Supergravity and Kähler geometries

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Contents

Chapter 1

Introduction

1.1 Background

1.1.1 Quantum gravity and string theory

Finding a theory of quantum gravity is a major goal of theoretical physics. Indeed the 20th century has seen the discovery of two great theories – quantum field theory (QFT) and general relativity (GR) – that both work extremely well in their respective domains of application but which cannot be reconciled on the overlap. The main difficulty resides in the fact that QFT rely heavily on the concept of renormalization in order to obtain sensible results from the computations that would otherwise yield divergences. On the other hand GR is non-renormalizable and leads to incurable divergences.

A theory of quantum gravity is needed in order to answer some of the most important questions concerning our universe. In particular primordial cosmology and the origin of the universe can be properly address only within this context as they touch the very nature of spacetime and the latter require a complete theory of quantum gravity to be properly understood. Similarly black holes are objects formed by a huge concentration of matter and they cannot be properly described in general relativity. For the moment these problems get only partial answers by using semi-classical methods. Both cases are linked to the presence of *singularities* (the Big-Bang and the center of the black hole) that should be resolved by a proper quantum treatment of gravity.

Another interesting quest is the unification of the forces and the understanding of the very nature of interactions and matter. The current knowledge culminates in the standard model of particle physics which describe all matter and non-gravitational forces that have been measured. But this theory is still unsatisfactory for several reasons: there are many free parameters (19 plus 7-8 neutrino masses) that are lacking theoretical interpretation. Similarly the hierarchy problem states that the Higgs mass should be of the same order of the cut-off scale where new physics appear (or the Planck mass otherwise), and in the current framework this value can be understand only by a very fine-tuning of the parameters, which is not natural. Another problem is the prediction of a huge value for the cosmological constant. The two last points are related to the question of naturalness which asks that parameters have natural values (in the correct units). Finally the standard model does not explain why there are three generations of fermions, the mass of the neutrinos nor why the gauge group is

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1). \tag{1.1.1}$$

A satisfying theory should be able to provide the derivation of the parameters from more fundamental properties (for examples through the dynamics of background fields) and to explain why one observes this field content. A first possibility is to unify the gauge group into one unique group at higher energy which would reduce the number of gauge couplings and unify matter families (through the embedding into representations of this group).

String theory is a promising candidate for a consistent quantum gravity theory which provides a grand unification framework at the same time. In this theory the fundamental constituents are strings and the usual fields appear as excitation modes of these strings. The interactions of the strings are non-local in spacetime and this smearing reduces the UV divergences as interactions cannot be concentrated at a point. The very existence of a fundamental string puts very stringent constraint on the structure of spacetime: supersymmetry is necessary for having a consistent theory, and spacetime should have 10 dimensions (for the five possible superstring theories). Hence one needs to hide these dimensions, either by compactification (with Kaluza–Klein dimensional reduction) or by using a braneworld scenario [13, 152, 153]. On the bright side string theory is unique and it describes quantum gravity unified to matter and interactions, and there are no free parameters (before compactification).

For decades the developments of string theory were limited to a perturbative analysis. Recently the understanding of string theory has been deepened by a series of discoveries concerning its non-perturbative structure: all five superstring theories (type II A and B, type I and two heterotic) are related by dualities to each other, and to an 11-dimensional theory called M-theory. The latter is unique and is believed to be the fundamental theory, but its definition is not known, and only some of its aspects are understood in some limits. Finally the previous analysis yielded the existence of branes which are extended objects generalizing particles and strings. They proved to be fundamental in the realization of black holes from string theory.

1.1.2 Supersymmetry and supergravity

In order to pursue the goal of unification one could ask if the internal gauge symmetry can be unified with spacetime symmetries. A no-go theorem from Coleman and Mandula [61] stated that it was impossible and the symmetry group is necessary a direct product

$$\text{conformal} \times \text{internal} \tag{1.1.2}$$

(in general one considers the Poincaré subgroup of the conformal group). But Haag, Łopuszański and Sohnius discovered a loophole in the argument [103]: the above group can be extended into the superconformal group (which includes the super-Poincaré group) by adding anticommuting generators. This group contains an automorphism subgroup called the R-symmetry group that acts both on the fermionic generators and as an internal symmetry.

Supersymmetry is generated by fermionic generators Q and it relates bosons to fermions, and conversely

$$Q\,|\text{fermion}\rangle = |\text{boson}\rangle\,, \qquad Q\,|\text{boson}\rangle = |\text{fermion}\rangle\,, \tag{1.1.3}$$

and the anticommutator of these generators is equivalent to a translation

$$\{Q, Q\} \sim P. \tag{1.1.4}$$

Fields of different spins are gathered into multiplets that transform irreducibly under super-Poincaré transformations. A theory with supersymmetry is characterized by the number N of fermionic generators; in $d = 4$ the condition that no spin higher than 2 are generated implies that $N \leq 8$ (when $N \geq 2$ one speaks about extended supersymmetry). This symmetry is very powerful and imposes constraints – the higher N is, the more severe they are – on the theory. For example $N = 1$ is already sufficient for curing some of the problems of the standard model (even if these extensions suffer from other problems): the Higgs mass is stabilized as it inherits the mass protection from its partner. For extended supersymmetry exact solutions could be derived, see for example the work of Seiberg and Witten on $N = 2$ [160, 161] and the integrability of $N = 4$ [12, 14, 20]. The reason is that the scalar fields ϕ^i parametrizes a non-linear sigma model

$$\mathcal{L} = -\frac{1}{2}\, g_{ij}(\phi)\, \partial\phi^i \partial\phi^j \tag{1.1.5}$$

whose target manifold with metric g_{ij} is very constrained by supersymmetry, and other fields of the multiplets inherits these properties. In particular the isometry group of this manifold translates (mostly) into the global symmetry of the Lagrangian.

Interestingly local supersymmetry includes general relativity: indeed the fact that the anticommutators of two supersymmetries close on the momentum implies that one cannot make local supersymmetry without making local the Poincaré group. This theory is called *supergravity*. In this context the R-symmetry group is made local and provides gauge interactions: this leads to a unification of spacetime and internal gauge symmetries!

As seen in the previous section, supersymmetry is necessary ingredient of string theory for including fermions in the spectrum and for removing inconsistencies (such as the tachyons). In this case supergravity corresponds to the low-energy approximation of superstring theories.

In this review we focus on $N = 2$ supergravity. The latter admits three main multiplets: the gravity multiplet (containing the metric and a vector field called the graviphoton), the vector multiplet (containing a vector field and a complex scalar field) and the hypermultiplet (containing four real scalar fields). This theory has more symmetries than $N = 1$ and the additional structures facilitate the computations, but it is also less constrained than higher N theories (such as the maximal $N = 8$ supergravity) and as a consequence it has a richer dynamics and admits more different models. The scalar manifold in $N = 1$ is only Kähler, while in $N = 2$ additional conditions imply that it is a direct product

$$\text{special Kähler} \times \text{quaternionic}, \tag{1.1.6}$$

and there is little freedom in their definition (for example a unique holomorphic function is sufficient to define a special Kähler manifold). Finally the scalar manifolds of $N > 2$ supergravity are all symmetric and fixed once the number of vector multiplets is given (hence the manifold is unique for $N > 4$). These spaces possess very interesting geometrical properties which all have an interpretation from supersymmetry.

Currently supersymmetry has not been found in nature, which means that it should be broken at an energy higher than those accessible in the current experiments. From the phenomenological point of view theories with a low number of supersymmetries ($N = 1, 2$) are preferable since they are closer to the standard model. Moreover $N = 2$ supergravity corresponds to the effective action of the low-energy limit of type II string theory compactified on a Calabi–Yau manifold. These models present some interest because they are very similar to the $N = 1$ theories resulting from the compactification of the heterotic string theory on a Calabi–Yau manifold [80, 180, 181].

The simplest version of these theories are called ungauged theories because the only local symmetry corresponds to the local super-Poincaré group. The $N = 2$ theory is quite simple in this case as some fields decouple from the others due to the absence of scalar potential (this also imply a vanishing cosmological constant). In order to get a richer dynamics one needs to deform the theory by using some of the vector fields as gauge fields for a local gauge symmetry – one then obtains a gauged supergravity. In the context of string compactification, this corresponds to some p-forms which are not vanishing along cycles of the internal manifold.

Finally supergravity is interesting by itself as a theory of quantum gravity: it is known that supersymmetry improves the ultraviolet behaviour of a theory. For example $N = 4$ super Yang–Mills is perturbatively finite. There is hope that a similar property is true for the maximal $N = 8$ supergravity: in particular recent studies have shown by explicit computations that expected loop divergences (from symmetry arguments) do not appear, for example at 3-loops in $N = 4$ (see for example [24–27]).

1.1.3 Black holes

General relativity is the theory of gravitational phenomena. It describes the dynamical evolution of spacetime through the Einstein–Hilbert action that leads to Einstein equations. The latter are highly non-linear differential equations and finding exact solutions is a notoriously difficult problem. There are different types of solutions but this review will cover only black-hole-like solutions (type-D in the Petrov classification) which can be described as particle-like objects that carry some charges, such as a mass or an electric charge.

Black holes are very specific entities that put a lot of strain on theories of quantum gravity, and as such they are useful sandboxes where one can test the properties and the predictions of the theory. Rotating black holes are the most relevant subcases for astrophysics as it is believed that most astrophysical black holes are rotating. These solutions may also provide exterior metric for rotating stars.

They resemble a lot a particle in the sense that they do not seem to have a structure: they are defined by few parameters – such as the mass, the electric charge or the angular momentum –, and any perturbation of a black hole dies off quickly. The most general solution of this type in pure Einstein–Maxwell gravity is the Plebański–Demiański metric [150, 151]: it possesses six charges: mass m, NUT charge n, electric charge q, magnetic charge p, rotation j and acceleration a.

Classically a black hole is a region delimited by an horizon where the gravitational field is so strong that nothing can escape from it (not even light), and they can be formed from the gravitational collapse of a supermassive star. At the center of the black hole is a singularity where the curvature of spacetime becomes infinite. Such divergence indicates a breakdown of the theory: indeed gravitational effects are so important close to the origin that classical GR is not sufficient and one needs a full quantization of gravity in order to account for quantum effects.

Bekenstein and Hawking discovered that a black hole behaves like a thermodynamical system in the sense that it has a temperature T, an entropy S, and each charge is associated to a potential. A black hole emits a perfect black body radiation at the temperature T which is related to the gravity on the horizon (called the surface gravity). Then the entropy can be derived from the first law using the relation between the mass and the energy. This picture explains the apparent simplicity of black holes: a statistical ensemble made of a great number of particles moving in a box is determined only by few parameters (temperature, pressure…). Statistical physics teaches us that entropy is related to the number of microstates of a system, and it is very natural to ask from a theory of quantum gravity what are these states for the black holes. A specific subclass consists of extremal black holes which have a vanishing temperature.

Usual systems have accustomed us to think that the entropy of a system should be proportional to its volume. This is not the case in gravity where the entropy follows an area law

$$S = \frac{A}{4}, \tag{1.1.7}$$

where A is the area of the horizon (in Planck units). This means that there is far less degrees of freedom than what one would think, and these would live on the horizon of the black hole. This suggests the existence of an holographic principle which states that (some) gravitational systems can be entirely described by data on their boundary. This principle has seen a nice realization within string theory under the adS/CFT correspondence.

Black holes are such special that it is always useful to classify all possible black hole solution that can be found in a given theory or in its low-energy limit. Hence studying black holes in supergravity gives indirect clues on the structure of string theory. In their seminal paper [165], Strominger and Vafa set up a framework where the microstates were identified with branes. The agreement between the microscopic counting and the macroscopic entropy computed in the corresponding supergravity have been shown to hold for many BPS or extremal black holes.

