{2\beta}\right), \qquad h \leq \frac{c-1}{24}; \tag{13.10}$$

- continuous imaginary
$$\omega \in i\mathbb{R} - \mathbb{D}, \qquad h \leq \frac{c-1}{24}. \tag{13.11}$$

States with $\omega \in i\mathbb{R}$ are called magnetic in the language of [52, sec. 2.2], and the states $\omega \in \mathbb{D}$ already play a special role. They are not bounded from below.

All states appear to be normalizable in the minisuperspace.

The sector of states $\omega \in i\mathbb{R}$ in the theory $c = 1$ (corresponding to Runkel–Watts) is unitary. Otherwise since $c < 1$ the same sector contains some states with negative weights, but they are bounded from below.

13.5 Minisuperspace

Minisuperspace study can be found in [27, 49, 57, 76][31].

We consider the constant mode of space of the timelike Liouville field

$$\chi(t,\sigma) = \chi_0(t). \tag{13.12}$$

Taking a flat fiducial metric and performing a spacetime Wick rotation of the action (13.3), we obtain the minisuperspace action

$$S_L = \int \mathrm{d}t\, L = \int \mathrm{d}t\,\Big(-\frac{\dot\chi_0^2}{2} - 2\pi\mu\, \mathrm{e}^{2\beta\chi_0}\Big). \tag{13.13}$$

The conjugate momentum is

$$\pi_0 = \frac{\delta S}{\delta\dot\chi_0} = -\dot\chi_0 \tag{13.14}$$

which provides the Hamiltonian

$$H_0 = \pi_0\chi_0 - L = -\frac{\pi_0^2}{2} + 2\pi\mu\, \mathrm{e}^{2\beta\chi_0} \tag{13.15}$$

13.5.1 Canonical quantization

Including the zero-point energy and proceeding to the canonical quantization

$$\pi_0 = -i\,\frac{\mathrm{d}}{\mathrm{d}\chi_0} \tag{13.16}$$

we obtain the Hamiltonian operator

$$H_0 = \frac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}\chi_0^2} + 2\pi\mu\, \mathrm{e}^{2\beta\chi_0}. \tag{13.17}$$

We are now able to consider the eigenfunctions of this operator

$$H_0\psi_\omega = -2\omega^2\psi_\omega. \tag{13.18}$$

We note that ω corresponds to the analytical continuation $p = -i\omega$. But due to the different sign the Hamiltonian is not self-adjoint [27, 30, 49, 57]. For this reason the naive solution that is obtained on the line of the spacelike computation is not sufficient for selecting a set of orthogonal states.

Using the change of variables

$$\ell = \mathrm{e}^{\beta\chi_0} \tag{13.19}$$

the Hamiltonian reads

$$\Big(\ell^2\,\frac{\mathrm{d}}{\mathrm{d}\ell^2} + \ell\,\frac{\mathrm{d}}{\mathrm{d}\ell} + 4(\hat\mu\,\ell^2 + \hat\omega^2)\Big)\,\psi_\omega = 0 \tag{13.20}$$

where we defined

$$\hat\mu = \frac{\pi\mu}{b^2}, \qquad \hat\omega = \frac{\omega}{b}. \tag{13.21}$$

[31] The first analysis of timelike Liouville has been done in [30], but with a different point of view, and hence different results.

13.5.2 Naive wave functions

In equation (13.20) one recognize the Bessel equation (E.15) (see appendix E.2.1 for more details). Depending on the value of $2i\hat{\omega}$ there are three possible solutions[32]

$$\psi_\omega(\ell) = \alpha_\omega\, J_{2i\hat{\omega}}(2\sqrt{\hat{\mu}}\,\ell) + \beta_\omega\, J_{-2i\hat{\omega}}(2\sqrt{\hat{\mu}}\,\ell), \qquad \hat{\omega} \in \mathbb{R}, \tag{13.22a}$$

$$\psi_p(\ell) = \alpha_p\, J_{2\hat{p}}(2\sqrt{\hat{\mu}}\,\ell) + \beta_p\, J_{-2\hat{p}}(2\sqrt{\hat{\mu}}\,\ell), \qquad \hat{p} \equiv i\hat{\omega} \in \mathbb{R} - \frac{\mathbb{Z}}{2}, \tag{13.22b}$$

$$\psi_n(\ell) = \alpha_n\, J_{\hat{n}}(2\sqrt{\hat{\mu}}\,\ell) + \beta_n\, Y_{\hat{n}}(2\sqrt{\hat{\mu}}\,\ell), \qquad \hat{n} \equiv 2i\hat{\omega} \equiv 2\hat{p} \in \mathbb{Z}, \tag{13.22c}$$

and we have distinguished the types of wave functions by changing the index. We consider only the case $n > 0$ due to the relations

$$J_{-\hat{n}}(\ell) = (-1)^{\hat{n}} J_{\hat{n}}(\ell), \qquad Y_{-\hat{n}}(\ell) = (-1)^{\hat{n}} Y_{\hat{n}}(\ell), \tag{13.23}$$

and also only the case $\hat{\omega}, \hat{p} > 0$ since the other can be simply obtained. The various hatted quantities are

$$\hat{\omega} = \frac{\omega}{b}, \qquad \hat{p} = \frac{p}{b}, \qquad \hat{n} = \frac{n}{b}. \tag{13.24}$$

For $\ell \to \infty$ one has

$$\psi_\omega(\ell) \sim_\infty \sqrt{\frac{1}{\pi\sqrt{\hat{\mu}}\,\ell}} \left[\alpha_\omega \cos\left(2\sqrt{\hat{\mu}}\,\ell - i\pi\hat{\omega} - \frac{\pi}{4}\right) + \beta_\omega \cos\left(2\sqrt{\hat{\mu}}\,\ell + i\pi\hat{\omega} - \frac{\pi}{4}\right)\right], \tag{13.25a}$$

$$\psi_p(\ell) \sim_\infty \sqrt{\frac{1}{\pi\sqrt{\hat{\mu}}\,\ell}} \left[\alpha_p \cos\left(2\sqrt{\hat{\mu}}\,\ell - \pi\hat{p} - \frac{\pi}{4}\right) + \beta_p \cos\left(2\sqrt{\hat{\mu}}\,\ell + \pi\hat{p} - \frac{\pi}{4}\right)\right], \tag{13.25b}$$

$$\psi_n(\ell) \sim_\infty \sqrt{\frac{1}{\pi\sqrt{\hat{\mu}}\,\ell}} \left[\alpha_n \cos\left(2\sqrt{\hat{\mu}}\,\ell - \frac{n\pi}{2} - \frac{\pi}{4}\right) + \beta_n \sin\left(2\sqrt{\hat{\mu}}\,\ell - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]. \tag{13.25c}$$

Due to the factor $\ell^{-1/2}$ all these functions tends to zero[33] as $\ell \to \infty$ and this does not give any conditions on the coefficients.

The behaviour of ψ_ω near zero reads

$$\psi_\omega \sim_0 \alpha_\omega\, \frac{\hat{\mu}^{i\hat{\omega}}}{\Gamma(1+2i\hat{\omega})}\, \ell^{2i\hat{\omega}} + \beta_\omega\, \frac{\hat{\mu}^{-i\hat{\omega}}}{\Gamma(1-2i\hat{\omega})}\, \ell^{-2i\hat{\omega}} \tag{13.26a}$$

$$= \alpha_\omega\, \frac{\hat{\mu}^{i\hat{\omega}}}{\Gamma(1+2i\hat{\omega})}\, \mathrm{e}^{2i\omega\chi_0} + \beta_\omega\, \frac{\hat{\mu}^{-i\hat{\omega}}}{\Gamma(1-2i\hat{\omega})}\, \mathrm{e}^{-2i\omega\chi_0}. \tag{13.26b}$$

In order to normalize the wave functions as a plane wave[34] $\mathrm{e}^{2i\omega\chi_0}$ we take

$$\alpha_\omega = \Gamma(1+2i\hat{\omega})\, \hat{\mu}^{-i\hat{\omega}}, \qquad \beta_\omega = \alpha_\omega\, \gamma_\omega \tag{13.27}$$

where we have rescaled β_ω in order to normalize the full wave functions. With these coefficients one has

$$\psi_\omega(\ell) = \Gamma(1+2i\hat{\omega})\, \hat{\mu}^{-i\hat{\omega}} \left(J_{2i\hat{\omega}}(2\sqrt{\hat{\mu}}\,\ell) + \gamma_\omega\, J_{-2i\hat{\omega}}(2\sqrt{\hat{\mu}}\,\ell)\right) \tag{13.28}$$

[32] The first two cases are formally the same, but we distinguish them for facilitating the analysis in later steps.

[33] But note that there are real exponentials for $\hat{\omega} \in \mathbb{R}$, and since this is growing faster than a power-law, the limit $\ell, \hat{\omega} \to \infty$ might be diverging; what does this mean?

[34] Note that if one interpret the Liouville mode as the time direction [27, 76] then ω corresponds to the energy and incoming waves would be $\mathrm{e}^{-2i\omega\phi_0}$, but we prefer to work with a plus sign.

with the asymptotic

$$\psi_\omega \sim_0 \mathrm{e}^{2i\omega\chi_0} + R_0(\omega)\,\mathrm{e}^{-2i\omega\chi_0}, \qquad R_0(\omega) = \gamma_\omega\,\frac{\Gamma(1+2i\hat{\omega})}{\Gamma(1-2i\hat{\omega})}\,\hat{\mu}^{-2i\hat{\omega}}. \tag{13.29}$$

There is no mathematical criteria to fix γ_ω, and one needs to additional principle to fix its value. This is a consequence of the attractive potential which gives an Hamiltonian unbounded from below [27, 76].

One could set $\gamma_\omega = 1$ to get a real wave functions. In [38, 80, 81] (see also [87, p. 24, 76, sec. 2]) the value was fixed to

$$\gamma_\omega = -\,\mathrm{e}^{2\pi\hat{\omega}} \tag{13.30}$$

for matching the analytic continuation of spacelike wave functions, using the formula (E.34). The interpretation is that there is only one outgoing wave.

Sometimes the choice

$$\alpha_\omega = \sqrt{\frac{2\omega}{\sinh 2\pi\omega}} \tag{13.31}$$

is made.

Now we turn to ψ_p whose expansion reads

$$\psi_p \sim_0 \alpha_p\,\frac{\hat{\mu}^{\hat{p}}}{\Gamma(1+2\hat{p})}\,\ell^{2\hat{p}} + \beta_p\,\frac{\hat{\mu}^{-\hat{p}}}{\Gamma(1-2\hat{p})}\,\ell^{-2\hat{p}}. \tag{13.32}$$

As $\ell \to 0$ for $p > 0$ the second term blows up and we need to take

$$\beta_p = 0. \tag{13.33}$$

Then α_p can be fixed by normalizing the integral

$$\int_0^\infty \frac{\mathrm{d}\ell}{\ell}\,\psi_p(\ell)^2 = \frac{\alpha_p}{4\hat{p}^2} = 1 \Longrightarrow \alpha_p = 2\sqrt{\hat{p}} \tag{13.34}$$

using (E.25). Hence the wave function is

$$\psi_p(\ell) = 2\sqrt{\hat{p}}\;J_{2\hat{p}}(2\sqrt{\hat{\mu}}\,\ell). \tag{13.35}$$

Finally we need to consider ψ_n for which

$$\psi_n(\ell) \sim_0 \alpha_n\,\frac{\hat{\mu}^{\hat{n}/2}}{\hat{n}!}\,\ell^{\hat{n}} - \beta_n\,\frac{(\hat{n}-1)!}{\pi}\,\hat{\mu}^{-\hat{n}/2}\,\ell^{-\hat{n}}. \tag{13.36}$$

For $\hat{n} > 0$ and $\ell \to 0$ the second term is divergent which forces to take

$$\beta_n = 0. \tag{13.37}$$

Then by asking for a unit normed one can fix α_n

$$\int_0^\infty \frac{\mathrm{d}\ell}{\ell}\,\psi_n(\ell)^2 = \frac{\alpha_n}{2\hat{n}^2} = 1 \Longrightarrow \alpha_n = \sqrt{2\hat{n}}. \tag{13.38}$$

giving the wave function

$$\psi_n(\ell) = \sqrt{2\hat{n}}\;J_{\hat{n}}(2\sqrt{\hat{\mu}}\,\ell). \tag{13.39}$$

As a conclusion we see that for $2i\hat{\omega} \in \mathbb{R}$ there is no difference between integer or non-integer parameters.

We cannot obtain a complete set of orthogonal wave functions, which is a consequence of the fact that the Hamiltonian is not self-adjoint.

13.5.3 Self-adjoint extension

For more details on the mathematical aspects, see appendix D.2.