1.1.4 BPS solutions and adS black holes

A BPS solution of supergravity is a solution of the equations of motion which preserves some supersymmetry (indicated as a fraction), i.e. it is annihilated by the action of some supersymmetry generators and it defines a background with its own supersymmetry algebra. Extremal black holes form long BPS representations and the action of supersymmetry is well defined, which is not the case for finite temperature black holes [10, p. 8], and for this reason they share similar properties.[1] These solutions are very useful because some of their properties are protected by non-renormalization theorem due to supersymmetry, and this makes it possible to infer their behaviour at strong coupling. In particular this last property is essential for comparing the entropy with the microstate counting.

Extremal black holes can be seen as solitons, i.e. solutions interpolating between two vacua, one sitting at the radial infinity (called the UV), the other being the near-horizon geometry (the IR) – both are solutions of the BPS equations. They are subject to the so-called *attractor mechanism* [81–83, 85, 167]: the scalar fields take on the horizon constant values which depends only of the electromagnetic charges of the solution. This is as if the fields were forgetting everything about their radial evolution outside the black hole, and in particular the corresponding values do not depend on the values at infinity.

We will mainly focus on adS black holes which have a negative cosmological constant. The first motivation is to provide solutions that can be used in the context of the adS/CFT correspondence, and in particular for the application to condensed matter through adS/CMT [106, 141, 156]. Moreover solutions with a negative cosmological constant are more natural in the context of gauged supergravity and string theory. AdS black holes present a richer thermodynamics [107, 154] than their asymptotically flat cousins; this results from the cosmological constant which acts as a space cut-off, the black hole does not feel the entire spacetime and is more stable as a consequence. Another interesting property of adS space is that a field can have a negative mass without being unstable if it satisfies the BreitenLohner–Freedman (BF) bound [31, 32].

Strictly speaking adS black holes are not asymptotic to adS space: if magnetic charges are present then the asymptotic space is deformed to the so-called magnetic adS (madS). It can be shown that to each madS vacuum is associated an adS vacuum. 1/2-BPS black holes are asymptotically adS but they correspond typically to a naked singularity, and for this reason we will concentrate on 1/4-BPS black holes.

1.1.5 Taub–NUT spacetime

The Taub–NUT spacetime is very peculiar and Misner said it was "a counterexample to almost anything" believed in general relativity. For example it can be BPS without being extremal. This solution is characterized by the NUT charge n which plays the same role as

[1] Moreover a static BPS black hole is necessarily extremal.

the magnetic charge in electromagnetism (in this analogy the usual mass corresponds to the electric charge) and for this reason one also refers to it as a magnetic mass.

This spacetime is a solution of the vacuum Einstein equation with no cosmological constant. In this case the space is not asymptotically flat and it is characterized by the value of n, the off-diagonal component of the metric giving a vector potential

$$A_\phi \sim g_{t\phi} = 2n\cos\theta. \tag{1.1.8}$$

This is recognized as being the potential of a magnetic-like monopole. On the other hand the solution can also include a mass m which asymptotically gives the usual scalar potential

$$\phi \sim \frac{1}{2}\,(1 - g_{tt}) = -\frac{m}{r} \tag{1.1.9}$$

which is the potential of an electric-like point source. Then the Taub–NUT solution with mass is a gravitational dyon.

The metric does not have any curvature singularity, in particular the space is regular at $r = 0$. But the metric suffers from a worse pathology which is the presence of Misner strings due to wire-like singularities (this is similar to the Dirac strings that one introduces with magnetic monopoles). These strings can be removed by using two patches of coordinates, but as a consequence closed timelike curves appear, with the periodicity of the time given by

$$\Delta t = 8\pi n. \tag{1.1.10}$$

Closed timelike curve may not appear for hyperbolic black holes if the NUT charge lies in some range [15, 17].

The solution is better behaved in Euclidean signature. There it corresponds to a gravitational instanton, which is a non-singular solution of the equations of motion with a finite action that contributes to the computation of the partition function in the saddle point approximation.

The NUT charge can be incorporated in more general solutions, for example in supergravity and with a non-vanishing cosmological constant.

1.2 Motivations

The last decades has seen a lot of works on $N = 2$ gauged supergravity for its applications on string phenomenology, holography and black holes. While many the ungauged theory has been deeply studied and understood, much less is known on the gauged version. For example a complete classification of BPS solutions exist [19, 115, 142, 148], the attractor mechanism has received a lot of attention [42, 46, 48]), and fairly general non-extremal solutions have been found [57, 58].

The first step is to study the vacua that can be obtained in this theory. In particular the most natural one is the $N = 2$ adS_4 vacua which have been discussed in [96, 110, 130, 178, 189], while adS_4 vacua with less supersymmetries were found in [40, 129, 130]. Another important type of vacua consists in the near-horizon geometries $\mathrm{adS}_2 \times \Sigma_g$ where Σ_g is a Riemann surface of genus g, and it has also received attention recently [76, 96, 104, 189]. Some steps towards a classification of the BPS solutions have been taken in [37, 123, 124, 143]. The equations for more specific ansatz have also been studied, for example static black holes [35, 52, 104, 111, 113, 155] or maximally supersymmetric solutions [110]. The supersymmetry algebras associated to BPS solutions were worked out in [108, 112]. Finally the attractor mechanism also takes place in these theories [22, 35, 53, 76, 114, 117, 120, 146].

As reviewed above the archetypal black hole of Einstein–Maxwell theory with cosmological constant is the Plebański–Demiański (PD) solution [150, 151] which contains six charges: mass m, NUT n, electric q and magnetic p charges, spin j and acceleration a. In the context of supergravity on adS space and of adS/CFT it is natural to consider topological horizons, which are not only spherical, but also flat or hyperbolic (or a compact Riemann surface obtained by quotienting with a discrete group) [121, 136, 175]; indeed the usual wisdom about horizon topology in asymptotically flat spaces does not hold for adS spaces [36]. The supersymmetry of the (topological) PD solution and its truncations has been studied in [4, 36, 122, 125, 154] by embedding it into pure $N = 2$ gauged supergravity, which is equivalent to taking constant scalars. Non-BPS solutions with running scalars have been studied in the STU model (which includes three vector multiplets) and its truncations [54, 56, 97, 98, 172]. Constructing the general solution with non-constant scalars in general $N = 2$ gauged supergravity is an outstanding goal, and a first step is to look at the BPS subclass which is simpler to study.

In ungauged supergravity static black holes are 1/2-BPS. The corresponding solutions in gauged supergravity are naked singularity (but there are regular 1/2-BPS rotating black holes) and cannot have magnetic charges [36, 111, 113, 155]. A static 1/4-BPS black hole with constant scalars was found in [52] where it was put forward that the solution is regular only if the horizon is hyperbolic. An important step has been taken by Cacciatori and Klemm who found the first regular 1/4-BPS black holes with running scalars in the STU model [35], and it was generalized to any symmetric very special manifold in [96] in the case of vanishing axions. In particular it was shown in [76, 113] that spherical horizons are possible if the scalars are non-trivial. These solutions have no flat space limit and are thus very different from the 1/2-BPS solutions [113]; as explained above they have a madS vacua. Finally the general analytic 1/4-BPS solution of Fayet–Iliopoulos (FI) gauged supergravity with a symmetric scalar manifold (with an arbitrary number of vector multiplets, running scalars and dyonic charges) was built in [105] using a formalism developed in [118] which rely heavily on the properties of very special Kähler manifolds. A 1/4-BPS black hole with NUT and magnetic charges was constructed in the case of only one vector multiplet [62]. All the previous discussion apply to FI gauged supergravity, but very few solutions with hypermultiplets have been found: recently an analytic BPS solution have been described in [53], while some numerical 1/4-BPS solutions were built in [104] (1/2-BPS solutions with pathological behaviour have been discussed in [111]). Finally 1/8-BPS solutions were classified in [143].

Solutions with a NUT charge are interesting in the fluid/gravity correspondence where a NUT charge in spacetime translates to vorticity in the dual fluid [38, 127, 149]. Another interesting path is to perform a Wick rotation and to compare the free energy with the result in the dual CFT using localization. Indeed it was put in evidence in a series of papers by Martelli and collaborators on minimal $N = 2$ gauged supergravity that the NUT charge and the acceleration correspond to the two squashing parameters of the boundary S^3 [137–140].

1.3 Content

An important motivation of our work is to study black holes which can be embedded into M-theory, such as the STU model with a specific choice of gaugings which is a dimensional reduction of $d = 11$ supergravity on S^7. In presence of the NUT charge the holographic duals correspond to the ABJM theory on a curved manifold. In particular after the Euclidean continuation these contain Seifert spaces (given by a U(1) bundle over Σ_g), including the Lens spaces $S^3/\mathbb{Z}_n$, where supersymmetry has been preserved by twisting the theory with respect to a general $\mathrm{U}(1) \subset \mathrm{SU}(4)_R \times \mathrm{U}(1)_R$. From an $N = 2$ point of view this includes flavour as well as R-symmetries.

The goal of this work is to deepen the understanding of BPS solutions in (matter-coupled) $N = 2$ gauged supergravity with abelian gaugings. When there are no hypermultiplets this corresponds to Fayet–Iliopoulos (FI) gauging.

In the case where hypermultiplets are present, the hyperscalars are the only scalar fields to be charged. Fortunately the isometries of homogeneous (symmetric or not) special quaternionic manifolds have been classified by de Wit and van Proeyen [180, 182–184]. These manifolds are constructed as a fibration over a special Kähler manifold through the c-map, and some isometries of the latter can be lifted to the full quaternionic spaces. In this work we are building on these results to provide symplectic covariant expressions for the Killing vectors and prepotentials for symmetric spaces only. This helps to clarify a conceptual point on the so-called hidden Killing vectors: they must act symplectically on the coordinates of the base special Kähler space and this was not evident in the analysis of de Wit and van Proeyen. Symmetric manifolds are coset spaces for which all possible isometries are realized and form a semi-simple Lie algebra.

The holonomy group of quaternionic manifolds contains a SU(2) factor which corresponds to the SU(2) R-symmetry of the $N = 2$ super-Poincaré algebra. A Killing vector does not need to preserve the SU(2) connections and it can induce a rotation given by a 3-vector called the compensator. It was already known that a necessary condition for getting a $N = 2$ adS_4 vacua is that at least one isometry with a non-trivial compensator be gauged [40, 130]. In particular we list the isometries with such compensators, and all of them are model-dependent (the isometries of the Heisenberg algebra associated to the Ramond scalars).

We also analyse $\mathrm{adS}_2 \times \Sigma_g$ vacua. In the case of FI gaugings this was solved in [104]. Since the equations for the vector and hyperscalars are decoupled we find that the entropy is given by the same formulas in both cases, except for the replacement of the FI parameters by the Killing prepotentials.

The idea in these two cases is to first solve the problem in FI supergravity by treating the prepotentials as constants. This provides a solution for the vector scalars in terms of the charges, gauging parameters and hyperscalars which can be fed into the other equations.

Solutions with less charges are easier to find and we focus on NUT charged ones. The addition of this charge is very natural because it preserves the SU(2) isometry and the hope is that BPS equations are not much different from the static case. The simple adS–NUT Schwarzschild black hole can be obtained from a limit of the PD solution, and there are two BPS branches preserving a half and quarter of the supersymmetry. An intriguing property in the presence of a NUT charge is the existence of BPS solutions that are not extremal and without horizons. On the other hand if there is an horizon then the solution is necessarily extremal. We discuss the root structure of the metric functions in order to clarify the different possibilities.

Then we compute the 1/4-BPS equations for NUT black hole in FI gauged supergravity and we look for solutions by using the techniques of [105]. In the case of extremal black hole we arrive at an analytic solution with running scalars and dyonic charges which generalize the one of [105]. In particular the near-horizon geometry does not feel the NUT charge. We were not able to find the general solution in the case where the black hole is not extremal, and it is not known if there are solutions with different near-horizon geometries or if they would simply be without horizons. Nonetheless we construct the constant scalar solutions in this formalism.

Symmetric Kähler manifolds are endowed with a invariant symmetric 4-tensor because the isometry group are of type E_7 [33, 84]. This quartic invariant appears in the expressions of the Killing vectors of symmetric special quaternionic manifolds, of the black hole entropy and the radius of adS_4 and of the BPS equations and of the analytic solutions for static and NUT-charged dyonic 1/4-BPS black holes [78, 79, 86, 104, 105, 184].

In conclusion the achievements of the current work are:

- symplectic covariant expressions for the quaternionic isometries;
- BPS equations with magnetic gaugings for matter-coupled $N = 2$ gauged supergravity;
- a framework for studying $N = 2$ adS_4 and $\mathrm{adS}_2 \times \Sigma_g$ vacua with abelian gaugings;
- quite generic solution for 1/4-BPS black holes with FI gaugings;

As a future direction one can extend the analysis of the BPS black holes (both static and with a NUT charge) in order to include hypermultiplets. A simpler intermediate goal would be to find an analytic solutions for the scalars in terms of the charges for the vacua. Another topic which has recently benefited from the study of quaternionic isometries is inflation in $N = 2$ supergravity where it was shown that at least one hidden isometries needs to be gauged in order to construct a physical model [51, 91].

Despite the fact that it would be very interesting to find the most general 1/4-BPS NUT solution when the horizon is not $\mathrm{adS}_2 \times \Sigma_g$, it may be more important to look first to solutions with rotation and acceleration[2] or at 1/2-BPS NUT solutions with running scalars. With more supersymmetry it would be easier to compute the microstates of these black holes.

It is not clear how the solution of Chow and Compère [56] is related to the known 1/4-BPS solutions and this point calls for an explanation. Finally computing the holographic free energy of the NUT charged solution is an interesting problem.

In all cases keeping the symplectic covariance of the equations by considering the general case was a key step in order to build the solutions by exploiting the power of the special geometry, and in particular of the quartic invariant. In the same idea it would be useful to extend the symplectic covariance of the Killing vectors to the case of homogeneous spaces and for non-abelian gaugings.

1.4 Structure

In part I we review the ungauged and gauged $N = 2$ supergravity: it describes the multiplets, the bosonic Lagrangian, the supersymmetry variations and the gauging procedure. These chapters are mostly self-contained and include a minimal description of the scalar manifolds. Next in parts II, III and IV we describe the properties of the scalar manifolds: this corresponds to a special Kähler manifold for the vector scalars, and to a quaternionic manifold for the hyperscalars. We describe the Riemannian properties of these manifolds and we build the isometries, focusing particularly on the symmetric spaces. Then in part V we look at the BPS equations and their static and NUT charged solutions. We start this part with a chapter on the general properties of adS–NUT black holes. Finally part VI is devoted to extended supergravities in general. Conventions, background informations, long formulas and computations are relegated in the appendix VII.

[2] In particular recently solutions with acceleration has been discovered [133, 134], and the rotating black holes from [54, 56] may give some intuitions. Also in this case the near-horizon geometries will certainly be different and a first analysis would be to look at these solutions.

Part I

$N = 2$ supergravity

Chapter 2

Introduction to $N = 2$ supergravity

Four-dimensional $N = 2$ supergravity can be obtained as the low-energy effective action of type II superstring theory compactified on Calabi–Yau 3-fold [90, sec. 21.4.3, 100, sec. 5] or on a $N = (2, 2)$ superconformal theory with $c = 9$ [166, 180, 181]. This case is interesting because heterotic string theory can be compactified on these manifolds and give rise to $N = 1$ supergravity in four dimensions, and some details of the resulting theory are independent of the number of supersymmetries [180, 181]. Finally $N = 2$ supergravity can also be found from M-theory on a 7-dimensional manifold with SU(3) structure [1, 39]. If fluxes are present then one gets gauged supergravity and we address this topic in the next chapter.

In this section we present the supersymmetry algebra and the corresponding multiplets. We then display the Lagrangian that describes the interaction of the hyper-, vector and gravity multiplets and we comment the electromagnetic duality of this theory. Finally we present the main details of the manifolds described by the scalar fields – the special Kähler and quaternionic geometries – which described in more details in later chapters.

General introductions can be found in the classical references [7, 8, 90].[1] Several thesis have been written recently on the topic [95, 109, 171].

[1] In particular an summary of the historical works may be found in [8, sec. 4].

2.1 Algebra and multiplets

The $N = 2$ supersymmetry algebra corresponds to [90, app. 6A]

$$[J_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\nu, \tag{2.1.1a}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\nu\sigma} J_{\mu\rho}, \tag{2.1.1b}$$

$$\{Q_\alpha, \bar{Q}^\beta\} = -\frac{i}{2}\, \delta_\alpha{}^\beta\, P_L \gamma_\mu\, P^\mu, \qquad \{Q^\alpha, \bar{Q}_\beta\} = -\frac{i}{2}\, \delta^\alpha{}_\beta\, P_R \gamma_\mu\, P^\mu, \tag{2.1.1c}$$

$$\{Q_\alpha, \bar{Q}_\beta\} = 0, \qquad \{Q^\alpha, \bar{Q}^\beta\} = 0, \tag{2.1.1d}$$

$$[P_\mu, Q_\alpha] = 0, \qquad [P_\mu, Q^\alpha] = 0, \tag{2.1.1e}$$

$$[J_{\mu\nu}, Q_\alpha] = -\frac{i}{2}\, \gamma_{\mu\nu} Q_\alpha, \quad [J_{\mu\nu}, Q^\alpha] = -\frac{i}{2}\, \gamma_{\mu\nu} Q^\alpha, \tag{2.1.1f}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\, \varepsilon_{\alpha\beta} P_L\, Z, \qquad \{Q^\alpha, Q^\beta\} = -\frac{1}{2}\, \varepsilon^{\alpha\beta} P_R\, \bar{Z}, \tag{2.1.1g}$$

$$\left[R^A, Q_\alpha\right] = (U^A)_\alpha{}^\beta Q_\beta, \qquad \left[R^A, Q^\alpha\right] = (U^A)^\alpha{}_\beta Q^\beta, \tag{2.1.1h}$$

$$\left[T^a, T^b\right] = f^{ab}{}_c T^c, \tag{2.1.1i}$$

where P_μ and $J_{\mu\nu}$ generate translation and Lorentz transformations and form the Poincaré algebra, Q_α are the fermionic generator of supersymmetry, R^A are the generator of the $\mathrm{U}(2)_R$ R-symmetry represented by the matrices U^A, T^a are generators of the internal symmetry, and finally Z is the central charge. The index α corresponds to the fundamental representation of $\mathrm{U}(2)_R$.

Note that $J_{\mu\nu}$ and P_μ describe the Poincaré subalgebra. The commutators of $J_{\mu\nu}$ with respectively itself, P_μ and Q_α show that they behave as an antisymmetric 2-tensor, a vector and a spinor. Two supersymmetric transformations close on a translation: as a consequence if supersymmetry is made local, so are the translations and one cannot have local supersymmetry without gravity. The R-symmetry group corresponds to the automorphism group: this is the only internal group that does not commute with the supersymmetry generators.

The algebra is given in terms of Weyl spinors (Q_α, Q^α) where the position of the index gives the chirality (see appendix A.5)

$$Q_\alpha = P_L\, Q_\alpha, \qquad Q^\alpha = P_R\, Q^\alpha. \tag{2.1.2}$$

Poincaré fields are organized into multiplets in this extended algebra. One of the constraint for building these representations is that the highest spin should not exceed $s = 2$ as interacting higher-spin theories (with a finite number of fields) are not consistent. The different multiplets are summarized in table 2.1. Using the table A.2 one can see that the bosonic and fermionic on-shell degrees of freedom match in each multiplets.

There are additional multiplets that we will not discuss, the tensor (or hypertensor, scalar-tensor) multiplet [11, 34, 73, 163, 164, 170, 187, 189], the double tensor multiplet [73] and the vector-tensor multiplet [11, 34, 59, 101]. While it is possible to always dualize the tensor into scalars in ungauged supergravity (where the vector-tensor and (double) tensor multiplets give respectively the vector and hyper-multiplets), this is not the case in gauged supergravity where the coupling of the multiplets with and without tensors are different. For example the masses of the tensor multiplets give magnetic gaugings. These multiplets have their interest in the context of flux compactifications where p-forms naturally arise.

We consider the following field content:

- Gravity multiplet

$$\{g_{\mu\nu}, \psi_{\alpha\mu}, \psi^\alpha_\mu, A^0_\mu\}. \tag{2.1.3}$$

multiplet	$s_{\max}$	$s=2$	$s=3/2$	$s=1$	$s=1/2$	$s=0$
gravity	2	1	2	1		
	3/2		1	2	1	
vector	1			1	2	2
hyper	1/2				2	4

Table 2.1: $N=2$ supergravity multiplets and spin content.

- n_v vector multiplets
$$\{A^i_\mu, \lambda^{\alpha i}, \lambda^{\bar\imath}_\alpha, \tau^i\}, \tag{2.1.4}$$
with $\tau^i \in \mathbb{C}$.
- n_h hypermultiplets
$$\{\zeta^{\mathcal{A}}, \zeta_{\mathcal{A}}, q^u\}, \tag{2.1.5}$$
with $q^u \in \mathbb{R}$.

The fields $\psi_{\alpha\mu}$, $\lambda^{\alpha i}$ and $\zeta^{\mathcal{A}}$ are respectively called gravitini, gaugini and hyperini. The ranges of the indices are

$$\alpha = 1, 2, \qquad i = 1, \ldots, n_v, \qquad u = 1, \ldots, 4n_h, \qquad \mathcal{A} = 1, \ldots, 2n_h. \tag{2.1.6}$$

The index α corresponds to the fundamental representation of $\mathrm{SU}(2) \sim \mathrm{Sp}(1)$ and $\mathcal{A}$ to the fundamental of $\mathrm{Sp}(n_h)$.

2.2 Lagrangian

It is natural to gather gauge fields into one vector of dimension $n_v + 1$

$$A^\Lambda = (A^0, A^i), \qquad \Lambda = 0, \ldots, n_v. \tag{2.2.1}$$

The bosonic part of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{bos}} = \frac{R}{2} + \frac{1}{4}\, \mathrm{Im}\,\mathcal{N}(\tau)_{\Lambda\Sigma}\, F^\Lambda_{\mu\nu}\, F^{\Sigma\mu\nu} - \frac{1}{8}\, \mathrm{Re}\,\mathcal{N}(\tau)_{\Lambda\Sigma}\, \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\, F^\Lambda_{\mu\nu}\, F^\Sigma_{\rho\sigma} \\ - g_{i\bar\jmath}(\tau)\, \partial_\mu\tau^i \partial^\mu \bar\tau^{\bar\jmath} - \frac{1}{2}\, h_{uv}(q)\, \partial_\mu q^u \partial^\mu q^v\end{aligned} \tag{2.2.2}$$

where the field strengths are defined by

$$F^\Lambda = \mathrm{d}A^\Lambda. \tag{2.2.3}$$

All fields are minimally coupled to gravity (through the factor $\sqrt{-g}$ in the action). Both vector- and hyperscalars describe a non-linear sigma model since the coefficient of the kinetic term is field-dependent. Moreover the gauge fields are coupled to the vector scalars through the period matrix $\mathcal{N}$: the imaginary and real parts correspond respectively to a generalization of the gauge coupling and of the topological θ-term. Finally the hyperscalars do not interact with the gauge fields nor the vector scalars.