From (D.18) the Hamiltonian (13.17)

$$H_0 = \frac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}\chi_0^2} + 2\pi\mu\,\mathrm{e}^{2\beta\chi_0} = \frac{\beta^2}{2}\left(\ell\,\frac{\mathrm{d}}{\mathrm{d}\ell}\right) + 2\hat{\mu}\,\ell^2 \tag{13.40}$$

is symmetric only if

$$\left[\frac{\mathrm{d}\chi(x)}{\mathrm{d}x}\,\psi(x)^* - \chi(x)\,\frac{\mathrm{d}\psi(x)^*}{\mathrm{d}x}\right]_{-\infty}^{\infty} = 0. \tag{13.41}$$

Once this condition is satisfied we want to find possible self-adjoint extensions. We need to determine the deficiency indices $d_\pm$ which correspond to the number of independent solutions to the equation

$$H\psi^\pm = \pm 2i\,\psi^\pm. \tag{13.42}$$

We define the roots of $i = \mathrm{e}^{\frac{i\pi}{2}}$ and $-i = \mathrm{e}^{\frac{3i\pi}{2}}$ (with some rescaling) by

$$\eta^+ = \frac{2}{\beta}\,\mathrm{e}^{\frac{3i\pi}{4}}, \qquad \eta^- = \frac{2}{\beta}\,\mathrm{e}^{\frac{i\pi}{4}}. \tag{13.43}$$

Since $\eta^\pm \notin \mathbb{Z}$ each equation has two independent solutions

$$\psi_1^\pm(\ell) = J_{\eta^\pm}(2\sqrt{\hat{\mu}}\ell), \qquad \psi_2^\pm(\ell) = J_{-\eta^\pm}(2\sqrt{\hat{\mu}}\ell). \tag{13.44}$$

The asymptotic behaviours are

$$J_{\pm\eta^\pm}(2\sqrt{\hat{\mu}}\ell) \sim_0 \ell^{\pm\eta^\pm} \sim \mathrm{e}^{\pm 2\beta\chi\,\mathrm{Re}\,\eta^\pm} \tag{13.45}$$

Since $\mathrm{Re}\,\eta^+ < 0$ and $\mathrm{Re}\,\eta^- > 0$ the solutions ψ_1^+ and ψ_2^- are blowing up as $\ell \to 0$ and they should be discarded. Hence the deficiency indices are

$$d_\pm = 1 \tag{13.46}$$

and there exists a 1-parameter self-adjoint extension. We will denote this parameter by ν_0 (and also $\hat{\nu}_0 = \nu_0/\beta$), and one has a U(1) unitary transformation

$$\psi_1^- = \mathrm{e}^{2i\pi\nu_0}\psi_2^+ \tag{13.47}$$

only if

$$\nu_0 \in [0, 1). \tag{13.48}$$

Consider first wave functions (13.35) with $p \in \mathbb{R}$. Then the condition (13.41) applied to ψ_{p_1} and ψ_{p_2} reads

$$\sin 2\pi(\hat{p}_1 - \hat{p}_2) = 0, \tag{13.49}$$

which implies that the p are quantized

$$p_n = \frac{\nu_0 + n}{2}, \qquad n \in \mathbb{N} \tag{13.50}$$

where ν_0 is identified with the smallest eigenvalue.[35] We denote the corresponding wave functions ψ_n

$$\psi_n(\ell) = \sqrt{2(\hat{n} + \hat{\nu}_0)}\;J_{\hat{n}}(2\sqrt{\hat{\mu}}\,\ell). \tag{13.51}$$

[35] This identification can be justified by a rigorous construction of the domains [27].

and they are correspond to the functions (13.39) shifted by ν_0, and one recovers these states for $\nu_0 = 0$. Note that these states form an orthogonal set from the relation (E.27)

$$\int_0^\infty \frac{\mathrm{d}\ell}{\ell}\, \psi_{n_1}(\ell)\psi_{n_2}(\ell) = \delta_{n_1,n_2}. \tag{13.52}$$

Let's turn to the wave functions with $\omega \in \mathbb{R}$. Then the condition (13.41) applied to ψ_ω and to ψ_n corresponds to

$$\gamma_\omega = \frac{\sinh \pi(\hat{\omega} + i\hat{\nu}_0)}{\sinh \pi(\hat{\omega} - i\hat{\nu}_0)}. \tag{13.53}$$

For $\nu_0 = 0$ one gets $\gamma_\omega = 1$. It is then possible to check that the condition (13.41) with ψ_{ω_1} and ψ_{ω_2} is fulfilled.[36] One can also check that these wave functions form are orthogonal

$$\int_0^\infty \frac{\mathrm{d}\ell}{\ell}\, \psi_{\omega_1}(\ell)^* \psi_{\omega_2}(\ell) = \delta(\omega_1 - \omega_2). \tag{13.54}$$

Finally for the states to be all orthogonal we need

$$\int_0^\infty \frac{\mathrm{d}\ell}{\ell}\, \psi_n(\ell)\psi_\omega(\ell) = 0. \tag{13.55}$$

A last check is that the states $\{\psi_\omega, \psi_n\}$ is complete

$$\sum_{n=0}^\infty \psi_n(\ell)\psi_n(\ell') + \int_0^\infty \mathrm{d}\omega\; \psi_\omega(\ell)\psi_\omega(\ell') = \delta(\ell - \ell'). \tag{13.56}$$

Hence the states of the form

$$p \in \mathbb{R} - \frac{\nu_0 + \mathbb{Z}}{2} \tag{13.57}$$

are excluded from the physical spectrum.

13.5.4 States

From the previous discussion it appears that the complete set of physical states is given by the wave functions ψ_ω and ψ_n. An important difference with the spacelike case is that all three types of wave functions are normalizable.[37]

13.5.5 Correlation functions

13.6 Correlation functions

First attempts to compute correlation functions can be found in [81], but the authors got only an expression of the 3-point function for $\beta = 1$ ($q = 0, c = 1$) by trying to continue analytically from the spacelike formula, while the 2-point function is given only for $\alpha = -q/2 + i\omega$.

The main question is which 3-point function one should use. Indeed if some quantities are obtained from spacelike Liouville by analytic continuation, the 3-point function can not be obtained by direct analytical continuation from (9.12) because this formula acquires an infinite accumulation of poles for $b \in i\mathbb{R}$ [81]. The point is that models with $c \leq 1$ does not depend smoothly on c, and the quantities are not analytic in a so we can not simply

[36] In fact only the condition at $\ell = \infty$ can be checked directly as the wave functions are only delta-function normalizable at $\ell = 0$, and they needs to be smeared with some distribution [27].

[37] In particular ψ_ω is normalizable only because it is the sum of two Bessel functions.

set $a = i\alpha$ [76]. At the level of the functions involved we can trace this to the fact that the Barnes double Γ function is not defined for $b \in i\mathbb{R}$ except for specific values of b. On the other hand the 2-point function and the Teschner's recurrence relations can be continued since they only involve normal Γ functions.

Hence there is two possibilities:

1. Use the second solution (10.3) to Teschner's relations. This formula will be valid for all values of the parameters, but it is difficult to interpret the fake identity and to satisfy crossing symmetry [72].

2. Use the analytic continuation (11.10) of RWS theories. There is no fake identity but the theory is defined only for some values of the parameters [40, p. 74].

14 Effective actions for $2d$ quantum gravity

Liouville effective action describes universally the coupling of conformal matter to gravity, but this is not anymore the case for more general matter [19, 20]. Other effective actions may appear, such as the Mabuchi and the Aubin–Yau actions.

To conform with the notation of [5, 19, 20] we change notations for this section: we will denote by g_0 the metric in the conformal gauge

$$g = \mathrm{e}^{2\phi} g_0. \tag{14.1}$$

Quantities constructed from the metric g_0 will have an index 0; for example we denote by A and A_0 the area measured with the metric g and g_0 respectively [38].

14.1 General properties

The effective action is defined by the relation

$$Z[g] = \mathrm{e}^{-S_{\text{eff}}[g_0, g]}\, Z[g_0] \tag{14.2}$$

where we write the dependence in ϕ in terms of g. In presence of non-conformal matter the effective action is not given by Liouville anymore, and it may be non-local.

This action is antisymmetric

$$S_{\text{eff}}[g, g'] = -S_{\text{eff}}[g', g] \tag{14.3}$$

and satisfy the cocycle identity

$$S_{\text{eff}}[g, g''] = S_{\text{eff}}[g, g'] + S_{\text{eff}}[g', g'']. \tag{14.4}$$

Other properties such as the relation with the trace of the energy tensor can be found in [19, sec. 2.2].

14.2 Kähler potential

In two dimensions every manifold is Kähler and the Kähler potential is sufficient to construct the full metric. For this reason it should be possible to trade the Liouville mode for the Kähler potential [5, 19, 20].

[38] In the main text the area associated to g_0 was denoted by A.

The Kähler potential K is obtained from the relation

$$e^{2\phi} = \frac{A}{A_0} \left(1 - \frac{1}{2} A_0 \Delta_0 K \right). \tag{14.5}$$

For a given ϕ this relation defines the pair (A, K) uniquely (up to constant shift of K), and positivity of the metric implies the inequality

$$\Delta_0 K < \frac{2}{A_0}. \tag{14.6}$$

The path integral measure is determined in [5].

14.3 Effective actions

The list of all interesting functionals and where they do appear is given in [19, sec. 3].

14.3.1 The area action

This is the simplest building block for the gravity action: if the action contains a function of the area $f(A)$, then the associated effective action will be [19, sec. 3.1.1]

$$S_f[g_0, g] = f(A) - f(A_0) \tag{14.7}$$

and the corresponding trace of the energy tensor is

$$t_f = 4\pi f'(A). \tag{14.8}$$

14.3.2 Liouville action

We recall the Liouville action without cosmological constant (5.15) (denoted previously by s_L) [5, sec. 2.2.1, 19, sec. 3.1.2]

$$S_L[g_0, g] = \frac{1}{4\pi} \int d^2\sigma \, \sqrt{h} \Big(h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + R_h \phi \Big). \tag{14.9}$$

The trace is

$$t_L = \frac{R}{2}. \tag{14.10}$$

14.3.3 Mabuchi action

The Mabuchi action reads [5, sec. 2.2.2, 19, sec. 3.1.3]

$$S_M[g_0, g] = \int d^2\sigma \, \sqrt{g_0} \left[-\pi\chi K \Delta_0 K + \left(\frac{4\chi}{A_0} - R_0 \right) K + \frac{4}{A} \, \phi \, e^{2\phi} \right], \tag{14.11}$$

χ being the Euler number of the surface. This action is invariant under constant shift of K. Note that the last exponential term also appears in logarithmic Liouville theory [4].

The energy tensor trace is

$$t_M = \frac{8\pi}{A} (\psi + 1) \tag{14.12}$$

where ψ is the Ricci potential

$$\Delta\psi = R - \frac{4\pi\chi}{A}. \tag{14.13}$$

The equation of motion for K is

$$R = \frac{4\pi\chi}{A} \tag{14.14}$$

and classical solutions correspond to metric of constant curvature, similarly to Liouville action.

The authors of [5, 19, 20] show that the effective action for a massive field is given by the Mabuchi action to leading order in a smal mass expansion.

Note that pure Mabuchi is not perturbatively renormalizable but it is believed that the theory admits a non-trivial UV fixed point [5, sec. 3.3].

14.3.4 Aubin–Yau action

The Aubin–Yau functional reads [19, sec. 3.1.4]

$$S_{AY}[g_0, \phi] = -\int \mathrm{d}^2\sigma \, \sqrt{g_0} \, \left(\frac{1}{4} \, K \Delta_0 K - \frac{K}{A_0} \right). \tag{14.15}$$

This is not a function of the metric g because it is not invariant under shift of K. But we can add a term proportionnal to K in order to build an invariant action.

15 Mabuchi action

We recall that Mabuchi action is given by (14.11)

$$S_M = \int \mathrm{d}^2\sigma \, \sqrt{g_0} \left[-2\pi(1-g) K \Delta_0 K + \left(\frac{8\pi(1-g)}{A_0} - R_0 \right) K + \frac{4}{A} \, \phi \, \mathrm{e}^{2\phi} \right] \tag{15.1}$$

where

$$\mathrm{e}^{2\phi} = \frac{A}{A_0} \left(1 - \frac{1}{2} \, A_0 \Delta_0 K \right). \tag{15.2}$$

Using this expression in the action we obtain a form without any ϕ

$$\begin{aligned} S_M = \int \mathrm{d}^2\sigma \, \sqrt{g_0} \, \bigg[& \pi\chi \, g_0^{\mu\nu} \partial_\mu K \partial_\nu K + \left(\frac{4\pi\chi}{A_0} - R_0 \right) K \\ & + \frac{2}{A_0} \left(1 - \frac{1}{2} \, A_0 \Delta_0 K \right) \ln \frac{A}{A_0} \left(1 - \frac{1}{2} \, A_0 \Delta_0 K \right) \bigg] \end{aligned} \tag{15.3}$$

where we have integrated by part the kinetic term.

15.1 Critical exponents

We consider the 1-loop effective action [5, sec. 2.2.3]

$$S_{\text{eff}}[g_0, g] = \frac{\kappa^2}{6} \, S_L[g_0, g] + \beta^2 \, S_M[g_0, g] \tag{15.4}$$

where $\kappa^2 = c_L$ for conformal matter.