Supersymmetry dictates the form of the various functions that appear. In particular the period matrix $\mathcal{N}$ and the metric $g_{i\bar\jmath}$ can be derived from a unique holomorphic function F called the prepotential (see section 2.4).[2]

[2]There are formulation of the theory without prepotential but we will not worry about this subtlety.

All the kinetic terms should be positive definite [69, sec. 2], and this imposes some restrictions on the scalar fields. The normalisation of the curvature term corresponds to a gauge choice.[3] Moreover the kinetic term for the gauge field has the correct signature because $\mathrm{Im}\,\mathcal{N}$ is negative definite (see section 6.4).

The Lagrangian is invariant under the local R-symmetry with gauge group $\mathrm{U}(2)_R$ for which there are two composite gauge fields $\mathcal{A}_\mu(\tau,\bar\tau)$ and $\mathcal{V}^x_\mu(q)$ with $x = 1, 2, 3$. Their origin can be seen most clearly from the superconformal tensor calculs. The scalar fields are neutral under this group.

We are not interested in the fermionic part of the Lagrangian but we will comment some of its properties. Fermions are coupled to the gauge fields through Pauli terms $F\psi\psi$ (and so on) which give rise to anomalous magnetic moments – in particular for the gaugini they are given by the quantity W_{ijk} (see section 6.5) [7, sec. 4.3]. Moreover the fermions are minimally coupled to the composite $\mathrm{U}(2)_R$ gauge fields. The Lagrangian includes four-fermion terms, but there are no mass terms.

The full Lagrangian is invariant under supersymmetry variations, we will give them only in the case of gauged supergravity (section 3.4).

2.3 Electromagnetic duality

Electromagnetic duality with and without scalars was studied in full generality by Gaillard and Zumino [92] (see also [10, sec. 3]). For a review of this topic see [8, sec. 2, 44, sec. 3, 45, 67, sec. 2, 90, sec. 4.2].

Recall that the field strength are determined from the gauge potential by

$$F^\Lambda = \mathrm{d}A^\Lambda. \tag{2.3.1}$$

Dual (magnetic) field strengths are given by

$$G_\Lambda = \star\left(\frac{\delta\mathcal{L}_{\text{bos}}}{\delta F^\Lambda}\right) = \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\, F^\Lambda + \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,\star F^\Lambda. \tag{2.3.2}$$

It is also possible to introduce magnetic gauge potential A_Λ such that

$$G_\Lambda = \mathrm{d}A_\Lambda. \tag{2.3.3}$$

Both types of field strengths and gauge fields form together a symplectic vector

$$\mathcal{F} = \mathrm{d}A = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}, \qquad A = \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}. \tag{2.3.4}$$

The self-dual and anti-self-dual field strength is defined by

$$F^\pm = \frac{1}{2}(F \mp i\star F), \tag{2.3.5}$$

and similarly for $G^\pm$. Using equation (6.3.4) one finds

$$G^+ = \mathcal{N} F^+, \qquad G^- = \bar{\mathcal{N}} F^-. \tag{2.3.6}$$

Using these fields the kinetic term for the gauge fields can be rewritten as [8, p. 5, 90, p. 446]

$$\mathcal{L}_{\text{vec}} = \frac{1}{2}\,\mathrm{Im}(\mathcal{N}_{\Lambda\Sigma}F^{+\Lambda}F^{+\Sigma}) = -\frac{i}{4}\,\mathcal{N}_{\Lambda\Sigma}F^{+\Lambda}F^{+\Sigma} + \text{c.c.} = -\frac{i}{4}\,G^+_\Lambda F^{+\Lambda} + \text{c.c.} \tag{2.3.7}$$

[3] In particular the term which appears before gauge fixing is $-i\left\langle \mathcal{V}, \bar{\mathcal{V}}\right\rangle R$, and we recover R by setting $\left\langle \mathcal{V}, \bar{\mathcal{V}}\right\rangle = i$ as in (6.2.25) [67, sec. 4].

This can be proven using the fact that

$$F^+_{\mu\nu} F^{+\mu\nu} = \frac{1}{2} (F_{\mu\nu} F^{\mu\nu} - i F_{\mu\nu} \star F^{\mu\nu}), \tag{2.3.8}$$

then one ends up with

$$\mathcal{L}_{\text{vec}} = -\frac{1}{4} \operatorname{Re} \Big(i \mathcal{N}_{\Lambda\Sigma} (F_{\mu\nu} F^{\mu\nu} - i F_{\mu\nu} \star F^{\mu\nu}) \Big). \tag{2.3.9}$$

Maxwell equations and Bianchi identities

$$\mathrm{d} F^\Lambda = 0, \qquad \mathrm{d} G_\Lambda = 0 \tag{2.3.10}$$

can be gathered as

$$\mathrm{d}\mathcal{F} = 0. \tag{2.3.11}$$

Note also that they can be traded for their dual

$$\mathrm{d} \star F^\Lambda = 0, \qquad \mathrm{d} \star G_\Lambda = 0 \quad \Longrightarrow \quad \mathrm{d} \star \mathcal{F} = 0. \tag{2.3.12}$$

They can also be rewritten as

$$\mathrm{d} \operatorname{Im} \mathcal{F}^\pm = 0. \tag{2.3.13}$$

The definition (2.3.2) of G_Λ implies the twisted self-duality condition [88, p. 5]

$$\star \mathcal{F} = \Omega \mathcal{M} \mathcal{F} \tag{2.3.14}$$

where $\mathcal{M}$ is a symplectic matrix built from $\mathcal{N}$, see section 6.4. Indeed using (2.3.2) one finds

$$G = R F - I \star F, \qquad \star G = R \star F - I F \tag{2.3.15}$$

which can be used to give $\star G$ and $\star F$ in terms of G and F. Taking a second time the Hodge operation is consistent with the fact that $\Omega\mathcal{M} = -1$.

These equations are invariant under linear transformations from $\mathrm{GL}(2n_v + 2, \mathbb{R})$, which reduces to symplectic transformations

$$\mathcal{F} \longrightarrow \mathcal{U}\mathcal{F}, \qquad \mathcal{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n_v + 2, \mathbb{R}) \tag{2.3.16}$$

if one wants to preserve the relation between F and G

$$G_\Lambda = \mathcal{N}_{\Lambda\Sigma} F^\Sigma \Longrightarrow G'_\Lambda = \mathcal{N}'_{\Lambda\Sigma} F'^\Sigma. \tag{2.3.17}$$

This is a consequence of the fact that a symplectic transformation of the various sections will induce a diffeomorphism of the scalar manifold, and the action will be of the same form only if both transformations are consistent together. The fact that both scalar and gauge fields transform can be seen as a consequence of supersymmetry which relates both fields: indeed if only the vector fields were transforming then the supersymmetry transformation would not be consistent anymore.

In presence of matter the dualities of the full equations of motion are restricted to a subgroup $G \subset \mathrm{Sp}(2n_v + 2, \mathbb{R})$, called the U-duality group, because the self-interaction terms are not invariant under the full symplectic group (see section 2.4).

It is important to note that the equations of motion – but not the action — are only covariant with respect to these symplectic transformations (called also duality-rotations or field-redefinitions), and as a consequence these are *not* symmetries of the action. [8, p. 7]. Symmetries of the equations of motion (and Bianchi identities) correspond to the subgroup of the symplectic transformation that leaves the equations invariant, and they are called duality transformations. We used this word duality because in general the action is not invariant, only the equations of motion are [90, p. 84].

The gauge field Lagrangian (2.3.7) transforms according to [8, p. 7, 67, p. 3]

$$2\mathcal{L}_{\text{vec}} = \operatorname{Im}(G^+_\Lambda F^{+\Lambda}) \longrightarrow \operatorname{Im}(G'^+_\Lambda F'^{+\Lambda}) = \operatorname{Im}\big(G^+_\Lambda F^{+\Lambda} + 2F^- C^t B G^- \\ + F^- C^t A F^- + G^- D B G^-\big). \tag{2.3.18}$$

Then a symmetry of the Lagrangian is possible only if $B = 0$ since the last term was not present in the original Lagrangian – these symmetries are called electric. Moreover it seems that we would have to require also $C = 0$, this is not necessary if one asks only for a symmetry of the action: the term $(C^t A)_{\Lambda\Sigma} F^{-\Lambda} F^{-\Sigma}$, which corresponds to a constant shift of $\mathcal{N}$

$$\mathcal{N} \longrightarrow A^{t-1} \mathcal{N} A^{-1} + C A^{t-1}, \tag{2.3.19}$$

is a topological density since the coefficient is constant. Nonetheless this term would have a quantum effect as it modifies the θ-angle of the theory. In particular the path integral is invariant only if the coefficients are integer multiples of 2π, which restricts the U-duality group G to a discrete subgroup [8, p. 27]. In the case $C \neq 0$ the prepotential is shifted [90, sec. 21.1.2], from (7.1.22)

$$\delta F = \frac{1}{2} X S^t Q X. \tag{2.3.20}$$

The transformation for which $B \neq 0$ are non-perturbative because they mix the electric and magnetic field strengths into the Lagrangian which does not involve the latter. From the microscopic point of view this is equivalent to exchanging the electric and magnetic currents, and then the elementary states with the soliton states [8, p. 28].

The electric and magnetic charges q_Λ and p^Λ contained in a volume V with boundary Σ are defined by

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \frac{1}{\operatorname{Vol}(\Sigma)} \int_\Sigma \mathcal{F}. \tag{2.3.21}$$

The charges are defined as densities to avoid infinite charges in the case of non-compact surfaces. For compact horizons one takes

$$\operatorname{Vol}(\Sigma) = \operatorname{Vol}(S^2) = 4\pi. \tag{2.3.22}$$

Note also that the charges are a priori not constant. Since the charges $\mathcal{Q}$ are obtained by integrating the field strengths $\mathcal{F}$, they also transform under symplectic transformations [67, sec. 2]. Let us stress that identifying charges as being magnetic or electric is a frame-dependent question as a consequence of the previous point.

The graviphoton dressed field strength T and its (anti-)self-dual parts are defined by

$$T^+ = -\left\langle \bar{\mathcal{V}}, \mathcal{F}^+ \right\rangle, \qquad T^- = -\left\langle \mathcal{V}, \mathcal{F}^- \right\rangle \tag{2.3.23}$$

since [90, p. 478]

$$\left\langle \mathcal{V}, \mathcal{F}^+ \right\rangle = \left\langle \bar{\mathcal{V}}, \mathcal{F}^- \right\rangle = 0. \tag{2.3.24}$$

Similarly one defines the dressed field strengths T^i of the vector multiplet fields as

$$T^+_i = -\left\langle U_j, \mathcal{F}^+ \right\rangle, \qquad T^-_{\bar{\imath}} = -\left\langle \bar{U}_{\bar{\imath}}, \mathcal{F}^- \right\rangle, \tag{2.3.25}$$

while the tensors with the upper index are $T^{\bar{\imath}+} = g^{\bar{\imath}j} T^+_j$ and $T^{i-} = g^{i\bar{\jmath}} T^-_{\bar{\jmath}}$.

Important quantities are the central and matter charges defined by

$$\mathcal{Z} = -\frac{1}{2} \int_\Sigma T^-, \qquad \mathcal{Z}_i = -\frac{1}{2} \int_\Sigma T^-_i. \tag{2.3.26}$$

If $\mathcal{V}$ does not depend on the coordinates on Σ, one can move $\mathcal{V}$ outside the integral in (2.3.26). Then the central and matter charges correspond to the components of $\mathcal{Q}$ along the basis $(\mathcal{V}, U_i)$ following (6.4.21)

$$\mathcal{Z} = \Gamma(\mathcal{Q}) = \langle \mathcal{V}, \mathcal{Q} \rangle , \qquad \mathcal{Z}_i = \mathrm{D}_i \mathcal{Z} = \langle U_i, \mathcal{Q} \rangle . \tag{2.3.27}$$

2.4 Scalar geometry

Scalar fields describe a non-linear sigma model with target space

$$\mathcal{M} = \mathcal{M}_v(\tau^i) \times \mathcal{M}_h(q^u) \tag{2.4.1}$$

where supergravity imposes constraints on the manifold holonomies which determine their types:[4]

- $\mathcal{M}_v$: special Kähler (SK) manifold (part III), $\dim_{\mathbb{R}} = 2n_v$ [166];
- $\mathcal{M}_h$: quaternionic Kähler (QK) manifold (part IV), $\dim_{\mathbb{R}} = 4n_h$ [16].

The R-symmetry group of the supersymmetry algebra can be split as

$$\mathrm{U}(2)_R = \mathrm{SU}(2)_R \times \mathrm{U}(1)_R, \tag{2.4.2}$$

and this is mirrored in the structure of the multiplets: SK manifolds have a U(1) bundle while QK manifolds have an SU(2) bundle. In particular if the manifolds $\mathcal{M}_v$ and $\mathcal{M}_h$ are cosets G/H, then their maximal compact subgroup H contains respectively a factor U(1) or SU(2).

In considering the fields as coordinates for the non-linear sigma model all relevant formulas are obtained through a pull-back, in particular

$$\mathrm{d}\tau^i = \partial_\mu \tau^i \,\mathrm{d}x^\mu, \qquad \mathrm{d}q^u = \partial_\mu q^u \,\mathrm{d}x^\mu. \tag{2.4.3}$$

2.4.1 Isometries

The isometry group[5]

$$G \equiv \mathrm{ISO}(\mathcal{M}) \tag{2.4.4}$$

of this manifold translates into an invariance of the scalar kinetic term which is just the pullback of the metric on $\mathcal{M}$. On the other hand through its embedding into the symplectic group (as explained in section 2.3) it defines the global symmetry group of the equations of motion and it is called the U-duality group. A subgroup of G can be gauged in order to generate new interactions, and this is the topic of chapter 3.

According to the discussion of section 2.3, an isometry can be of one of the three following types [8, sec. 6, 45]:

- Classical symmetries: the matrix $\mathcal{U}$ is block diagonal

$$\mathcal{U} = \begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}, \tag{2.4.5}$$

(where the lower component follows from the constraints (7.1.3)), and it is a true symmetry of the Lagrangian.

[4] The manifold described by the scalars of n_t vector-tensor multiplets is real.

[5] We will also use the notations $G_v \equiv \mathrm{ISO}(\mathcal{M}_v)$ and $G_h \equiv \mathrm{ISO}(\mathcal{M}_h)$.

- Perturbative symmetries: the matrix $\mathcal{U}$ is lower triangular

$$\mathcal{U} = \begin{pmatrix} A & 0 \\ C & A^{t-1} \end{pmatrix}. \tag{2.4.6}$$

At the classical level the action is invariant, while at the quantum level only the path integral is invariant for a subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$.

- Non-perturbative symmetries: the matrix U has the general form (2.3.16)

$$\mathcal{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.4.7}$$

and they are symmetries of the quantum theory but they cannot be defined perturbatively.

Isometries of the scalar manifold extend to a symmetry of the Lagrangian if all couplings are diffeomorphism invariant, which means that they depend only on the metric, the curvature and Christoffel symbols [8, sec. 7.1].

In $d = 4$ all symmetries of the scalar manifold extend to symmetries of the full Lagrangian (as opposed to $d = 5$) [184, 188] and this is a consequence of supersymmetry.[6]

If one considers models obtained from compactification of type II, then the corresponding SK manifold $\mathcal{M}_v$ is symmetric and the QK is special, which means that it entirely specified by another SK manifold $\mathcal{M}_z$ which is also symmetric. Moreover the manifolds $\mathcal{M}_v$ and $\mathcal{M}_z$ are interchanged when compactifying type II A and B on the same manifold [184].

We review the main properties of these manifolds and we refer the reader to part II for more details.

2.4.2 Special Kähler manifolds

A special Kähler manifold is a Kähler manifold with a bundle with group $\mathrm{Sp}(2n_v + 2, \mathbb{R})$.

SK manifolds are better described in terms of projective coordinates X^Λ where

$$\tau^i = \frac{X^i}{X^0}. \tag{2.4.8}$$

Then the prepotential is a holomorphic function $F = F(X^\Lambda)$ of weight 2. The gradient of the prepotential gives a set of functions

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda} \tag{2.4.9}$$

that together with X^Λ form a section of the symplectic bundle

$$v = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}. \tag{2.4.10}$$

Then the Kähler potential reads

$$K = -\ln i\big(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda\big) \tag{2.4.11}$$

from which derives the metric

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K. \tag{2.4.12}$$

[6]This was proved only for cubic prepotentials, but no counter-example is known [184, p. 15].

It is always possible to describe the SK manifold in terms of a prepotential and we will focus on this case [67]. But this does not mean that symplectically rotated theories are equivalent (for example different theories with the same geometry may have different gauge groups, and partial symmetry breaking from $N = 2$ to $N = 1$ in FI gauged supergravity is impossible if a superpotential exists) [67, sec. 4.2].

The pull-back of the U(1) connection (5.4.9) is

$$\mathcal{A}_\mu = -\frac{i}{2}\big(K_i \partial_\mu \tau^i - K_{\bar\imath} \partial_\mu \bar\tau^{\bar\imath}\big). \tag{2.4.13}$$

2.4.3 Quaternionic manifolds

The quaternionic manifold with metric h_{uv} has a triplet of structures J^x satisfying the quaternionic algebran $\mathrm{SU}(2) \sim \mathrm{Sp}(1)$

$$J^x J^y = -\delta^{xy} + \varepsilon^{xyz} J^z \tag{2.4.14}$$

where $x = 1, 2, 3$ is the vector representation of $\mathrm{SO}(3) \sim \mathrm{SU}(2)$. They define a triplet of 2-forms

$$K^x = J^x_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v, \qquad J^x_{uv} = h_{uw} (J^x)_v{}^w. \tag{2.4.15}$$

The manifold has an SU(2) bundle with connection ω^x and a curvature proportional to the quaternionic 2-forms

$$\Omega^x = \nabla \omega^x = \lambda K^x \tag{2.4.16}$$

These forms are covariantly closed

$$\nabla \Omega^x = \nabla K^x = 0. \tag{2.4.17}$$

Finally one can introduce vielbeine

$$h_{uv} = \mathbb{C}_{\mathcal{A}\mathcal{B}} \varepsilon_{\alpha\beta} U_u^{\alpha\mathcal{A}} U_v^{\mathcal{B}\beta}, \tag{2.4.18}$$

where the indices $\mathcal{A}$ and α run respectively in the fundamental representations of $\mathrm{Sp}(n_h)$ and $\mathrm{Sp}(1)$, where the corresponding symplectic metrics are $\mathbb{C}$ and ε. This splitting of the indices is a consequence of the holonomy of the manifold.

In supergravity one has the restriction [7, p. 6, 72, p. 719]

$$\lambda = -1 \tag{2.4.19}$$

which implies that the quaternionic spaces have negative curvature

$$R = -8 n_h (n_h + 2). \tag{2.4.20}$$

The pull-back of the SU(2) connection corresponding to the composite $\mathrm{SU}(2)_R$ gauge field is

$$\mathcal{V}^x_\mu = -\omega^x_u\, \partial_\mu q^u. \tag{2.4.21}$$

In most of the cases that are of interest to us the quaternionic manifold is special (see chapter 11) and all its properties are given by a special Kähler manifold $\mathcal{M}_z$ of dimension $2(n_h - 1)$ with prepotential G. These manifolds are constructed from the c-map: $d = 4$ supergravity is reduced to $d = 3$ where all vectors can be dualized to scalar fields. Since there are only scalar fields (coming from the original vector and hypermultiplets, and from the reduction) the geometry can only be quaternionic. Then the manifold that are constructed in this way can be used for $\mathcal{M}_h$ in $d = 4$ [180–182]. The idea is that dualities of the $d = 4$

equations of motion will translate into invariance of the $d = 3$ Lagrangian since there are no more gauge fields [180, 184, sec. 2.3].

In this case the fields are denoted by $(\phi, \sigma, \xi^A, \tilde{\xi}_A)$ where $A = 0, \ldots, n_h - 1$. Physically ϕ is the dilaton (coming from the metric), σ is the axion (coming from dualization of the NS B-field) and the $(\xi^A, \tilde{\xi}_A)$ corresponds to the RR scalars (coming from the reduction of the RR forms) [40, p. 5].

Chapter 3

Gauged supergravity

A gauged supergravity is obtained from an ungauged theory by using some of the gauge fields in order to introduce a local gauge symmetry. In this chapter we describe the two main possibilities which consists in gauging a subgroup of the isometry group of the scalar manifolds or in introducing Fayet–Iliopoulos gaugings (both are not exclusive). The gauging procedure is described in [8, sec. 7, 90, chap. 21, 109, chap. 2, 171, chap. 1] (see also [143, 173]).

Gauged supergravities typically appear in flux compactifications which refers to compactifications where some p-form field of the higher-dimensional theory has a value along a (non-trivial) cycle of the internal manifold [100, sec. 5, 158, sec. 4] (see also [128]).

In order to understand the details of the gauging one needs to understand the isometries of the SK and QK scalar manifolds, which are the topics of chapters 9 and 9. Our study of the BPS solutions will rely heavily on a symplectic covariant formalism: this requires us to introduce magnetic gaugings in order to treat equally electric and magnetic field strengths. Constructing a Lagrangian with magnetic gaugings is a difficult task and we will restrict ourselves to a simple case involving only the equations of motion/BPS.

3.1 Generalities

Since the Lagrangian (2.2.2) is invariant under the global isometry group G of the scalar manifold $\mathcal{M}$ (section 2.4) one can gauge a subgroup K of the global symmetry group G such that part of the symmetries are made local

$$K \subset G. \tag{3.1.1}$$

The group should be at most $n_v + 1$, which corresponds to the number of gauge fields

$$m = \dim K \leq n_v + 1. \tag{3.1.2}$$

This produces typically a non-abelian theory with gauge fields A^Λ in the adjoint representation, and by supersymmetry the fields X^Λ also sits in the adjoint representation. Vector scalar and hyperscalars are minimally coupled to the gauge fields through the Killing vectors of SK and QK geometries respectively, and they are in some representation of the gauge group. The fermions are coupled through the Killing prepotentials (or moment maps) acting as a deformation of the composite $\mathrm{U}(2)_R$ connections and derivatives of the SK/QK Killing vectors for the gaugini/hyperini. If the SK P^0_Λ and QK P^x_Λ moment maps are non-zero then the fermions are charged respectively under the $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)_R$ factors of the

R-symmetry which are gauged by physical gauge fields (in particular this is the only coupling for the gravitini), while only non-dynamical gauge fields were gauging it in ungauged supergravity [90, sec. 19.5, 173].

If only QK isometries are made local then the gauge group is necessarily abelian

$$K = \mathrm{U}(1)^m, \qquad m \leq \dim G_h. \tag{3.1.3}$$

Indeed since the fields X^Λ are in the adjoint representation, non-abelian gaugings are possible only if a subgroup of G_v is gauged.

If there are hypermultiplets then the quaternionic moment maps are fully determined from Killing vectors. On the other hand if $n_h = 0$ then the quaternionic moment maps can still be (non-vanishing) constants called Fayet–Iliopoulos parameters. They correspond to the coupling constants of the gravitini to the gauge fields using the R-symmetry group $\mathrm{SU}(2)_R$.[1] If one is not gauging a subgroup of G_v then the resulting group is abelian and for each gauge field this amounts to consider a U(1) inside the $\mathrm{SU}(2)_R$

$$\mathrm{U}(1) \subset \mathrm{SU}(2)_R. \tag{3.1.4}$$

Then one often considers the maximal case with

$$K = \mathrm{U}(1)^{n_v+1}, \tag{3.1.5}$$

(it is convenient to consider the diagonal U(1) inside $\mathrm{SU}(2)_R$), which is referred to as *Fayet–Iliopoulos gauging*. Minimal gauged supergravity is constructed in this way.