It is natural to define first the partition function at fixed area

$$Z_{\text{eff}}[A] = \frac{1}{\sqrt{A}} \int \mathrm{d}K \; \mathrm{e}^{-S_{\text{eff}}} \tag{15.5}$$

and then the gravitational partition function

$$Z = \int \mathrm{d}A\; \mathrm{e}^{-\mu A} Z_{\text{eff}}[A]. \tag{15.6}$$

Here we define the susceptibility by

$$Z[A] \sim \mathrm{e}^{\gamma - 3}. \tag{15.7}$$

At tree level it is given by

$$\gamma_{\text{tree}} = (g-1)\frac{\kappa^2}{6} - 2\beta^2 \tag{15.8}$$

while at the next order it is

$$\gamma_{\text{1-loop}} = (g-1)\frac{\kappa^2}{6} - 2\beta^2 + \frac{19-7g}{6} - \frac{12\beta^2}{\kappa^2}. \tag{15.9}$$

Example 15.1 (Massive scalar fields) We consider the action for a massive scalar field with conformal coupling [20, p. 7]

$$S_m = \frac{1}{4\pi}\int \mathrm{d}^2\sigma \sqrt{g}\,(g^{\mu\nu}\partial_\mu X \partial_\nu X + qRX + m^2 X^2). \tag{15.10}$$

We add another action of conformal matter with central charge c_m.

We will be interested in the effective action to leading order in a small mass expansion (details on the conditions are given in the above paper). We have

$$\kappa^2 = \frac{25 - 3q^2 - c_m}{6}, \qquad \beta^2 = \begin{cases} \frac{q^2}{4} & q \neq 0, \\ \frac{m^2 A}{16\pi} & q = 0. \end{cases} \tag{15.11}$$

---

16 Boundary Liouville theory

16.1 Action

The action of the boundary Liouville theory reads [52, sec. 2.1]

$$S = \frac{1}{4\pi}\int_{\mathcal{M}} \mathrm{d}^2\sigma \sqrt{h}\Big(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + QR\phi + 4\pi\mu\,\mathrm{e}^{2b\phi}\Big) + \frac{1}{2\pi}\int_{\partial\mathcal{M}} \mathrm{d}\sigma \sqrt{h}\Big(QK\phi + 2\pi\lambda\,\mathrm{e}^{b\phi}\Big) \tag{16.1}$$

where K is the curvature of the boundary $\partial\mathcal{M}$ and λ is the boundary cosmological constant. The normalization is such that

$$\int_{\mathcal{M}} R + 2\int_{\partial\mathcal{M}} K = 4\pi\chi. \tag{16.2}$$

The bulk and boundary cosmological constants couple respectively to the area of the space-time and to the length of the boundary

$$A = \int_{\mathcal{M}} \mathrm{d}^2\sigma\; \mathrm{e}^{2b\phi}, \qquad \ell = \int_{\partial\mathcal{M}} \mathrm{d}\sigma\; \mathrm{e}^{b\phi} \tag{16.3}$$

In order to work in complex coordinates, one maps the disc $\mathcal{M}$ the the upper-half plane, for which $\operatorname{Im} z \geq 0$ and the boundary is given by $\operatorname{Im} z = 0$; we will denote by x the coordinates on the boundary.

Usually Neumann boundary conditions are imposed to the Liouville field

$$i(\partial - \bar{\partial})\phi = 4\pi\lambda\,\mathrm{e}^{b\phi(x)}. \tag{16.4}$$

16.2 Correlation functions

See [52].

Part IV
Applications

17 Cosmology

Many questions are still open in quantum cosmology and one may hope to address them in the simpler context of $2d$ quantum gravity. In this case one needs to consider timelike Liouville theory because it is closer to the $4d$ gravity; in particular it admits de Sitter solutions. The first applications of Liouville to cosmology can be found in [8, 67].

More recently timelike Liouville theory has been used to study inflation in $2d$ [55, 60, 87]. Several models of inflation have been studied in [87]. It was then shown in [55, 60] that the scalar sector of $4d$ perturbations can be exactly matched to $2d$ perturbations.

Part V
Appendices

A Conventions

A.1 General notations

We denote by $\mu = 0, 1$ the 2-dimensional indices and $a = 0, \ldots, d-1$ the d-dimensional ones.

Flat euclidean and Lorentzian metrics are denoted by

$$\text{euclidean:} \qquad \delta_{\mu\nu} = \operatorname{diag}(1, 1) \tag{A.1a}$$

$$\text{Lorentzian:} \qquad \eta_{\mu\nu} = \operatorname{diag}(-1, 1) \tag{A.1b}$$

Given an action S with metric $g_{\mu\nu}$ we define the stress–energy (or energy–momentum) tensor by

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \tag{A.2}$$

The contravariant form is obtained from

$$T^{\mu\nu} = \frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} \tag{A.3}$$

(note the minus sign of difference).

We define by g the absolute value of the determinant of the metric $g_{\mu\nu}$

$$g = |\det g_{\mu\nu}| = \pm \det g_{\mu\nu} \tag{A.4}$$

where the plus and minus signs correspond respectively to Euclidean and Lorentzian signatures.

We will also denote

$$\dot{\phi} = \partial_\tau \phi, \qquad \phi' = \partial_\sigma \phi. \tag{A.5}$$

A.2 Complex coordinates

In the case of Lorentz signature, we have $\tau = it$, such that

$$z = \tau + i\sigma = i(t + \sigma) \tag{A.6}$$

and we see that $z, \bar{z}$ correspond to light-cone coordinates $\sigma^\pm$ times a factor i.

A.3 Light-cone coordinates

Light-cone coordinates are defined by

$$\sigma^\pm = t \pm \sigma. \tag{A.7}$$

The line element is

$$\mathrm{d}s^2 = -\mathrm{d}\sigma^+ \mathrm{d}\sigma^- \tag{A.8}$$

such that the metric and its inverse are

$$\eta = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \eta^{-1} = -2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.9}$$

A.4 Wick rotation

We consider a d-dimensional curved spacetime with metric $g_{\mu\nu}$ and coordinates x^μ.

Wick rotation is a useful procedure which allows to replace a Lorentzian metric by an euclidean metric in order to avoid complications that may arise in the first case due to the fact that it is not positive-definite.

We can continue analytically from real to complex coordinates. Then Lorentzian and euclidean coordinates correspond to different real sections of this complex spacetime [12]. An action that was originally invariant under real coordinate transformations will still be invariant if the coordinates are complex, thus the form of the action will not change along the different sections.

Euclidean quantities will be have an index E.

Locally the passage from Minkowski metric

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2 \tag{A.10}$$

to Euclidean metric

$$\mathrm{d}s_E^2 = \delta_{\mu\nu}\mathrm{d}x_E^\mu \mathrm{d}x_E^\nu = \mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{x}^2 \tag{A.11}$$

is done through the substitution of the real time $x^0 = t$ by the euclidean time $x^4 = \tau$ [86, sec. 3.4]

$$t = -i\tau. \tag{A.12}$$

The volume element is invariant under coordinate transformations [39]

$$\sqrt{-g}\,\mathrm{d}^d x = \sqrt{-g_E}\,\mathrm{d}^d x_E. \tag{A.13}$$

The differential element and the determinant transform separately as

$$\mathrm{d}^d x = -i\,\mathrm{d}^d x_E, \qquad \sqrt{-g} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\sqrt{-g_E} = i\sqrt{-g_E}. \tag{A.14}$$

The argument of the last square root is negative since $g_E > 0$: this happens because we use the formula that is adapted to Lorentzian metrics, but not for positive definite metric; the solution is to insert the i into the square root [40]. We thus get that

$$\sqrt{-g} = \sqrt{g_E}. \tag{A.15}$$

This can also be understood by seeing that $g = -g_E$, and then by transforming only the differential. At the end we get

$$\sqrt{-g}\,\mathrm{d}^d x = -i\sqrt{g_E}\,\mathrm{d}^d x_E. \tag{A.16}$$

As we said the action and the Lagrangian are invariant under (A.12) because general coordinate invariance is valid even for complex coordinates. Then the action becomes

$$\begin{aligned} S &= \int \mathrm{d}^d x\,\sqrt{-g}\,\mathcal{L} = \int \mathrm{d}^d x_E\,\sqrt{-g_E}\,\mathcal{L} = -i\int \mathrm{d}^d x_E\,\sqrt{g_E}\,\mathcal{L} \\ &= i\int \mathrm{d}^d x_E\,\sqrt{g_E}\,\mathcal{L}_E = iS_E \end{aligned}$$

where we introduced the new quantities [67, p. 126]

$$S = iS_E, \qquad \mathcal{L} = -\mathcal{L}_E. \tag{A.17}$$

[39] We write explicitly the sign of the determinant here.

[40] Or equivalently to remove the minus sign in the square root by using the branch $\sqrt{-1} = -i$ [12].

The euclidean Lagrangian $\mathcal{L}_E$ is positive definite and will be interpreted as an energy [41].

Since the Lorentzian and Euclidean Lagrangians differ by a sign, it will be the same for the associated stress–energy tensor. We thus obtain the euclidean stress–energy tensor from (A.2)

$$T_{E,\mu\nu} = \frac{4\pi}{\sqrt{g_E}} \frac{\delta S}{\delta g_E^{\mu\nu}}. \tag{A.18}$$

In this way the expression for the two tensors (and especially for their traces) will agree in both signatures.

Finally the partition function is

$$Z = \int \mathrm{d}\phi \; \mathrm{e}^{iS} = \int \mathrm{d}\phi \; \mathrm{e}^{-S_E}. \tag{A.19}$$

Since the Euclidean action is now positive definite, the minus sign in the partition function gives exponential damping.

Example A.1 (Scalar field with potential) As an example look at the scalar Lagrangian with potential

$$\mathcal{L} = -\frac{1}{2}\Big((\partial^\mu \phi)^2 + m^2 \phi^2 \Big) - V(\phi) \tag{A.20}$$

which gives the equation of motion

$$(-\Delta + m^2)\phi = V'(\phi). \tag{A.21}$$

Plugging plane-waves into the free equation ($V = 0$) gives the mass-shell condition

$$p^2 = -m^2 \tag{A.22}$$

and the Green function

$$G(p) = \frac{1}{p^2 + m^2} \tag{A.23}$$

has a singularity.

Applying the Wick rotation gives

$$\mathcal{L}_E = \frac{1}{2}\Big((\partial_E^\mu \phi)^2 + m^2 \phi^2 \Big) + V(\phi), \tag{A.24}$$

which is positive definite, and the equation of motion

$$(-\Delta_E + m^2)\phi = -V'(\phi). \tag{A.25}$$

with Green function

$$G(p) = \frac{1}{p_E^2 + m^2}. \tag{A.26}$$

This function has no singularity since plane-waves are not anymore solutions of the Klein-Gordon equation (said another way, there is no particle in Euclidean space).

---

[41] Recall that $L = T - V$; the Wick rotation changes the sign of T, such that $E = -L$. For scalar fields this will be equal to the Hamiltonian of the system, but this will not be the case for fermions [22, sec. 2.3.2].

Example A.2 (Schrödinger equation) In Lorentzian signature the Schrödinger equation reads

$$i\,\frac{\partial}{\partial t}\psi = H\psi \tag{A.27}$$

where H is the Hamiltonian operator. Using plane waves in time

$$\psi = \mathrm{e}^{-iEt}\psi_E \tag{A.28}$$

we obtain the time-independent equation

$$H\psi = E\psi. \tag{A.29}$$

Spatial momentum in H is replaced with the rule

$$p = -i\frac{\mathrm{d}}{\mathrm{d}x}. \tag{A.30}$$

Doing the Wick rotation (A.12)

$$t = -i\tau \tag{A.31}$$

the Schrödinger equation is transformed into the heat equation

$$-\frac{\partial}{\partial\tau}\psi = H\psi. \tag{A.32}$$

Plane waves are

$$\psi = \mathrm{e}^{-E\tau}\psi_E \tag{A.33}$$

and the time-independent equation is the same (note that we did *not* transform the energy). If we had chosen the other sign for the time in (A.12) then the wave would blow up. The new momentum is related to the previous one by

$$p = -i\pi. \tag{A.34}$$

(as can be seen by looking at the example $p = \dot\phi$).

A.5 CFT parametrization

In this section we summarize various parametizations for the quantities of interest in CFTs.

The central charge can be parametrized by

$$c = 1 + 6Q^2 \tag{A.35}$$

where

$$Q = \frac{1}{b} + b. \tag{A.36}$$

Momenta can be written

$$a = \frac{Q}{2} + ip \tag{A.37}$$

and the corresponding weights are

$$\Delta = a(Q-a) = \frac{Q^2}{4} + p^2. \tag{A.38}$$

When $c \leq 1$ is it useful to make an analytical continuation

$$Q = i\Gamma, \qquad b = -i\beta, \qquad a = -i\alpha, \qquad p = -i\omega. \tag{A.39}$$

We will term by rational values the theories such that

$$\beta^2 = -b^2 = \frac{q}{p} \tag{A.40}$$

where p and q are coprime. In this case $c \leq 1$.