Gauging adds complexity to the theory and additional terms are generated in order to preserve supersymmetry:

- a scalar potential $V(\tau, q)$;
- (scalar-dependent) fermion masses;
- Chern–Simons terms for A^Λ.

The hypermultiplets are not spectators anymore and the dynamics is much richer. Moreover a non-trivial potential is necessary for obtaining AdS_4 vacua.

In section 2.4.1 we explained that the isometry group is embedded into the symplectic group, and that different types of symmetries can be distinguished. In particular within the current formalism it is possible to gauge only isometries which correspond to perturbative (or electric) symmetries, i.e. those which have a lower triangular embedding into the symplectic group; this issue will discussed further in section 3.5.

Hence the choice of the symplectic frame is important for determining the gauging. In particular it is always possible to find a frame where the gaugings are electric. On the other hand a prepotential may not exist in this frame, or it can be ugly, and there is a trade-of between having electric gaugings and the existence of a prepotential [90, sec. 21.2.2].

As soon as the theory is gauged, models related by symplectic transformations are not equivalent anymore because the gauging breaks the symplectic invariance. Indeed even if the bosonic part of the Lagrangian is invariant, minimal coupling of the gauge fields to the fermions breaks this duality invariance [132].

[1] We stress that this is compatible with the previous option of gaugings a sugroup of G_v. This procedure amounts to gauge the R-symmetry by physical gauge fields furthermore with constant couplings.

3.2 Gaugings

3.2.1 Isometries

Except in the FI case, the gauging is encoded by $n_v + 1$ Killing vectors

$$k_\Lambda = k^i_\Lambda(\tau)\, \partial_i + k^{\bar\imath}_\Lambda(\bar\tau)\, \partial_{\bar\imath} + k^u_\Lambda(q)\, \partial_u \tag{3.2.1}$$

which act on the fields as

$$\delta\tau^i = \alpha^\Lambda k^i_\Lambda(\tau), \qquad \delta q^u = \alpha^\Lambda k^u_\Lambda(q). \tag{3.2.2}$$

where α^Λ are the parameters of the gauge transformation. The vectors $\{k^i_\Lambda, k^{\bar\imath}_\Lambda, k^u_\Lambda\}$ correspond to linear combinations of the Killing vectors generating the isometries of $\mathcal{M}_v$ and $\mathcal{M}_h$

$$k_\Lambda = \theta^\alpha_\Lambda\, k_\alpha, \qquad \alpha = 1, \ldots, \dim G. \tag{3.2.3}$$

The coefficients θ^α_Λ of the linear combination are called the gauging parameters and the vectors k_α span the algebra of the full isometry group.

The Killing vectors form a Lie algebra

$$[k_\Lambda, k_\Sigma] = f_{\Lambda\Sigma}{}^\Omega\, k_\Omega \tag{3.2.4}$$

where $f_{\Lambda\Sigma}{}^\Omega$ are the structure constants. This provides constraints for the gauging parameters which are not all independent [158, sec. 3.1, 187, sec. 3]: the constraints can be worked out by using the explicit algebras $\mathfrak{g}_v$ and $\mathfrak{g}_h$ on the LHS and by identifying the coefficients with the RHS. In particular if no isometries of $\mathcal{M}_v$ are gauged then the Killing vector algebra is necessarily abelian (but this does not mean that the isometries of the manifolds are abelian: only their linear combination needs to be abelian, see section 3.6 for an example).

The isometry induces a symplectic $T = \alpha^\Lambda T_\Lambda$ and a Kähler $f = \alpha^\Lambda f_\Lambda$ transformation

$$\delta\mathcal{V} = T\mathcal{V} + f(\tau)\mathcal{V}, \tag{3.2.5}$$

where T_Λ is lower triangular

$$T_\Lambda = \begin{pmatrix} A_\Lambda & 0 \\ C_\Lambda & A^{t-1}_\Lambda \end{pmatrix}, \tag{3.2.6}$$

and C_Λ is symmetric. This transformation needs to be consistent with the transformation of the field strength F^Λ under a non-abelian gauge transformation [90, p. 474]

$$\delta F^\Lambda = \alpha^\Omega F^\Sigma f_{\Sigma\Omega}{}^\Lambda. \tag{3.2.7}$$

In particular this justifies the restriction to electric gaugings with $B_\Lambda = 0$, and this indicates that T_Λ should be

$$T_\Lambda = \begin{pmatrix} -f_\Lambda & 0 \\ C_\Lambda & f^t_\Lambda \end{pmatrix} = \begin{pmatrix} -f_{\Lambda\Sigma}{}^\Omega & 0 \\ C_{\Lambda\Sigma\Omega} & f_{\Lambda\Omega}{}^\Sigma \end{pmatrix}. \tag{3.2.8}$$

These generators satisfy the Lie algebra under the conditions

$$C_{(\Lambda\Sigma\Omega)} = 0, \tag{3.2.9a}$$

$$f_{\Xi\Omega}{}^\Gamma C_{\Gamma\Lambda\Sigma} = 2 f_{\Lambda[\Xi}{}^\Gamma C_{\Omega]\Sigma\Gamma} + 2 f_{\Sigma[\Xi}{}^\Gamma C_{\Omega]\Lambda\Gamma}. \tag{3.2.9b}$$

If the second term is present it induces a Kähler transformation

$$\delta K = \alpha^\Lambda (f_\Lambda + \bar f_\Lambda). \tag{3.2.10}$$

This implies the constraint

$$k_\Lambda^i \partial_i f_\Sigma - k_\Sigma^i \partial_i f_\Lambda = f_{\Lambda\Sigma}{}^\Omega f_\Omega. \tag{3.2.11}$$

Generically if $B_\Lambda = 0$ then one also has $f_\Lambda = 0$ [8, p. 33].

In the kinetic term of the scalar fields the partial derivatives are modified to covariant derivatives through minimal coupling

$$\mathrm{D}_\mu = \partial_\mu - A_\mu^\Lambda k_\Lambda. \tag{3.2.12}$$

The fact that only the electric gauge field A^Λ are introduced implies that one breaks the symplectic covariance. Moreover the field strengths of the gauge fields are modified by a non-abelian piece

$$F_{\mu\nu}^\Lambda = \partial_\mu A_\nu^\Lambda - \partial_\nu A_\mu^\Lambda + f_{\Sigma\Omega}{}^\Lambda A_\mu^\Sigma A_\nu^\Omega. \tag{3.2.13}$$

Moment maps are real functions that can be built from special and quaternionic Killing vector

$$P_\Lambda^0 = i\big(k_\Lambda^i \partial_i K - f_\Lambda\big), \qquad P_\Lambda^x = k_\Lambda^u \omega_u^x + W_\Lambda^x \tag{3.2.14}$$

where f_Λ is the shift of the Kähler potential and W_Λ^x the SU(2) rotation of the triplet of hyperkähler structures induced by the isometry.

There are two important relations

$$k_\Lambda^i L^\Lambda = 0, \qquad P_\Lambda^0 L^\Lambda = 0. \tag{3.2.15}$$

The Kähler U(1) connection (2.4.13) is modified to

$$\mathcal{A}_\mu = -\frac{i}{2}\big(K_i \mathrm{D}_\mu \tau^i - K_{\bar\imath} \mathrm{D}_\mu \bar\tau^{\bar\imath}\big) - \frac{1}{4}\, A_\mu^\Lambda (f_\Lambda - \bar f_\Lambda) \tag{3.2.16a}$$

$$= -\frac{i}{2}\big(K_i \partial_\mu \tau^i - K_{\bar\imath} \partial_\mu \bar\tau^{\bar\imath}\big) - \frac{i}{2}\, A_\mu^\Lambda P_\Lambda^0, \tag{3.2.16b}$$

while the SU(2) connection becomes

$$\mathcal{V}_\mu^x = -\omega_u^x\, \mathrm{D}_\mu q^u + \frac{1}{2}\, A_\mu^\Lambda W_\Lambda^x \tag{3.2.17a}$$

$$= -\omega_u^x\, \partial_\mu q^u - \frac{1}{2}\, A_\mu^\Lambda P_\Lambda^x. \tag{3.2.17b}$$

The fact that spinors are charged implies Dirac-like quantization conditions on the Killing prepotentials

$$p^\Lambda P_\Lambda^0 \in \mathbb{Z}, \qquad p^\Lambda P_\Lambda^x \in \mathbb{Z}. \tag{3.2.18}$$

where p^Λ are the magnetic charges.

One defines the prepotential charges (also called the superpotential)

$$\mathcal{L}^x = -P_\Lambda^x L^\Lambda \tag{3.2.19}$$

(see (3.5.5) for a symplectic covariant definition).

3.2.2 Fayet–Iliopoulos gauging

A good reference is [172, sec. 2] (see also [90, sec. 21.5.1]).

In Fayet–Iliopoulos (FI) gauging the fermions become charged under a subgroup K_{FI} of the R-symmetry group

$$K_{\mathrm{FI}} \subset \mathrm{SU}(2)_R \tag{3.2.20}$$

This corresponds to constant quaternionic moment maps ξ^x_Λ called the FI parameters

$$\xi^x_\Lambda \equiv P^x_\Lambda = \text{cst}, \tag{3.2.21}$$

which is possible only if $n_h = 0$ (otherwise they are determined by the quaternionic geometry and they are non-constant). These moment maps can be non-vanishing even if $n_h = 0$ because there is always a compensating hypermultiplet, which was fixed during the construction of the theory. If one gauges also a subgroup $K \subset G_v$, then a necessary condition is [109, p. 35]

$$K_{\text{FI}} \subset K. \tag{3.2.22}$$

If one considers abelian isometries, then the equivariance condition (10.3.26) reads

$$\varepsilon^{xyz} \xi^y_\Lambda \xi^z_\Sigma = 0. \tag{3.2.23}$$

As a consequence it is possible to choose a direction for the SU(2) vector

$$\xi^x_\Lambda = (0, 0, g_\Lambda) \tag{3.2.24}$$

which corresponds to

$$\mathrm{U}(1) \subset \mathrm{SU}(2)_R \tag{3.2.25}$$

(U(1) being the diagonal subgroup). The parameters g_Λ are the electric charges of the gravitini under this U(1) symmetry: the gauge fields are coupled to the gravitini through the linear combinations $g_\Lambda A^\Lambda$, and the two gravitini have opposite charges $\pm g_\Lambda$. Note that the vector scalars are neutral. In general speaking about FI gauging refers to this latter case.

Pure supergravity is a subcase of (abelian) FI gauged supergravity.

3.3 Lagrangian

3.3.1 General case

The bosonic part of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{bos}} = {}& \frac{R}{2} + \frac{1}{4}\,\mathrm{Im}\,\mathcal{N}(\tau)_{\Lambda\Sigma}\, F^\Lambda_{\mu\nu}\, F^{\Sigma\mu\nu} - \frac{1}{8}\,\mathrm{Re}\,\mathcal{N}(\tau)_{\Lambda\Sigma}\, \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\, F^\Lambda_{\mu\nu}\, F^\Sigma_{\rho\sigma} \\ & - g_{i\bar\jmath}(\tau)\, \mathrm{D}_\mu \tau^i \mathrm{D}^\mu \bar\tau^{\bar\jmath} - \frac{1}{2}\, h_{uv}(q)\, \mathrm{D}_\mu q^u \mathrm{D}^\mu q^v \\ & + \frac{2}{3}\, C_{\Lambda,\Sigma\Xi}\, \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\, A^\Lambda_\mu A^\Sigma_\nu \left(\partial_\rho A^\Xi_\sigma + \frac{3}{8}\, f_{\Omega\Gamma}{}^\Xi A^\Omega_\rho A^\Gamma_\sigma \right) - V(\tau, \bar\tau, q).\end{aligned} \tag{3.3.1}$$

The term proportional to $C_{\Lambda\Sigma\Omega}$ is necessary to compensate the transformation of the matrix $\mathcal{N}$

$$\delta \mathcal{N}_{\Lambda\Sigma} = -\alpha^\Gamma \big(f_{\Gamma\Lambda}{}^\Omega \mathcal{N}_{\Sigma\Omega} + f_{\Gamma\Sigma}{}^\Omega \mathcal{N}_{\Lambda\Omega} + C_{\Gamma\Lambda\Sigma} \big). \tag{3.3.2}$$

under a gauge transformation.

The scalar potential reads

$$V = \big(f^\Lambda_i g^{i\bar\jmath} \bar f^\Sigma_{\bar\jmath} - 3 L^\Lambda \bar L^\Sigma \big) P^x_\Lambda P^x_\Sigma + \bar L^\Lambda L^\Sigma \big(2\, h_{uv} k^u_\Lambda k^v_\Sigma + g_{i\bar\jmath} k^i_\Lambda k^{\bar\jmath}_\Sigma \big). \tag{3.3.3}$$

Note that there is only one negative term in the potential. Another expression for the potential is

$$V = \left(-\frac{1}{2}\,\mathrm{Im}\,\mathcal{N}^{\Lambda\Sigma} - 4 L^\Lambda \bar L^\Sigma \right) P^x_\Lambda P^x_\Sigma + 2 \bar L^\Lambda L^\Sigma\, h_{uv} k^u_\Lambda k^v_\Sigma + 2\,\mathrm{Im}\, F^{\Lambda\Sigma} P^0_\Lambda P^0_\Sigma \tag{3.3.4}$$

using (6.3.7) to rewrite the first term and writing the SK Killing vectors in terms of their prepotentials [90, p. 475].

We will not describe the full Lagrangian which is complicated and instead we refer the reader to [8, sec. 8, 90, sec. 21.3]. We are only interested in the mass terms of the fermions

$$\mathcal{L}_m = \frac{1}{2}\, S_{\alpha\beta}\, \bar{\psi}^{\alpha}_{\mu}\gamma^{\mu\nu}\psi^{\beta}_{\nu} - \frac{1}{2}\, m^{\alpha\beta}_{ij}\, \bar{\lambda}^{i}_{\alpha}\lambda^{j}_{\beta} - m^{\mathcal{A}}_{\alpha\bar{\imath}}\, \bar{\lambda}^{\alpha\bar{\imath}}\zeta_{\mathcal{A}} - \frac{1}{2}\, m_{\mathcal{A}\mathcal{B}}\, \bar{\zeta}^{\mathcal{A}}\zeta^{\mathcal{B}} - \bar{\psi}_{\mu\alpha}\gamma^{\mu}\chi^{\alpha} + \text{c.c.} \tag{3.3.5}$$

In the last term χ^α corresponds to the gravitini

$$\chi^{\alpha} = \frac{1}{2}\, W^{\alpha\beta}_{i}\, \lambda^{i}_{\beta} + 2\, N^{\alpha}_{\mathcal{A}}\zeta^{\mathcal{A}}. \tag{3.3.6}$$

The various mass matrices are given by

$$S_{\alpha\beta} = i\, \bar{L}^{\Lambda} P^{x}_{\Lambda}\sigma^{x}{}_{\alpha}{}^{\gamma}\varepsilon_{\gamma\beta}, \tag{3.3.7a}$$

$$m^{\alpha\beta}_{ij} = \frac{i}{2}\, C_{ijk} g^{k\bar{k}}\, \bar{f}^{\Lambda}_{\bar{k}} P^{x}_{\Lambda}\, \varepsilon^{\alpha\gamma}\sigma^{x}{}_{\gamma}{}^{\beta} + \varepsilon^{\alpha\beta} g_{j\bar{\imath}} k^{\bar{\imath}}_{\Lambda} f^{\Lambda}_{i}, \tag{3.3.7b}$$

$$m^{\mathcal{A}}_{\alpha\bar{\imath}} = 2ik^{u}_{\Lambda}\, \varepsilon_{\alpha\beta} U^{\beta\mathcal{A}}_{u} \bar{f}^{\Lambda}_{\bar{\imath}}, \tag{3.3.7c}$$

$$m_{\mathcal{A}\mathcal{B}} = -2L^{\Lambda}\, \varepsilon^{\alpha\beta} U^{v}_{\alpha\mathcal{A}} U^{u}_{\beta\mathcal{B}} \nabla_{v} k_{u\Lambda}, \tag{3.3.7d}$$

$$W^{\alpha\beta}_{i} = i\big(\varepsilon^{\alpha\beta} P^{0}_{\Lambda} - P^{x}_{\Lambda}\, \varepsilon^{\alpha\gamma}\sigma^{x}{}_{\gamma}{}^{\beta}\big) f^{\Lambda}_{i}, \tag{3.3.7e}$$

$$N^{\alpha}_{\mathcal{A}} = -i\, \mathbb{C}_{\mathcal{A}\mathcal{B}}\, U^{\alpha\mathcal{B}}_{u} k^{u}_{\Lambda} L^{\Lambda}. \tag{3.3.7f}$$

Another expression for $W^{\alpha\beta}_i$ is

$$W^{\alpha\beta}_{i} = -\varepsilon^{\alpha\beta} g_{i\bar{\jmath}} k^{\bar{\jmath}}_{\Lambda} L^{\Lambda} - P^{x}_{\Lambda}\, \varepsilon^{\alpha\gamma}\sigma^{x}{}_{\gamma}{}^{\beta} f^{\Lambda}_{i} \tag{3.3.8}$$

These masses are related to the fermion shift that appears in the supersymmetric variations. Through Ward identities for supersymmetry the superpotential is also given by [8, sec. 9]

$$V\, \delta^{\alpha}{}_{\beta} = -3\, S^{\alpha\gamma} S_{\gamma\beta} + W^{\alpha\gamma}_{i} g^{i\bar{\jmath}} W_{\bar{\jmath}\beta\gamma} + 4\, N^{\alpha}_{\mathcal{A}} \bar{N}^{\mathcal{A}}_{\beta}. \tag{3.3.9}$$

3.3.2 Fayet-Iliopoulos gaugings

The scalar potential reads

$$V(\tau, \bar{\tau}) = \big(g^{i\bar{\jmath}} f^{\Lambda}_{i} \bar{f}^{\Sigma}_{\bar{\jmath}} - 3\bar{L}^{\Lambda} L^{\Sigma}\big)\, g_{\Lambda} g_{\Sigma}. \tag{3.3.10}$$

3.3.3 Minimal gauged sugra

Pure supergravity corresponds to $n_v = n_h = 0$. Its bosonic action is equivalent to Einstein–Maxwell theory. Its prepotential reads [90, ex. 21.3]

$$F = -\frac{i}{2}\, (X^0)^2. \tag{3.3.11}$$

Gauge fixing gives

$$X^0 = \frac{1}{\sqrt{2}} \tag{3.3.12}$$

which gives the value of $\mathcal{N}$

$$\mathcal{N} = -i, \tag{3.3.13}$$

which implies in particular

$$G = -\star F. \tag{3.3.14}$$

The $T_{\mu\nu}$ tensor equals simply the field strength up to a factor

$$T_{\mu\nu} = 2\sqrt{2}\, F_{\mu\nu}. \tag{3.3.15}$$

The scalar potential is constant

$$V = \Lambda = -6g^2 \tag{3.3.16}$$

with Λ the cosmological constant.

3.4 Supersymmetry variations

The bosonic part of the supersymmetry variations with parameter ε^α of the fermionic fields is given by

$$\delta\psi^\alpha_\mu = \hat{\mathrm{D}}_\mu \varepsilon^\alpha = \mathrm{D}_\mu \varepsilon^\alpha - \frac{i}{8}\, T^-_{ab}\, \gamma^{ab}\gamma_\mu \varepsilon^{\alpha\beta}\varepsilon_\beta + \frac{1}{2}\,\gamma_\mu S^{\alpha\beta}\varepsilon_\beta, \tag{3.4.1a}$$

$$\delta\lambda^i_\alpha = \mathrm{D}_\mu \tau^i\, \varepsilon_\alpha + \frac{1}{4}\, T^{-i}_{ab}\, \gamma^{ab}\varepsilon_{\alpha\beta}\varepsilon^\beta + g^{i\bar{\jmath}}\bar{W}_{\bar{\jmath}\beta\alpha}\varepsilon^\beta, \tag{3.4.1b}$$

$$\delta\zeta^{\mathcal{A}} = \frac{i}{2}\, U^{\alpha\mathcal{A}}_u\, \mathrm{D}_\mu q^u\, \varepsilon_\alpha + \bar{N}^{\mathcal{A}}_\alpha \varepsilon^\alpha. \tag{3.4.1c}$$

The additional terms are quadratic in the fermions and can be found in [8, sec. 8].

We denote by $\hat{\mathrm{D}}_\mu$ the supercovariant derivative. The gauge and spacetime covariant derivatives are

$$\mathrm{D}_\mu \varepsilon^\alpha = \nabla_\mu \varepsilon^\alpha + i\, \mathcal{V}^x_\mu \sigma^{x\alpha}{}_\beta \varepsilon^\beta, \tag{3.4.2a}$$

$$\nabla_\mu \varepsilon^\alpha = \left(\partial_\mu + \frac{1}{4}\, \omega_{\mu ab}\gamma^{ab} - i\mathcal{A}_\mu\right)\varepsilon^\alpha. \tag{3.4.2b}$$

The (bosonic part of) the anti-self-dual field strengths T_{ab} and T^i_{ab} were defined in (2.3.23)

$$T^- = -\left\langle \mathcal{V}, \mathcal{F}^- \right\rangle, \qquad T^-_i = -g^{i\bar{\jmath}} \left\langle \bar{U}_{\bar{\jmath}}, \mathcal{F}^- \right\rangle. \tag{3.4.3}$$

Finally the composite U(1) and SU(2) connections were given in (3.2.16) and (3.2.17).

A BPS solution is a field configuration that solves the equations of motion and which preserves some amount of supersymmetry, which is equivalent to the invariance of the configuration under supersymmetry variations. Moreover for classical solutions the fermionic fields typically vanish which ensures that the variations of the bosonic fields are zero. Then we just need to compute the variations of the fermionic fields (if they were not vanishing they would acquire a non-zero value after a supersymmetry transformation)

$$\delta\psi_{\alpha\mu} = \delta\lambda^{\alpha i} = \delta\zeta^{\mathcal{A}} = 0. \tag{3.4.4}$$

These equations will typically separate into matrix equations, which project out some components of the parameter ε_α, and scalar equations, which can be differential or algebraic.

BPS equations imply a part of the equations of motion [117, 143]. In particular full-BPS configurations solve all equations of motion [110, p. 15], while in other cases it is sufficient to solve Maxwell equations to show that it is a solution [111, sec. 4].

The condition for ε^α to be a Killing spinor is equivalent to ε^α being covariantly constant with respect to the supercovariant derivative. In particular by taking the commutator of this equation one obtains the integrability condition

$$\left[\hat{\mathrm{D}}_\mu, \hat{\mathrm{D}}_\nu\right] \varepsilon^\alpha = \hat{R}_{\mu\nu}\varepsilon^\alpha = 0 \tag{3.4.5}$$

which is necessary but not sufficient. This equation is non-differential and gives constraints and projectors.