Degenerate fields are such that

$$\alpha_{r,s} = \frac{Q}{2} - \frac{rb}{2} - \frac{s}{2b}, \tag{A.41}$$

or in terms of the momentum

$$p_{r,s} = i\left(\frac{rb}{2} + \frac{s}{2b}\right). \tag{A.42}$$

B General relativity

The Laplacian on curved space (Laplace–Beltrami operator) is defined by

$$\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{|g|}}\left(\partial_\mu\sqrt{|g|}g^{\mu\nu}\partial_\nu\right). \tag{B.1}$$

The negative of the Laplacian is positive definite

$$(\phi, -\Delta\phi) \geq 0 \tag{B.2}$$

and in terms of exterior derivatives it reads

$$-\Delta = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}. \tag{B.3}$$

The Green function for the Laplacian Δ at finite area is defined by

$$\Delta G(\sigma, \sigma') = \frac{1}{\sqrt{g}}\,\delta(\sigma - \sigma') - \frac{1}{A}, \qquad \int \mathrm{d}^2\sigma'\,\sqrt{g}\,\Delta G(\sigma, \sigma') = 0. \tag{B.4}$$

The second term is essential in order to take into account the zero mode, or equivalently to verify that the integral is vanishing since there is no boundary.

B.1 Einstein–Hilbert action

We have the formula

$$\delta\sqrt{g} = -\frac{1}{2}\,\sqrt{g}\,g_{\mu\nu}\delta g^{\mu\nu} \tag{B.5}$$

$$\int \mathrm{d}^2\sigma\,\sqrt{g}\,\phi\,\delta R_{\mu\nu} = -\int \mathrm{d}^2\sigma\,\sqrt{g}\,\big(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Delta\phi\big)\delta g^{\mu\nu} \tag{B.6}$$

B.2 Weyl transformation

Under a Weyl transformation

$$g_{\mu\nu} = \mathrm{e}^{2\omega}h_{\mu\nu} \tag{B.7}$$

we have

$$g^{\mu\nu} = \mathrm{e}^{-2\omega}h^{\mu\nu}, \tag{B.8a}$$

$$\sqrt{g} = \mathrm{e}^{2\omega}\sqrt{h}, \tag{B.8b}$$

$$\Delta_g = \mathrm{e}^{-2\omega}\,\Delta_h, \tag{B.8c}$$

$$R_g = (R_h - 2\Delta_h\omega)\,\mathrm{e}^{-2\omega}. \tag{B.8d}$$

The Laplacian is not invariant because of the $g^{-1/2}$ factor [41].

B.3 Conformal gauge

Using diffeomorphisms we can fix the gauge. In most of this review we will choose the conformal gauge

$$g_{\mu\nu} \longrightarrow g'_{\mu\nu} = \mathrm{e}^{2\phi} h_{\mu\nu}. \tag{B.9}$$

where $h_{\mu\nu}$ is some fixed (non-dynamical) metric. The above decomposition is left unchanged under the transformation (emerging Weyl symmetry)

$$h_{\mu\nu} = \mathrm{e}^{2\omega} h'_{\mu\nu}, \qquad \phi = \phi' - \omega. \tag{B.10}$$

We now look how to rewrite $\delta g^{\mu\nu}$ in term of $\delta\phi$. We have

$$\delta g^{\mu\nu} = -2\, \mathrm{e}^{-2\phi}\, h^{\mu\nu} \delta\phi = -2\, g^{\mu\nu} \delta\phi \tag{B.11}$$

from which we get

$$\frac{\delta\phi}{\delta g^{\mu\nu}} = -\frac{1}{4}\, g_{\mu\nu}, \qquad g^{\mu\nu} \frac{\delta\phi}{\delta g^{\mu\nu}} = -\frac{1}{2}. \tag{B.12}$$

Hence if we have a variation with respect to $g^{\mu\nu}$ we can replace it by one with ϕ using the chain rule

$$\frac{\delta}{\delta g^{\mu\nu}} = \frac{\delta\phi}{\delta g^{\mu\nu}} \frac{\delta}{\delta\phi} = -\frac{1}{4}\, g_{\mu\nu}\, \frac{\delta}{\delta\phi}. \tag{B.13}$$

C Matter models

In this appendix we will list various models for $2d$ matter coupled to gravity. Most of these models are non-conformal and we will give the limit under which the conformal symmetry is recovered. We will use the euclidean signature; Lorentzian models can be found using a Wick rotation (app. A.4).

C.1 Minimal models

We consider models for which the central charge is parametrized as [85, sec. 3]

$$c = 1 - 6q^2 = 13 - 6\left(\frac{1}{\beta^2} + \beta^2\right), \qquad q = \frac{1}{\beta} - \beta. \tag{C.1}$$

The minimal models $M_{p,p'}$ are defined for rational β^2 which corresponds to a pair of integers (p, p')

$$\beta^2 = \frac{p'}{p} \in \mathbb{Q}, \qquad p' < p \tag{C.2}$$

where the second condition comes from the fact that $\beta < 1$. Then the central charge reads

$$c = 1 - 6\, \frac{(p - p')^2}{pp'} = 13 - 6\, \frac{p^2 + p'^2}{pp'}. \tag{C.3}$$

Minimal models have the property to possess only a finite number of primary fields $\Phi_{m,n}$ where

$$1 \le m \le p' - 1, \qquad 1 \le n \le p - 1 \tag{C.4}$$

and whose conformal dimensions are given by Kac formula and read

$$h_{m,n} = \alpha_{m,n}(q + \alpha_{m,n}) = \frac{1}{4}\left[\left(\frac{m}{\beta} - n\beta\right)^2 - q^2\right] \tag{C.5}$$

and

$$\alpha_{m,n} = \frac{(n-1)\beta}{2} - \frac{m-1}{2\beta} \tag{C.6}$$

Virasoro representations are degenerated and have a null vector at level mn. There are

$$\frac{1}{2}(p-1)(p'-1) \tag{C.7}$$

independent fields after the identification of $\Phi_{m,n}$ and $\Phi_{p'-m,p-n}$.

Two special classes exist for special values of p':

- $p' = 1$: topological series [57]

$$c = 13 - 6\left(\frac{1}{p} + p\right), \tag{C.8}$$

these models contains logarithmic operators [21]. For $p = 2$ the central charge is $c = -2$ which appears in several places and is the simplest logarithmic theory.

- $m \equiv p' = p + 1$: unitary (or principal) series $M_m \equiv M_{p,p+1}$

$$c = 1 - 6\,\frac{1}{m(m+1)}, \qquad m \geq 3, \tag{C.9}$$

and in particular $c \in [1/2, 1)$.

Zamolodchikov conjectured the existence of generalized minimal models for which $\beta^2 \in \mathbb{R} - \mathbb{Q}$ [85, sec. 3]. In this case the labels m and n of the fields $\Phi_{m,n}$ do not have any bound. The conformal bootstrap for these models have been verified numerically in [72].

C.2 Models with a Lagrangian

We will consider action S derived from a Lagrangian $\mathcal{L}$ (we will omit the subscript m that is present in the main text)

$$S[g,\psi] = \frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{g}\,\mathcal{L}. \tag{C.10}$$

We recall that the energy tensor is given by

$$T_{\mu\nu} = \frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{C.11}$$

and using the previous action it reads

$$T_{\mu\nu} = 2\,\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} - g_{\mu\nu}\,\mathcal{L} \tag{C.12}$$

where the second term comes from the variation of $\sqrt{g}$.

The trace is obtaiend by contracting with $g^{\mu\nu}$

$$T = 2\,g^{\mu\nu}\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} - 2\,\mathcal{L}. \tag{C.13}$$

In the conformal gauge $g = \mathrm{e}^{2\phi}h$ this last equation simplifies to (see app. B.3)

$$T = -\frac{\delta \mathcal{L}}{\delta \phi} - 2\,\mathcal{L}. \tag{C.14}$$

C.3 Scalar fields

For a scalar field X we will consider Lagrangian made from the usual kinetic term and a potential

$$\mathcal{L} = \frac{1}{2}\, g^{\mu\nu} \partial_\mu X \partial_\nu X + V(X) \tag{C.15}$$

and under integration we have

$$g^{\mu\nu} \partial_\mu X \partial_\nu X = -X \Delta X. \tag{C.16}$$

The equation of motion is

$$-\Delta X + V'(X) = 0. \tag{C.17}$$

Using the expression (C.12) for the energy tensor we obtain

$$T_{\mu\nu} = \partial_\mu X \partial_\nu X - \frac{1}{2} g_{\mu\nu}\, g^{\rho\sigma} \partial_\rho X \partial_\sigma X + 2\, \frac{\delta V}{\delta g^{\mu\nu}} - g_{\mu\nu}\, V, \tag{C.18a}$$

$$T = -\frac{\delta V}{\delta \phi} - 2\, V. \tag{C.18b}$$

The kinetic term is conformal and the trace depends only of the potential. If the potential does not depend on the conformal factor ϕ, then the non-invariance comes from the lonely factor coming from $\sqrt{g}$.

Below we consider various potential: from each of these pieces it is possible to build a bigger potential.

C.3.1 Massive field

For a massive scalar the potential is

$$V = \frac{m^2}{2}\, X^2. \tag{C.19}$$

The energy tensor reads [42] [19, sec. 4.2]

$$T_{\mu\nu} = -\frac{m^2}{2}\, g_{\mu\nu} \tag{C.20}$$

and its trace of the energy tensor reads

$$T = -m^2 X^2 \tag{C.21}$$

and we see directly that the conformal limit is $m = 0$ which corresponds to a massless scalar.

C.3.2 Exponential potential

We look at the potential

$$V = \mu\, \mathrm{e}^{2aX}. \tag{C.22}$$

If X is a conformal field, then the vertex operator e^{2aX} has conformal dimension

$$h = -a^2. \tag{C.23}$$

If we choose $a = i$ then the conformal dimension is 1 and this potential corresponds to a marginal deformation of the conformal theory.

[42] Note that Ferrari et al. [19] define the energy tensor with the Lorentzian formula (A.2) instead of (A.18).

C.3.3 Sine–Gordon model

The potential for the Sine–Gordon model is [34, sec. 2.1, 58, 78, sec. 4.3]

$$V = \mu(1 - \cos aX). \tag{C.24}$$

We added the 1 to remove the constant from the cosine Taylor series: this amounts to redefine the cosmological constant. For $a \sim 0$ this model reduces to a massive scalar with mass $m^2 = a^2\mu$ (we need to send $\mu \to \infty$ such that m^2 is fixed).

Since this potential is made from exponential and that $(-a)^2 = a^2$, at the quantum level this term has conformal dimension 1 if $a = 1$. In this case it is just a marginal deformation.

The model name comes from the equation of motion

$$-\Delta X + \mu \sin aX = 0. \tag{C.25}$$

Since this potential is made from vertex operators it can be coupled to gravity for any value of a and it will receive a gravitational dressing (as discussed in sec. 5.6)

$$V = \mu(1 - \mathrm{e}^{2\xi\phi} \cos aX) \tag{C.26}$$

where ϕ is the Liouville field and ξ is such that V has conformal dimension 1. If this field is the only one, all parameters are fixed [58]. In presence of other matter this operator can be relevant, irrelevant or marginal depending on the value of a.

C.3.4 Sinh–Gordon model

This potential corresponds to the analytical continuation in a of the previous model

$$V = -\mu(1 - \cosh \alpha X). \tag{C.27}$$

It also reduces to the massive scalar field with mass $m^2 = \alpha^2\mu$ for $\alpha \sim 0$.

C.3.5 Non-minimal coupling

The non-minimal (or conformal) coupling to the metric is given by [20]

$$V = \frac{q}{2}\, RX. \tag{C.28}$$

The trace of the classical energy tensor in the conformal gauge is [19, sec. 4.2]

$$T = q\Delta_g X. \tag{C.29}$$

using the fact that

$$R_g = (R_h - 2\Delta_h\phi)\,\mathrm{e}^{-2\phi}, \qquad \Delta_g = \mathrm{e}^{-2\omega}\,\Delta_h \tag{C.30}$$

and

$$T = -\frac{q}{2}\, X\, \frac{\delta R}{\delta\phi} - q\, RX = qX(R_h - 2\Delta_h\phi)\,\mathrm{e}^{-2\phi} + q\,\mathrm{e}^{-2\phi}\Delta_h X - q\, RX \tag{C.31}$$

(after two integrations by part).

We obtain the full energy tensor from (B.6)

$$T_{\mu\nu} = -q\big(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Delta\phi\big). \tag{C.32}$$

We will consider the quantum version of this potential below (sec. C.5).

C.4 Polyakov action

We want to construct the most general action for X^a ($a = 1, \ldots, d$) scalar fields coupled to gravity – it is called the Polyakov action. These fields are in the vector representation of the (global) euclidean group (rotations and translation)

$$\mathrm{E}(d) = \mathrm{O}(d) \ltimes \mathbb{R}^d. \tag{C.33}$$

In addition to renormalizability, diffeomorphism invariance and at most second order in derivatives, the total action should be invariant under these (internal) euclidean transformations.

C.4.1 Symmetries and action

Under a transformation $(R, c) \in \mathrm{E}(d)$, where R is a rotation and c a translation, the scalar fields transform as

$$X'^a = R_a{}^b X^b + c^a. \tag{C.34}$$

X^a are scalars under diffeomorphisms (4.2) and do not transform

$$X'^a(\sigma'^\mu) = X^a(\sigma^\mu). \tag{C.35}$$

We also recall that they do not transform under Weyl symmetry (4.3).

The matter action can depend only on X^a derivatives since it is invariant under translations and the only possibility is [48]

$$S_m[g, X] = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \sqrt{g}\, g^{\mu\nu} \partial_\mu X^a \partial_\nu X_a. \tag{C.36}$$

We will generally discard d-dimensional indices. It is invariant under Weyl transformation (4.3)

$$S_m[g, X] = S_m[\mathrm{e}^{2\omega} g, X]. \tag{C.37}$$

Since the action S_m is quadratic we can evaluate (4.15) as a Gaussian integral, but we need to take care of the zero mode; we will defer its treatment to a later section.

C.4.2 Equations of motion and stress–energy tensor

Using (4.16) the matter stress–energy tensor reads

$$T^{(m)}_{\mu\nu} = \partial_\mu X \cdot \partial_\nu X - \frac{1}{2} g_{\mu\nu}\, g^{\rho\sigma} \partial_\rho X \cdot \partial_\sigma X \tag{C.38}$$

and it is traceless. The equation of motion for $g_{\mu\nu}$ is just

$$T^{(m)}_{\mu\nu} = 0. \tag{C.39}$$

The variation of S_m (C.36) with respect to X gives

$$\Delta X^a = 0 \tag{C.40}$$

where Δ is the Laplacian on the space $\mathcal{M}$.

C.5 Coulomb gas

We consider a free scalar field ϕ in the presence of a background charge q [22, chap. 9, 34, sec. 1.4]. Its action reads

$$S = \frac{1}{4\pi} \int \mathrm{d}^2 z \, \sqrt{g} \, (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + iqR\phi) \, . \tag{C.41}$$

The corresponding stress–energy tensor $T = T_{zz}$ gets a new term

$$T = -(\partial\phi)^2 + iq\partial^2\phi, \tag{C.42}$$

and the associated central charge is

$$c = 1 - 6q^2. \tag{C.43}$$

As an effect of the charge, the central charge is shift to $c < 1$ for $q \in \mathbb{R}$. Because T is imaginary, the theory is not unitary for arbitrary value of q, but the spectrum is unitary for specific values of q (especially it contains the minimal models); they can be found using the Kac table.

By computing the OPE between $\mathrm{e}^{2a\phi}$ and T we can prove that

$$h_a = a(q - a). \tag{C.44}$$

The momentum of the state created by such a vertex operators is

$$ip_\phi = a - \frac{iq}{2}. \tag{C.45}$$

Vertex operators are not anymore invariant under $a \to -a$ but they are under

$$a \longrightarrow q - a. \tag{C.46}$$

C.6 Fermionic fields

The Lagrangian for a massive fermion (massive Ising model) is

$$\mathcal{L} = i\,\bar{\psi}\gamma^\mu \partial_\mu \psi + m\bar{\psi}\psi. \tag{C.47}$$

When coupling it to gravity we need to use a covariant derivative

$$\mathrm{D}_\mu = \partial_\mu + \omega_\mu \sigma^3 \tag{C.48}$$

where ω_μ is the spin connection.

D Mathematical tools

D.1 Gaussian integrals

The Gaussian integral for one scalar is

$$\int_{-\infty}^{\infty} \mathrm{d}x \; \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}. \tag{D.1}$$

For a n-dimensional Gaussian integral where the measure is not normalised one has

$$\int \mathrm{d}^n x \;\; \mathrm{e}^{x_a M_{ab} x_b} = (\det M)^{-1/2} \int \mathrm{d}x^n \;\; \mathrm{e}^{-x_a x_a}. \tag{D.2}$$

This can be written as

$$\int d\mu(x_a)\ e^{x_a M_{ab} x_b} = (\det M)^{-1/2} \tag{D.3}$$

where the canonical measure is

$$d\mu(x_a) = \frac{d^n x}{\int dx^n\ e^{-x_a x_a}}. \tag{D.4}$$

In particular this ensures that the integral is one when $M = 1$.

D.2 Self-adjoint extension

A good reference on self-adjoint extension is [49, app. A] (see also [27, sec. 2.2, 57, sec. 3]).

Let $\mathcal{H}$ be an Hilbert space with scalar product $\langle\cdot|\cdot\rangle$, and A a linear operator on a domain $D(A) \subset \mathcal{H}$.

Considering $\chi \in \mathcal{H}$, the adjoint $A^\dagger$ is defined by

$$\langle\chi|A\psi\rangle = \left\langle A^\dagger\chi\middle|\psi\right\rangle, \qquad \forall\psi \in D(A). \tag{D.5}$$

The domain $D(A^\dagger)$ of the adjoint is defined by all $\chi \in \mathcal{H}$ that satisfy this relation. Another way to define the adjoint is

$$\langle\chi|A\psi\rangle^* = \left\langle\psi\middle|A^\dagger\chi\right\rangle. \tag{D.6}$$

The operator A is said to be symmetric (or Hermitian) if

$$D(A) \subset D(A^\dagger), \qquad A^\dagger|_{D(A)} = A. \tag{D.7}$$

A symmetric operator is called self-adjoint if

$$D(A) = D(A^\dagger). \tag{D.8}$$

An operator B is an extension of A if

$$D(A) \subset D(B), \qquad B|_{D(A)} = A. \tag{D.9}$$

A symmetric operator A can be extended to a self-adjoint operator $\tilde{A}$ if there exists a domain such that

$$D(A^\dagger) \supset D(\tilde{A}^\dagger) = D(\tilde{A}) \supset D(A). \tag{D.10}$$

We now review when conditions are met for such an extension.

First the domain $D(A^\dagger)$ can be decomposed as

$$D(A^\dagger) = D(A) + K_+ + K_- \tag{D.11}$$

where[43]

$$K_\pm(A^\dagger) = \ker(A^\dagger \pm i). \tag{D.12}$$

The deficiency indices are defined

$$d_\pm = \dim K_\pm, \tag{D.13}$$

and hence they correspond to the number of linearly independent solutions to the equations

$$A^\dagger\psi = \pm i\,\psi. \tag{D.14}$$

[43] The number $\pm i$ can be replaced by any pair of complex conjugate numbers.

The operator A admits a self-adjoint extension if

$$d \equiv= d_+ = d_-, \tag{D.15}$$

in which case there the extension is characterized by d parameters.

In the following we compute the conditions such that the momentum and Hamiltonian operators

$$P = -i \frac{\mathrm{d}}{\mathrm{d}x}, \qquad H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}. \tag{D.16}$$

are symmetric for wave functions defined on some interval $[a, b]$. We will compute $P^\dagger$ and $H^\dagger$ using the formula (D.6) using $\psi, \chi \in \mathcal{H}$.

Consider first P

$$\begin{aligned}
\langle \chi | P\psi \rangle^* &= \left(\int_a^b \mathrm{d}x\, \chi(x)^* \left(-i \frac{\mathrm{d}}{\mathrm{d}x} \right) \psi(x) \right)^* \\
&= \int_a^b \mathrm{d}x\, \left(-i \frac{\mathrm{d}}{\mathrm{d}x} \right) \chi(x)\, \psi(x)^* + i \big[\chi(x) \psi(x)^* \big]_a^b
\end{aligned}$$

after integrating by part, giving Since the first term is recognized to be $\langle \psi | P\chi \rangle$, P is symmetric if

$$\big[\chi(x) \psi(x)^* \big]_a^b = 0. \tag{D.17}$$

We now turn to H

$$\begin{aligned}
\langle \chi | H\psi \rangle^* &= \left(\int_a^b \mathrm{d}x\, \chi(x)^* \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right) \psi(x) \right)^* \\
&= \int_a^b \mathrm{d}x\, \frac{\mathrm{d}\chi(x)}{\mathrm{d}x} \frac{\mathrm{d}\psi(x)^*}{\mathrm{d}x} - \left[\chi(x) \frac{\mathrm{d}\psi(x)^*}{\mathrm{d}x} \right]_a^b \\
&= \int_a^b \mathrm{d}x\, \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right) \chi(x)\, \psi(x)^* + \left[\frac{\mathrm{d}\chi(x)}{\mathrm{d}x} \psi(x)^* - \chi(x) \frac{\mathrm{d}\psi(x)^*}{\mathrm{d}x} \right]_a^b
\end{aligned}$$

using two integration by parts. Since the first term is $\langle \psi | H\chi \rangle$, the Hamiltonian is symmetric if

$$\left[\frac{\mathrm{d}\chi(x)}{\mathrm{d}x} \psi(x)^* - \chi(x) \frac{\mathrm{d}\psi(x)^*}{\mathrm{d}x} \right]_a^b = 0. \tag{D.18}$$

Note that an Hamiltonian that does not contain any other derivative term will require the same condition.

Example D.1 (Particle on a line) Consider a particle on the line $[0, 2\pi]$ with wave function $\psi(x) \in L^2(\mathbb{C})$

$$\int_0^{2\pi} \mathrm{d}x\, |\psi(x)|^2 < \infty. \tag{D.19}$$

From the condition (D.17) P is symmetric if

$$\chi(2\pi)\psi(2\pi)^* - \chi(0)\psi(0)^* = 0. \tag{D.20}$$

One needs to define a domain for each function; then the boundary term vanishes if one takes

$$D(P) = \{ \psi \in L^2 \mid \psi(2\pi) = \psi(0) = 0 \}, \tag{D.21}$$

but then there is no need to impose any condition on functions in the dual space

$$D(P^\dagger) = \{\chi \in L^2\}. \tag{D.22}$$

Then in some sense $D(P^\dagger) > D(P)$ and the operator is not self-adjoint $P \neq P^\dagger$. The physical interpretation is that the boundary conditions break invariance by translation.

One can check that $H = H^\dagger$ because there are two integration by parts, and the boundary term vanishes only if $D(P^\dagger) = D(P)$.

Example D.2 (Particle on a circle) On a circle we have $P^\dagger = P$ since one can impose periodic boundary conditions. In particular the operator P admits a one-parameter extension if one takes the domain to be

$$D(P) = \{\psi \mid \psi(2\pi) = \mathrm{e}^{i\theta}\psi(0)\}, \qquad D(P) = D(P^\dagger). \tag{D.23}$$

Physically this is due to a magnetic flux inside the circle (Aharonov–Bohm effect).

E Special functions

E.1 Gamma and beta functions

The gamma function reads

$$\Gamma(x) = \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-t} t^{x-1}. \tag{E.1}$$

For integer argument it reduces to the factorial

$$\Gamma(n) = (n-1)! \tag{E.2}$$

It satisfies the recursion relation

$$\Gamma(x+1) = x\,\Gamma(x). \tag{E.3}$$

The asymptotic formulas are

$$\Gamma(x) \sim_0 \frac{1}{x} - \gamma_e + O(x), \tag{E.4a}$$

$$\Gamma(x) \sim_\infty \sqrt{2\pi}\,\mathrm{e}^{-x} x^{x-\frac{1}{2}} \tag{E.4b}$$

where γ_e is the Euler constant.

An analytic continuation can be obtained from the relation

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x}. \tag{E.5}$$

For $y \in \mathbb{R}$, the gamma function satisfies

$$\Gamma(z)^* = \Gamma(z^*). \tag{E.6}$$

The beta function is

$$B(x,y) = \int_0^1 \mathrm{d}t\ t^{x-1}(1-t)^{y-1} \tag{E.7}$$

We can obtain the equivalent forms

$$B(x,y) = \int_0^\infty \mathrm{d}t\, \frac{t^{x-1}}{(1+t)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{E.8}$$

We define the little gamma function

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} = \frac{\sin \pi x}{\pi}\, \Gamma(x)^2. \tag{E.9}$$

From the properties of the Γ-function one can deduce

$$\gamma(x+1) = -x^2\, \gamma(x). \tag{E.10}$$

One also has

$$\gamma(x)\gamma(-x) = -\frac{1}{x^2}. \tag{E.11}$$

It has the following asymptotic

$$\gamma(x) \sim_0 \frac{1}{x} - 2\gamma_e + 2\gamma_e^2 x + O(x^2), \tag{E.12a}$$

$$\gamma(x) \sim_\infty \frac{\sin \pi x}{x} \left(\frac{1}{x}\right)^{1-2x} \exp\left(\frac{1}{12x} - 2x\right). \tag{E.12b}$$

It has zeros at $x = n+1$, $n \in \mathbb{N}^*$, and poles at $x = -n$, $n \in \mathbb{N}$ with residue

$$\operatorname{Res} \gamma(-n) = \frac{(-1)^n}{(n!)^2}. \tag{E.13}$$

There is an integral representation for $|\operatorname{Re} x| < 1/2$

$$\gamma(x+1/2) = \int_0^\infty \frac{\mathrm{d}t}{t} \left[2x\, \mathrm{e}^{-t} - \frac{\sinh xt}{\sinh t/2}\right]. \tag{E.14}$$

E.2 Bessel function family

Properties of Bessel functions can be found in [61, chap. 10].

E.2.1 Bessel functions

Bessel differential equation is

$$x^2 f'' + x f' + (x^2 - \nu^2) f = 0 \tag{E.15}$$

admits $J_{\pm\nu}(x)$ (Bessel functions of first kind) as solutions if $\nu \notin \mathbb{N}$. For integer parameter, they are associated to the generating function

$$\exp \frac{x}{2}\left(t - \frac{1}{t}\right) = \sum_{n\in\mathbb{Z}} J_n(x) t^n. \tag{E.16}$$

If we have the equation

$$x^2 f'' + x f' + (a^2 x^2 - \nu^2) f = 0 \tag{E.17}$$

then the change of variable $y = ax$ does not change the derivative terms as the first two terms are invariant under rescaling of x, and we get the solution $J_{\pm\nu}(ax)$.

If $n = \nu \in \mathbb{Z}$ then $J_{\pm n}(x)$ are not linearly independent since

$$J_{-n} = (-1)^n J_n \qquad n \in \mathbb{Z}. \tag{E.18}$$

In this case one has to introduce Bessel functions of second kind $Y_\nu(x)$ (denoted sometime N_ν)

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu x) - J_{-\nu}(x)}{\sin(\nu x)}, \tag{E.19}$$

the limit where ν is an integer being regular. This function also satisfies

$$Y_{-\nu} = (-1)^\nu Y_\nu \qquad \nu \in \mathbb{Z}. \tag{E.20}$$

Note that the pair (J_ν, Y_ν) are always independent solutions, but it is not of common usage to use this pair for $\nu \notin \mathbb{Z}$.

Asymptotic forms at infinity are pure imaginary exponentials

$$J_\nu(x) \sim_\infty \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right), \qquad Y_\nu(x) \sim_\infty \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \tag{E.21a}$$

while near the origin one has

$$J_\nu(x) \sim_0 \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \qquad Y_\nu(x) \sim_0 -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu - \frac{\Gamma(-\nu)}{\pi}\left(\frac{x}{2}\right)^\nu \cos\pi\nu. \tag{E.21b}$$

The latter are not valid for $x \in -\mathbb{N}^*$, in which case one can use the relation (E.18). The leading term in $Y_\nu(x)$ is the first if

$$\operatorname{Re}\nu > 0 \text{ or } \nu \in -\frac{\mathbb{N}+1}{2}, \tag{E.22}$$

otherwise it is the second one.

Note that for complex ν and real x one has

$$\overline{J_\nu(x)} = J_{\overline{\nu}}(x). \tag{E.23}$$

Hence for pure imaginary ν and real x the (independent) functions $J_{\pm\nu}(x)$ are complex conjugate. This also holds for $Y_\nu(x)$.

The integral of two functions reads

$$\int_0^\infty \frac{\mathrm{d}x}{x}\, J_\nu(x)J_{\nu'}(x) = \frac{1}{\nu+\nu'}\,\frac{\sin\frac{\pi}{2}(\nu-\nu')}{\frac{\pi}{2}(\nu-\nu')} \tag{E.24}$$

if $\operatorname{Re}(\nu+\nu') > 0$, otherwise the integral does not converge (for example with $\nu, \nu' \in i\mathbb{R}$). If $\nu = \nu'$ then one finds

$$\int_0^\infty \frac{\mathrm{d}x}{x}\, J_\nu(x)^2 = \frac{1}{2\nu}. \tag{E.25}$$

Moreover for

$$\nu = \nu_0 + 2m + 1, \qquad \nu = \nu_0 + 2n + 1 \tag{E.26}$$

we find that

$$\int_0^\infty \frac{\mathrm{d}x}{x}\, J_\nu(x)J_{\nu'}(x) = \frac{\delta_{m,n}}{2(2m+\nu_0+1)}. \tag{E.27}$$

E.2.2 Hankel functions

Hankel functions are to Bessel functions what exponentials are to trigonometric functions

$$H_\nu^{(\pm)}(x) = J_\nu(x) \pm i\,Y_\nu(x), \qquad H_\nu^{(2)}(x) = J_\nu(x) - i\,Y_\nu(x) \tag{E.28}$$

where we have defined $H_\nu^{(+)}(x) \equiv H_\nu^{(1)}(x)$ and $H_\nu^{(-)}(x) \equiv H_\nu^{(2)}(x)$. They behave as

$$H_\nu^{(\pm)}(x) \sim \sqrt{\frac{2}{\pi x}}\, \exp\left[\pm i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right]. \tag{E.29}$$

E.2.3 Modified Bessel functions

Modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ are solutions of the differential equation

$$x^2 f'' + x f' - (x^2 + \nu^2) f = 0. \tag{E.30}$$

The modified Bessel functions have exponential behaviour.

One has the asymptotic forms

$$I_\nu(x) \sim_0 \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \qquad K_\nu(x) \sim_0 \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu + \frac{\Gamma(-\nu)}{2}\left(\frac{x}{2}\right)^\nu, \tag{E.31a}$$

$$I_\nu(x) \sim_\infty \frac{\mathrm{e}^x}{\sqrt{2\pi x}}, \qquad K_\nu(x) \sim_\infty \sqrt{\frac{\pi}{2x}}\, \mathrm{e}^{-x} \tag{E.31b}$$

If ν is integer, then there is an extra factor $\ln x$ in the second term for the limit of K_ν at 0 (note also that the second term is subleading with respect to the first one).

The derivative of the Bessel function is

$$\frac{\mathrm{d}K_\nu(x)}{\mathrm{d}x} = -\frac{1}{2}\Big(K_{\nu-1}(x) + K_{\nu+1}(x)\Big), \tag{E.32}$$

and

$$x\,\frac{\mathrm{d}K_\nu(x)}{\mathrm{d}x} = \nu K_\nu(x) - x\,K_{\nu+1}(x). \tag{E.33}$$

In terms of Bessel functions they are given as

$$\begin{aligned} I_\nu(x) &= i^{-\nu} J_\nu(ix), \\ K_\nu(x) &= \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}. \end{aligned} \tag{E.34}$$

E.3 Υ_b function

References on the $\Upsilon_b(x)$ function may be found in [40, app. A]. For the moment we just list the properties we need.

E.3.1 Definitions

It depends on a parameter b and we also define

$$Q = b + \frac{1}{b}. \tag{E.35}$$

The function is defined by

$$\ln \Upsilon_b(x) = \int_0^\infty \frac{\mathrm{d}t}{t}\left((Q/2 - x)^2\, \mathrm{e}^{-t} - \frac{\sinh^2 \frac{t}{2}(Q/2 - x)}{\sinh\frac{tb}{2}\,\sinh\frac{t}{2b}}\right). \tag{E.36}$$

This formula is defined only for $\operatorname{Re} x \in [0, Q]$ but it admits an analytical extension over the entire complex plane. It is not convergent for $b \in i\mathbb{R}$.

We have the special values

$$\Upsilon_b(0) = \Upsilon_b(Q/2) = 1. \tag{E.37}$$

If one write

$$x = \frac{Q}{2} + ip \tag{E.38}$$

the

$$\ln \Upsilon_b(p) = \int_0^\infty \frac{\mathrm{d}t}{t} \left(-p^2 \, \mathrm{e}^{-t} + \frac{\sin^2 \frac{tp}{2}}{\sinh \frac{tb}{2} \sinh \frac{t}{2b}} \right). \tag{E.39}$$

The first thing to note is that $\Upsilon_b(p)$ is an even function of p.

E.3.2 Symmetries and recursion relations

This function is symmetric around $Q/2$

$$\Upsilon_b(Q/2 - x) = \Upsilon_b(x) \tag{E.40}$$

and satisfies recursion relations

$$\Upsilon_b(x + b) = \gamma(bx)\, b^{1-2bx}\, \Upsilon_b(x), \tag{E.41a}$$

$$\Upsilon_b(x + b^{-1}) = \gamma(b^{-1}x)\, b^{\frac{2x}{b}-1}\, \Upsilon_b(x). \tag{E.41b}$$

where $\gamma(x)$ is defined in (E.9).

Using these relation we note that

$$\Upsilon_b(x + Q) = b^{-2+2x\left(\frac{1}{b}-b\right)}\, \gamma(xb+1)\gamma(b^{-1}x)\, \Upsilon_b(x). \tag{E.42}$$

E.3.3 Zeros and limits

The zeros of the function are at

$$x = mb + \frac{n}{b}, \qquad mn \leq 0, \quad m, n \in \mathbb{Z}. \tag{E.43}$$

Especially we see that $\Upsilon_b(0) = 0$ and that the Taylor series at $x = 0$ is

$$\Upsilon_b(x) \sim x\, \Upsilon_b'(0) + O(x^2). \tag{E.44}$$

The function has no poles.

One has the asymptotic formula for small b [57, p. 7]

$$\Upsilon_b(bx) = \Upsilon_b(b) \frac{b^{1-x}}{\Gamma(x)}. \tag{E.45}$$

E.3.4 Analytical continuation

In order to study the analytical continuation of Υ_b we define the function [40, sec. 7.1]

$$H_b(x) = \Upsilon_b(x) \Upsilon_{ib}(-ix + ib) \tag{E.46}$$

with $\operatorname{Re} b > 0, \operatorname{Im} b < 0$ so that we can use the integral formula (E.36). This function is entire and has simple zeros on a lattice generated by b and $1/b$, and from the recursion relations (E.41) it satisfies

$$H_b(x+b) = \mathrm{e}^{\frac{i\pi}{2}(2bx-1)} H_b(x), \qquad H_b(x+1/b) = \mathrm{e}^{\frac{i\pi}{2}(1-2x/b)} H_b(x). \tag{E.47}$$

We can observe from the Jacobi ϑ_1 function (E.54) and its recursion equations (E.59) that

$$\mathrm{e}^{\frac{i\pi}{2}(x^2+x/b-xb)} \vartheta_1(x/b, 1/b^2) \tag{E.48}$$

satisfies the same recursion relations as $H_b(x)$. The ratio of the two functions is doubly periodic and entire in x so that it depends only on b. We can find this function by setting $x = b/2 + 1/2b$. The end result is

$$H_b(x) = \exp\left[\frac{i\pi}{2}\left(x^2 + \frac{x}{b} - xb + \frac{b^2}{4} - \frac{3}{4b^2} - \frac{1}{4}\right)\right] \frac{\vartheta_1(x/b, 1/b^2)}{\vartheta_1(1/2 + 1/2b^2, 1/b^2)}. \tag{E.49}$$

The exponent can also be written

$$\left(x - \frac{Q}{2}\right)^2 + \frac{2x}{b} - \frac{1}{b^2} - \frac{3}{4}. \tag{E.50}$$

Then with this formula we can study the upsilon function for imaginary b

$$\Upsilon_{ib}(-ix + ib) = \frac{H_b(x)}{\Upsilon_b(x)}. \tag{E.51}$$

The argument function ϑ_1 in the denominator of H_b reaches the real τ-axis where there is a violent singularity running all along, and this is the boundary of the analytical extension of ϑ_1. Then for generic values of x and b the function Υ_b can not be continued to $b \in i\mathbb{R}$.

We ave seen that for $q = 1$ the function ϑ_1 is a periodic Dirac distribution, and products and quotients of Dirac distributions is not defined.

E.4 Jacobi ϑ functions

Jacobi ϑ functions are quasi-periodic functions of two complex variables (z, τ) and they appear in the context of elliptic functions [44] [22, app. 10.A, 36, sec. XII.4.12, 37, app. 8.A, 40, sec. 7.1, 57, sec. 6, 61, chap. 20].

We introduce the variables

$$q = \mathrm{e}^{i\pi\tau}, \qquad w = \mathrm{e}^{2i\pi z}. \tag{E.52}$$

E.4.1 Definitions

The Jacobi θ-functions denoted by $\vartheta_k(z, \tau)$, with $k = 1, \ldots, 4$, are solutions of the heat equation

$$\frac{i}{\pi}\frac{\partial^2 \vartheta_k}{\partial z^2} + 4\frac{\partial \vartheta_k}{\partial \tau} = 0. \tag{E.53}$$

[44] We will follow Witten's convention; for example $\vartheta_1(z, \tau)$ corresponds to `EllipticTheta[1, πz, q]` for Mathematica.

We will consider mainly the first theta function

$$\vartheta_1(z,\tau) = i \sum_{n\in\mathbb{Z}} (-1)^n \, \mathrm{e}^{i\pi\tau(n-1/2)^2} \, \mathrm{e}^{i\pi z(2n-1)} \tag{E.54a}$$

$$= i \sum_{n\in\mathbb{Z}} (-1)^n q^{(n-1/2)^2} w^{n-1/2} \tag{E.54b}$$

$$= -2 \sum_{n\in\mathbb{N}} (-1)^n q^{(n-1/2)^2} \sin(2n\pi z). \tag{E.54c}$$

Shifting the sum gives also

$$\vartheta_1(z,\tau) = -i \sum_{n\in\mathbb{Z}} (-1)^n q^{(n+1/2)^2} w^{n+1/2} \tag{E.54d}$$

$$= 2 \sum_{n\in\mathbb{N}} (-1)^n q^{(n+1/2)^2} \sin\big(2(n+1)\pi z\big). \tag{E.54e}$$

Using this last expression we can note that ϑ_1 is odd for its first argument [57, p. 21]

$$\vartheta_1(-z,\tau) = -\vartheta_1(z,\tau) \tag{E.55}$$

and that

$$\vartheta_1(0,\tau) = 0. \tag{E.56}$$

It is defined only for $\operatorname{Im}\tau > 0$ or $|q| < 1$. To see it we consider the partial series $\sum u_n$ and we use D'Alembert test

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{q^{(n+1/2)^2} w^{n+1/2}}{q^{(n-1/2)^2} w^{n-1/2}}\right| = |q|^{2n} |w| = \mathrm{e}^{-2\pi n \operatorname{Im}\tau} \, \mathrm{e}^{-2\pi \operatorname{Im} z}. \tag{E.57}$$

The ratio tends to zero if $|q| < 1$ or if $\operatorname{Im}\tau > 0$, and the unit circle $|q| = 1$ is a natural boundary of analyticity (we can not impose any condition on z because of the periodicity – see next section). The value $\operatorname{Im}\tau = 0$ is very singular because it means that the torus degenerates to a line, and all zeros collapse to the real axis of z [12].

There is also a branch cut for $q \in [-1, 0[$ due to the factor $q^{1/4}$; this corresponds to $z = 1 + iy$ with $y \in \mathbb{R}$.

For fixed q the function is entire.

This function admits the following infinite product representation

$$\vartheta_1(z,\tau) = q^{1/4} \prod_{n>0} (1-q^{2n})(1-q^{2n}w)\big(1-q^{2(n-1)}w^{-1}\big) \tag{E.58a}$$

$$= 2q^{1/4} \sin\pi z \prod_{n>0} (1-q^{2n})(1-q^{2n}\,\mathrm{e}^{2\pi i z})(1-q^{2n}\,\mathrm{e}^{-2\pi i z}) \tag{E.58b}$$

$$= 2q^{1/4} \sin\pi z \prod_{n>0} (1-q^{2n})(1-2q^{2n}\cos 2\pi z + q^{4n}). \tag{E.58c}$$

E.4.2 Periodicities and zeros

It satisfies [36, p. 347, 37, p. 70, 72, 61, sec. 20.7]

$$\vartheta_1(z+1,\tau) = \mathrm{e}^{-i\pi} \vartheta_1(z,\tau) = -\vartheta_1(z,\tau), \tag{E.59a}$$

$$\vartheta_1(z+\tau,\tau) = \mathrm{e}^{i\pi(1-\tau-2z)} \vartheta_1(z,\tau) = -\,\mathrm{e}^{-i\pi(\tau+2z)} \vartheta_1(z,\tau), \tag{E.59b}$$

$$\vartheta_1(z,\tau+1) = \mathrm{e}^{\frac{i\pi}{4}} \vartheta_1(z,\tau), \tag{E.59c}$$

$$\vartheta_1(z,-1/\tau) = -i\sqrt{-i\tau}\;\mathrm{e}^{i\pi\tau z^2} \vartheta_1(z\tau,\tau). \tag{E.59d}$$

The first two relations come from the double quasi-periodicity in z, and the two others from the modular properties of τ. We have explicitly written the phases in order to compare later with other θ-functions.

Under a general modular transformation [37, p. 72]

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \tag{E.60}$$

the ϑ_1 and its derivative transforms as

$$\vartheta_1\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \omega\sqrt{c\tau+d}\ \exp\left(\frac{i\pi c z^2}{c\tau+d}\right)\vartheta_1(z,\tau), \tag{E.61a}$$

$$\vartheta_1'\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \omega(c\tau+d)^{3/2}\ \exp\left(\frac{i\pi c z^2}{c\tau+d}\right)\left[\vartheta_1'(z,\tau) + \frac{2i\pi z}{c\tau+d}\vartheta_1(z,\tau)\right] \tag{E.61b}$$

where ω is an eighth-root of unity.

Its (simple) zeros for z lie at

$$z = m + n\tau, \qquad m, n \in \mathbb{Z}. \tag{E.62}$$

E.4.3 Derivatives

By convention a prime denotes the derivative with respect to the first variable.

The logarithmic derivative is

$$\frac{\vartheta_1'(z,\tau)}{\vartheta_1(z,\tau)} - \cot\pi z = 4\sum_{n>0}\frac{q^{2n}}{1-q^{2n}}\ \sin(2\pi n z) \tag{E.63}$$

valid for $|\mathrm{Im}\, z| < \mathrm{Im}\,\tau$.

The value of the derivative at $z = 0$ is [36, p. 347]

$$\vartheta_1'(0,\tau) = 2\pi q^{1/4}\prod(1-q^{2n})^3 = 2\pi\eta(z)^3 = \pi\,\vartheta_2(0,\tau)\vartheta_3(0,\tau)\vartheta_4(0,\tau) \tag{E.64}$$

where $\eta(z)$ is the Dedekind function.

Under a modular transformation one gets

$$\vartheta_1'\left(0, \frac{a\tau+b}{c\tau+d}\right) = \mathrm{e}^{\frac{i\pi}{4}}(c\tau+d)^{3/2}\ \vartheta_1'(0,\tau). \tag{E.65}$$

E.4.4 Limits

At $\tau = 0$ or $q = 1$ we have (after shifting $2n-1$ to $2n$)

$$\vartheta_1(z,0) = -\sum_{n\in\mathbb{Z}}(-1)^n\,\mathrm{e}^{2i\pi n z}, \tag{E.66}$$

and for $z \in \mathbb{R}$. Similarly for $\tau = 1$ or $q = -1$ we get

$$\vartheta_1(z,1) = -\,\mathrm{e}^{\frac{i\pi}{4}}\sum_{n\in\mathbb{Z}}(-1)^n\,\mathrm{e}^{2i\pi n z}. \tag{E.67}$$

These sum represents (periodic) Dirac distributions.

Defining the periodic step function

$$\theta(x) = \begin{cases} -1 & -1/2 < x < 0 \\ +1 & 0 < x < 1/2 \end{cases} \tag{E.68}$$

we have [76, sec. 8]

$$\lim_{\varepsilon\to 0^+}\varepsilon\,\frac{\vartheta_1'(z,1+i\varepsilon)}{\vartheta_1(z,1+i\varepsilon)} = -2\pi z + \pi\,\theta(z/2), \qquad z \in\,]-1,1[. \tag{E.69}$$

E.5 Distributions

E.5.1 Periodic distributions

The periodic delta function (also called the Dirac comb) is denoted by

$$\delta(x) \equiv \text{III}(x) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{2i\pi nx} = \sum_{n\in\mathbb{Z}} \delta(x-n) \tag{E.70}$$

(we sometimes use the same symbol for both the usual and periodic delta functions, their sense will be clear from the context).

References

[1] Luis F. Alday, Davide Gaiotto, and Yuji Tachikawa. "Liouville Correlation Functions from Four-dimensional Gauge Theories". *Letters in Mathematical Physics* 91.2 (Feb. 2010), pp. 167–197.
DOI: 10.1007/s11005-010-0369-5.
arXiv: 0906.3219.

[2] Kenichiro Aoki and Eric D'Hoker. "On the Liouville Approach to Correlation Functions for 2-D Quantum Gravity". *Modern Physics Letters A* 07.03 (Jan. 1992), pp. 235–249.
DOI: 10.1142/S0217732392000185.
arXiv: hep-th/9109024.

[3] Katrin Becker, Melanie Becker, and John H. Schwarz. *String Theory and M-Theory: A Modern Introduction*. 1st ed. Cambridge University Press, Dec. 2006.

[4] Carl M. Bender et al. "Infinitely many inequivalent field theories from one Lagrangian" (Aug. 2014).
arXiv: 1408.2432.

[5] Adel Bilal, Frank Ferrari, and Semyon Klevtsov. "2D Quantum Gravity at One Loop with Liouville and Mabuchi Actions" (Oct. 2013).
arXiv: 1310.1951.

[6] T. T. Burwick and A. H. Chamseddine. "Classical and Quantum Considerations of Two-dimensional Gravity" (Apr. 1992).
DOI: 10.1016/0550-3213(92)90473-O.
arXiv: hep-th/9204002.

[7] Sean M Carroll. *Spacetime and geometry: an introduction to general relativity*. English. Addison Wesley, 2004.

[8] Adrian Cooper, Leonard Susskind, and Lárus Thorlacius. "Two-dimensional quantum cosmology". *Nuclear Physics B* 363.1 (Sept. 1991), pp. 132–162.
DOI: 10.1016/0550-3213(91)90238-S.

[9] E. D'Hoker and R. Jackiw. "Classical and quantal Liouville field theory". *Physical Review D* 26.12 (Dec. 1982), pp. 3517–3542.
DOI: 10.1103/PhysRevD.26.3517.

[10] Eric D'Hoker. "Equivalence of Liouville Theory and 2-D Quantum Gravity". *Modern Physics Letters A* 06.09 (Mar. 1991), pp. 745–767.
DOI: 10.1142/S0217732391000774.

[11] Eric D'Hoker and P.S. Kurzepa. "2D Quantum Gravity and Liouville Theory". *Modern Physics Letters A* 05.18 (July 1990), pp. 1411–1421.
DOI: 10.1142/S0217732390001608.

[12] Atish Dabholkar. *Lectures on Liouville theory.* 2013.

[13] F. David. “Conformal Field Theories Coupled to 2-D Gravity in the Conformal Gauge”. *Modern Physics Letters A* 03.17 (Dec. 1988), pp. 1651–1656.
DOI: 10.1142/S0217732388001975.

[14] Corinne De Lacroix. “Gravité quantique à deux dimensions. Régularisation de l’action de Liouville sur la sphère”. PhD thesis. École Normale Supérieure, 2013.
URL: https://mail-attachment.googleusercontent.com/attachment/?ui=2\&ik=9fcaf12d66\&view=att\&th=13f7b3755bf87d6b\&attid=0.1\&disp=safe\&zw\&saduie=AG9B_P-7MXTfx_HYoJbZ8BUbReCC\&sadet=1372163209928\&sads=tZXjVuv4TWjEf5a8QnaZA4YjUxk\&sadssc=1.

[15] P. Di Francesco and D. Kutasov. “Correlation functions in 2D string theory”. *Physics Letters B* 261.4 (June 1991), pp. 385–390.
DOI: 10.1016/0370-2693(91)90444-U.

[16] Jacques Distler and Hikaru Kawai. “Conformal field theory and 2D quantum gravity”. *Nuclear Physics B* 321.2 (July 1989), pp. 509–527.
DOI: 10.1016/0550-3213(89)90354-4.

[17] H. Dorn and H.-J. Otto. “Two and three-point functions in Liouville theory” (Mar. 1994).
DOI: 10.1016/0550-3213(94)00352-1.
arXiv: hep-th/9403141.

[18] Vladimir Dotsenko. *Série de Cours sur la Théorie Conforme.* français. Sept. 2006.
URL: https://hal.archives-ouvertes.fr/cel-00092929.

[19] Frank Ferrari, Semyon Klevtsov, and Steve Zelditch. “Gravitational Actions in Two Dimensions and the Mabuchi Functional” (Dec. 2011).
DOI: 10.1016/j.nuclphysb.2012.02.003.
arXiv: 1112.1352.

[20] Frank Ferrari, Semyon Klevtsov, and Steve Zelditch. “Random Geometry, Quantum Gravity and the Kähler Potential” (July 2011).
DOI: 10.1016/j.physletb.2011.09.098.
arXiv: 1107.4022.

[21] Michael Flohr. “On Modular Invariant Partition Functions of Conformal Field Theories with Logarithmic Operators”. *International Journal of Modern Physics A* 11.22 (Sept. 1996), pp. 4147–4172.
DOI: 10.1142/S0217751X96001954.
arXiv: hep-th/9509166.

[22] Philippe Di Francesco, Pierre Mathieu, and David Senechal. *Conformal Field Theory.* Français. 2nd. Springer, Jan. 1999.

[23] Stefan Fredenhagen. “Boundary conditions in Toda theories and minimal models”. *Journal of High Energy Physics* 2011.2 (Feb. 2011).
DOI: 10.1007/JHEP02(2011)052.
arXiv: 1012.0485.

[24] Stefan Fredenhagen and Cosimo Restuccia. “The geometry of the limit of N=2 minimal models” (Aug. 2012).
arXiv: 1208.6136.

[25] Stefan Fredenhagen and Cosimo Restuccia. “The large level limit of Kazama-Suzuki models” (Aug. 2014).
arXiv: 1408.0416.

[26] Stefan Fredenhagen, Cosimo Restuccia, and Rui Sun. “The limit of N=(2,2) superconformal minimal models” (Apr. 2012).
arXiv: 1204.0446.

[27] Stefan Fredenhagen and Volker Schomerus. “On Minisuperspace Models of S-branes”. *Journal of High Energy Physics* 2003.12 (Dec. 2003), pp. 003–003.
DOI: 10.1088/1126-6708/2003/12/003.
arXiv: hep-th/0308205.

[28] Stefan Fredenhagen and David Wellig. “A common limit of super Liouville theory and minimal models”. *Journal of High Energy Physics* 2007.09 (Sept. 2007), pp. 098–098.
DOI: 10.1088/1126-6708/2007/09/098.
arXiv: 0706.1650.

[29] Daniel H. Friedan. “Introduction to Polyakov’s string theory”. Dec. 1982.
URL: http://www.physics.rutgers.edu/~friedan/papers/Les_Houches_1982.pdf.

[30] Tamas Fulop. “Reduced SL(2,R) WZNW Quantum Mechanics”. *Journal of Mathematical Physics* 37.4 (1996), p. 1617.
DOI: 10.1063/1.531472.
arXiv: hep-th/9502145.

[31] Matthias R. Gaberdiel and Paulina Suchanek. “Limits of minimal models and continuous orbifolds”. *Journal of High Energy Physics* 2012.3 (Mar. 2012).
DOI: 10.1007/JHEP03(2012)104.
arXiv: 1112.1708.

[32] Jens A. Gesser. “Non-compact Geometries in 2D Euclidean Quantum Geometries”. PhD thesis. University of Copenhagen, Apr. 2008.
URL: http://www.nbi.ku.dk/english/research/phd_theses/phd_theses_2008/jens_gesser/jens_gesser.pdf.

[33] P. Ginsparg. “Matrix models of 2d gravity” (Dec. 1991).
arXiv: hep-th/9112013.

[34] P. Ginsparg and Gregory Moore. “Lectures on 2D gravity and 2D string theory (TASI 1992)” (Apr. 1993).
arXiv: hep-th/9304011.

[35] Gaston Giribet. “On the timelike Liouville three-point function”. *Physical Review D* 85.8 (Apr. 2012).
DOI: 10.1103/PhysRevD.85.086009.
arXiv: 1110.6118.

[36] Roger Godement. *Analyse mathématique IV : intégration et théorie spectrale, analyse harmonique, le jardin des délices modulaires*. Français. Vol. IV. Springer, Mar. 2008.

[37] Michael B. Green, John H. Schwarz, and Edward Witten. *Superstring Theory: Introduction*. Vol. 1. Cambridge University Press, July 1988.

[38] M. Gutperle and A. Strominger. “Timelike Boundary Liouville Theory”. *Physical Review D* 67.12 (June 2003).
DOI: 10.1103/PhysRevD.67.126002.
arXiv: hep-th/0301038.

[39] Leszek Hadasz, Zbigniew Jaskolski, and Paulina Suchanek. “Elliptic recurrence representation of the N=1 Neveu-Schwarz blocks”. *Nuclear Physics B* 798.3 (Aug. 2008), pp. 363–378.
DOI: 10.1016/j.nuclphysb.2007.12.015.
arXiv: 0711.1619.

[40] Daniel Harlow, Jonathan Maltz, and Edward Witten. “Analytic Continuation of Liouville Theory” (Aug. 2011).
arXiv: 1108.4417.

[41] Brian Hatfield. *Quantum Field Theory of Point Particles and Strings*. English. New Ed edition. Addison Wesley, Apr. 1998.

[42] Shoichi Ichinose. “Renormalization of Polyakov’s two-dimensional quantum gravity”. *Physics Letters B* 251.1 (Nov. 1990), pp. 49–53.
DOI: 10.1016/0370-2693(90)90230-4.

[43] Yacine Ikhlef, Jesper Lykke Jacobsen, and Hubert Saleur. “Three-point functions in c <= 1 Liouville theory and conformal loop ensembles” (Sept. 2015).
arXiv: 1509.03538.

[44] R. Jackiw. “Another View on Massless Matter-Gravity Fields in Two Dimensions” (Jan. 1995).
arXiv: hep-th/9501016.

[45] D. R. Karakhanyan, R. P. Manvelyan, and R. L. Mkrtchyan. “Area-preserving structure of 2d-gravity” (Jan. 1994).
arXiv: hep-th/9401031.

[46] H. Kawai. “Quantum gravity and random surfaces”. *Nuclear Physics B - Proceedings Supplements* 26 (1992), pp. 93–110.
DOI: 10.1016/0920-5632(92)90231-G.

[47] Hikaru Kawai and Masao Ninomiya. “Renormalization group and quantum gravity”. *Nuclear Physics B* 336.1 (May 1990), pp. 115–145.
DOI: 10.1016/0550-3213(90)90345-E.

[48] Elias Kiritsis. *String Theory in a Nutshell*. English. Princeton University Press, 2007.

[49] Hiroyuki Kobayashi and Izumi Tsutsui. “Quantum mechanical Liouville model with attractive potential”. *Nuclear Physics B* 472.1-2 (July 1996), pp. 409–426.
DOI: 10.1016/0550-3213(96)00230-1.
arXiv: hep-th/9601111.

[50] I. K. Kostov and V. B. Petkova. “Bulk correlation functions in 2D quantum gravity”. *Theoretical and Mathematical Physics* 146.1 (Jan. 2006), pp. 108–118.
DOI: 10.1007/s11232-006-0011-y.
arXiv: hep-th/0505078.

[51] I. K. Kostov and V. B. Petkova. “Non-Rational 2D Quantum Gravity: I. World Sheet CFT”. *Nuclear Physics B* 770.3 (May 2007), pp. 273–331.
DOI: 10.1016/j.nuclphysb.2007.02.014.
arXiv: hep-th/0512346.

[52] Ivan K. Kostov, Benedicte Ponsot, and Didina Serban. “Boundary Liouville Theory and 2D Quantum Gravity” (July 2003).
arXiv: hep-th/0307189.

[53] D. Kutasov. “Some Properties of (Non) Critical Strings” (Oct. 1991).
arXiv: hep-th/9110041.

[54] Robert Marnelius. “Canonical quantization of polyakov’s string in arbitrary dimensions”. *Nuclear Physics B* 211.1 (Jan. 1983), pp. 14–28.
DOI: 10.1016/0550-3213(83)90183-9.

[55] Emil J. Martinec and Wynton E. Moore. “Modeling Quantum Gravity Effects in Inflation” (Jan. 2014).
arXiv: 1401.7681.

[56] N.E. Mavromatos and J.L. Miramontes. “Regularizing the Functional Integral in 2D-Quantum Gravity”. *Modern Physics Letters A* 04.19 (Sept. 1989), pp. 1847–1853.
DOI: 10.1142/S0217732389002082.

[57] Will McElgin. “Notes on Liouville Theory at $c \leq 1$” (June 2007).
DOI: 10.1103/PhysRevD.77.066009.
arXiv: 0706.0365.

[58] Gregory Moore. “Gravitational Phase Transitions and the Sine-Gordon Model” (Mar. 1992).
arXiv: hep-th/9203061.

[59] Gregory Moore and Philip Nelson. “Measure for moduli The Polyakov string has no nonlocal anomalies”. *Nuclear Physics B* 266.1 (Mar. 1986), pp. 58–74.
DOI: 10.1016/0550-3213(86)90177-X.

[60] Wynton E. Moore. “Primordial fluctuations in extended Liouville theory” (Nov. 2014).
arXiv: 1411.2612.

[61] *NIST Digital Library of Mathematical Functions*.
URL: http://dlmf.nist.gov/.

[62] Mikio Nakahara. *Geometry, Topology and Physics*. 2nd edition. Institute of Physics Publishing, June 2003.

[63] Yu Nakayama. “Liouville Field Theory – A decade after the revolution” (Feb. 2004).
DOI: 10.1142/S0217751X04019500.
arXiv: hep-th/0402009.

[64] L. O’Raifeartaigh, J. M. Pawlowski, and V. V. Sreedhar. “The Two-exponential Liouville Theory and the Uniqueness of the Three-point Function”. *Physics Letters B* 481.2-4 (May 2000), pp. 436–444.
DOI: 10.1016/S0370-2693(00)00448-2.
arXiv: hep-th/0003247.

[65] L. O’Raifeartaigh and V. V. Sreedhar. “Duality in Liouville Theory as a Reduced Symmetry”. *Physics Letters B* 461.1-2 (Aug. 1999), pp. 66–70.
DOI: 10.1016/S0370-2693(99)00816-3.
arXiv: hep-th/9906116.

[66] Ari Pakman. “Liouville theory without an action”. *Physics Letters B* 642.3 (Nov. 2006), pp. 263–269.
DOI: 10.1016/j.physletb.2006.09.064.
arXiv: hep-th/0601197.

[67] Joseph Polchinski. “A two-dimensional model for quantum gravity”. *Nuclear Physics B* 324.1 (Sept. 1989), pp. 123–140.
DOI: 10.1016/0550-3213(89)90184-3.

[68] Joseph Polchinski. *String Theory: Volume 1, An Introduction to the Bosonic String*. Cambridge University Press, June 2005.

[69] A. M. Polyakov. “Quantum geometry of bosonic strings”. *Physics Letters B* 103.3 (July 1981), pp. 207–210.
DOI: 10.1016/0370-2693(81)90743-7.

[70] Cosimo Restuccia. “Limit theories and continuous orbifolds” (Oct. 2013).
arXiv: 1310.6857.

[71] Sylvain Ribault. “Conformal field theory on the plane” (June 2014).
arXiv: 1406.4290.

[72] Sylvain Ribault and Raoul Santachiara. “Liouville theory with a central charge less than one” (Mar. 2015).
arXiv: 1503.02067.

[73] I. Runkel and G. M. T. Watts. “A non-rational CFT with c=1 as a limit of minimal models”. *Journal of High Energy Physics* 2001.09 (Sept. 2001), pp. 006–006.
DOI: 10.1088/1126-6708/2001/09/006.
arXiv: hep-th/0107118.

[74] I. Runkel and G. M. T. Watts. “A non-rational CFT with central charge 1” (Jan. 2002).
arXiv: hep-th/0201231.

[75] Norisuke Sakai. “c=1 Two dimensional quantum gravity”. *Vistas in Astronomy* 37 (1993), pp. 585–599.
DOI: 10.1016/0083-6656(93)90098-5.

[76] Volker Schomerus. “Rolling Tachyons from Liouville theory”. *Journal of High Energy Physics* 2003.11 (Nov. 2003), pp. 043–043.
DOI: 10.1088/1126-6708/2003/11/043.
arXiv: hep-th/0306026.

[77] Volker Schomerus. “Non-compact String Backgrounds and Non-rational CFT”. *Physics Reports* 431.2 (Aug. 2006), pp. 39–86.
DOI: 10.1016/j.physrep.2006.05.001.
arXiv: hep-th/0509155.

[78] Nathan Seiberg. “Notes on Quantum Liouville Theory and Quantum Gravity”. *Progress of Theoretical Physics Supplement* 102 (1990), pp. 319–349.
DOI: 10.1143/PTPS.102.319.

[79] Nathan Seiberg and David Shih. “Branes, Rings and Matrix Models in Minimal (Super)string Theory”. *Journal of High Energy Physics* 2004.02 (Feb. 2004), pp. 021–021.
DOI: 10.1088/1126-6708/2004/02/021.
arXiv: hep-th/0312170.

[80] Andrew Strominger. “Open String Creation by S-Branes” (Sept. 2002).
arXiv: hep-th/0209090.

[81] Andrew Strominger and Tadashi Takayanagi. “Correlators in Timelike Bulk Liouville Theory” (Mar. 2003).
arXiv: hep-th/0303221.

[82] J. Teschner. “Liouville theory revisited” (Apr. 2001).
DOI: 10.1088/0264-9381/18/23/201.
arXiv: hep-th/0104158.

[83] David Tong. “Lectures on String Theory” (Aug. 2009).
arXiv: 0908.0333.

[84] A. B. Zamolodchikov and Al B. Zamolodchikov. “Structure Constants and Conformal Bootstrap in Liouville Field Theory” (June 1995).
DOI: 10.1016/0550-3213(96)00351-3.
arXiv: hep-th/9506136.

[85] Al Zamolodchikov. “On the Three-point Function in Minimal Liouville Gravity” (May 2005).
arXiv: hep-th/0505063.

[86] Alexei Zamolodchikov and Alexander Zamolodchikov. *Lectures on Liouville Theory and Matrix Models*. 2007.
URL: http://qft.itp.ac.ru/ZZ.pdf.

[87] Bruno Carneiro da Cunha and Emil J. Martinec. “Closed String Tachyon Condensation and Worldsheet Inflation”. *Physical Review D* 68.6 (Sept. 2003).
DOI: 10.1103/PhysRevD.68.063502.
arXiv: hep-th/0303087.