3.5 Magnetic gaugings

In order to obtain symplectic covariant expressions it is also possible to introduce magnetic gauging parameters such that the magnetic gauge fields A_Λ from (2.3.4) will be coupled to the scalars through the covariant derivatives. A Lagrangian description of this theory is quite involved as one needs to introduce new (tensor) fields and gauge invariances, and this is better formulated with the embedding tensor formalism [158, 187, 189]. When gaugings are abelian another possibility is to work directly with the BPS equations and the equations of motion since on-shell quantities are easier to deal with: these equations are completed such that they become symplectic covariant [76, 78]. For other works on magnetic gaugings, see also [11, 73, 144, 164].

3.5.1 Generalities

Introducing magnetic Killing vectors k^Λ that are paired with the electric ones k_Λ into a symplectic vector

$$\mathcal{K} = \begin{pmatrix} k^\Lambda \\ k_\Lambda \end{pmatrix}, \qquad \mathcal{K} = \mathcal{K}^i\, \partial_i + \mathcal{K}^{\bar\imath}\, \partial_{\bar\imath} + \mathcal{K}^u\, \partial_u, \tag{3.5.1}$$

the covariant derivative of the scalar fields becomes

$$\mathrm{D}_\mu = \partial_\mu - A_\mu \Omega \mathcal{K} = \partial_\mu - A_\mu^\Lambda k_\Lambda + A_{\Lambda\mu} k^\Lambda \tag{3.5.2}$$

in order to respect symplectic covariance [39, sec. 4.2, 144, sec. 3]. The Killing vectors can be expanded on the set of Killing vectors k_α generating the isometries of $\mathcal{M}$ (these are the same as the one appearing at the beginning of section 3.2.1)

$$\mathcal{K} = \Theta^\alpha\, k_\alpha, \qquad \Theta^\alpha = \begin{pmatrix} \theta^{\alpha\Lambda} \\ \theta^\alpha_\Lambda \end{pmatrix}. \tag{3.5.3}$$

Hence the coefficients of the linear combination are symplectic vectors, and $\theta^{\alpha\Lambda}$ and θ^α_Λ being respectively the magnetic and electric gauging parameters.

The Killing vectors satisfy constraints from closure of the algebra. There are three possibilities, depending if the vectors are both electric, both magnetic, or one electric and one magnetic.

The symplectic Killing prepotentials are given by

$$\mathcal{P}^x = \mathcal{K}^u \omega_u^x - \mathcal{W}^x, \tag{3.5.4a}$$

or in components

$$P^{x\Lambda} = k^{\Lambda u} \omega_u^x - W^{x\Lambda}, \qquad P^x_\Lambda = k^u_\Lambda \omega^x_u - W^x_\Lambda, \tag{3.5.4b}$$

One defines the prepotential charges (also called the superpotential)

$$\mathcal{L}^x = \langle \mathcal{V}, \mathcal{P}^x \rangle, \qquad \mathcal{L}^x_i = \langle U_i, \mathcal{P}^x \rangle. \tag{3.5.5}$$

In the case of FI gauging (section 3.2.2), one adds the constants g^Λ which correspond to the magnetic charges of the gravitini under the local U(1). The symplectic vector is denoted by

$$\mathcal{G} \equiv \mathcal{P}^3 = \begin{pmatrix} g^\Lambda \\ g_\Lambda \end{pmatrix}. \tag{3.5.6}$$

3.5.2 Scalar potential

For FI gauging $\mathcal{G} = (g^\Lambda, g_\Lambda)$ the scalar potential reduces to [97, sec. 2.1]

$$V = g^{i\bar{\jmath}} \mathrm{D}_i \mathcal{L} \mathrm{D}_{\bar{\jmath}} \bar{\mathcal{L}} - 3|\mathcal{L}|^2, \tag{3.5.7}$$

where

$$\mathcal{L} = \langle \mathcal{V}, \mathcal{G} \rangle, \qquad \mathrm{D}_i \mathcal{L} = \partial_i \mathcal{L} + \frac{1}{2}\, \partial_i K\, \mathcal{L}. \tag{3.5.8}$$

3.5.3 Constraints from locality

To ensure the existence of a Lagrangian and, more importantly, of an electric frame (since we derived the BPS equations from an electric frame, before doing a symplectic rotation), we must impose locality conditions on the parameters [187, sec. 3]. Then the locality constraints read [78, sec. 6.1, app. C] (see also [39, sec. 2])

$$\left\langle \Theta^\alpha, \Theta^\beta \right\rangle = 0. \tag{3.5.9}$$

It is necessary to impose this condition only when the gauge group is abelian, which is the case here [189, sec. 5]. This constraint is also a consequence of the Ward identity from which the scalar potential (3.3.9) is obtained [164].

The constraints imply that

$$\langle \mathcal{K}^u, \mathcal{P}^x \rangle = 0. \tag{3.5.10}$$

First we denote by k_α^u the generic set of Killing vectors such that

$$\mathcal{K}^u = \Theta^\alpha\, k_\alpha^u, \qquad \mathcal{W}^x = \Theta^\alpha\, w_\alpha^x, \tag{3.5.11}$$

then using the formula (3.5.4) for the prepotential we have

$$\begin{aligned} \langle \mathcal{K}^u, \mathcal{P}^x \rangle &= \langle \mathcal{K}^u, \mathcal{K}^v \omega_v^x - \mathcal{W}^x \rangle \\ &= k_\alpha^u (\omega_v^x k_\beta^v - w_\beta^x) \left\langle \Theta^\alpha, \Theta^\beta \right\rangle, \end{aligned}$$

and this vanishes from the locality constraint.

3.6 Quaternionic gaugings

In this section we consider only abelian gaugings of the isometries of special quaternionic manifolds [78].

The Killing vector $k_\Lambda^u \partial_u$ can be expanded on the basis of Killing vectors on $\mathcal{M}_h$ (studied in section 12.1)

$$k_\alpha = \{k_{\mathbb{U}}, k_\xi, \widehat{k}_\xi, k_+, k_0, k_-\} \tag{3.6.1}$$

with the coefficients

$$\theta_\Lambda^\alpha = \{\mathbb{U}_\Lambda, \alpha_\Lambda, \widehat{\alpha}_\Lambda, \epsilon_{\Lambda+}, \epsilon_{\Lambda 0}, \epsilon_{\Lambda -}\}, \tag{3.6.2}$$

using the notations of [180, 182, 184] for the parameters. Note that α_Λ and $\widehat{\alpha}_\Lambda$ are symplectic vectors (of the base SK space $\mathcal{M}_z$) of dimensions $2n_h$

$$\alpha_\Lambda = \begin{pmatrix} \alpha_\Lambda^A \\ \alpha_{A\Lambda} \end{pmatrix}, \qquad \widehat{\alpha}_\Lambda = \left(\widehat{\alpha}_\Lambda^A \_ \widehat{\alpha}_{A\Lambda}\right). \tag{3.6.3}$$

Explicitly this reads

$$k_\Lambda = k_\Lambda^u\, \partial_u = k_{\mathbb{U}_\Lambda} + \alpha_\Lambda^t \mathbb{C}\, k_\xi + \widehat{\alpha}_\Lambda^t \mathbb{C}\, \widehat{k}_\xi + \epsilon_{+\Lambda} k_+ + \epsilon_{0\Lambda} k_0 + \epsilon_{-\Lambda} k_-. \tag{3.6.4}$$

Similarly the magnetic Killing vector is written $k^{u\Lambda}$ and all the magnetic parameters have the index Λ up.

All these parameters are not independent and consistency conditions impose relations between them (see also appendix D.2). The number of constraints can be much greater than the number of parameters, showing that some of these constraints are redundant.

3.6.1 Some expressions without hidden isometries

Explicit expressions with all isometries can be cumbersome and we will restrict explicit formula to the case where there are no hidden gauged isometries.

Taking the Killing vectors from (12.1.37), we can give the various components of k_Λ [39]

$$k^\sigma_\Lambda = \epsilon_{+,\Lambda} - 2\sigma\epsilon_{0,\Lambda} + \frac{1}{2}\big(\alpha_{\Lambda A}\xi^A - \alpha^A_\Lambda\widetilde{\xi}_A\big), \tag{3.6.5a}$$

$$k^A_\Lambda = \alpha^A_\Lambda + (\mathbb{U}_\Lambda\xi)^A - \epsilon_{0,\Lambda}\xi^A, \tag{3.6.5b}$$

$$k_{\Lambda A} = \alpha_{\Lambda A} + (\mathbb{U}_\Lambda\xi)_A - \epsilon_{0,\Lambda}\widetilde{\xi}_A, \tag{3.6.5c}$$

$$k^{\dot{A}}_\Lambda = (\mathbb{U}_\Lambda Z)^{\dot{A}}, \tag{3.6.5d}$$

$$k^\phi_\Lambda = \epsilon_{0,\Lambda}. \tag{3.6.5e}$$

Recall that there is also the complex conjugate of the penultimate. The second and third can be written as

$$\begin{pmatrix} k^A_\Lambda \\ k_{\Lambda A} \end{pmatrix} = \alpha_\Lambda + (\mathbb{U}_\Lambda - \epsilon_{0,\Lambda})\xi, \tag{3.6.6}$$

while the first is

$$k^\sigma_\Lambda = \epsilon_{+,\Lambda} - 2\sigma\epsilon_{0,\Lambda} - \frac{1}{2}\alpha^t_\Lambda\mathbb{C}\xi. \tag{3.6.7}$$

3.6.2 Constraints from algebra closure

The Killing algebra is abelian if the right hand side of (3.2.4) vanishes. From the algebra with electric/electric Killing vectors we derive the following constraints [78, sec. 6.1, app. C]

$$0 = \mathbb{T}(\alpha_\Lambda, \hat{\alpha}_\Sigma) - \mathbb{T}(\alpha_\Sigma, \hat{\alpha}_\Lambda), \tag{3.6.8a}$$

$$0 = -(\mathbb{U}_\Lambda\alpha_\Sigma - \mathbb{U}_\Sigma\alpha_\Lambda) + (\epsilon_{0\Lambda}\alpha_\Sigma - \epsilon_{0\Sigma}\alpha_\Lambda) + (\epsilon_{+\Lambda}\widehat{\alpha}_\Sigma - \epsilon_{+\Sigma}\widehat{\alpha}_\Lambda), \tag{3.6.8b}$$

$$0 = (\mathbb{U}_\Lambda\widehat{\alpha}_\Sigma - \mathbb{U}_\Sigma\widehat{\alpha}_\Lambda) + (\epsilon_{-\Lambda}\alpha_\Sigma - \epsilon_{-\Sigma}\alpha_\Lambda) + (\epsilon_{0\Lambda}\widehat{\alpha}_\Sigma - \epsilon_{0\Sigma}\widehat{\alpha}_\Lambda), \tag{3.6.8c}$$

$$0 = \alpha^t_\Lambda\mathbb{C}\alpha_\Sigma + 2(\epsilon_{+\Sigma}\epsilon_{0\Lambda} - \epsilon_{+\Lambda}\epsilon_{0\Sigma}), \tag{3.6.8d}$$

$$0 = (\widehat{\alpha}^t_\Lambda\mathbb{C}\alpha_\Sigma - \alpha^t_\Lambda\mathbb{C}\widehat{\alpha}_\Sigma) + 2(\epsilon_{+\Sigma}\epsilon_{-\Lambda} - \epsilon_{+\Lambda}\epsilon_{-\Sigma}), \tag{3.6.8e}$$

$$0 = \widehat{\alpha}^t_\Lambda\mathbb{C}\widehat{\alpha}_\Sigma + 2(\epsilon_{0\Lambda}\epsilon_{-\Sigma} - \epsilon_{0\Sigma}\epsilon_{-\Lambda}). \tag{3.6.8f}$$

And we recall the definition of $\mathbb{T}_{\alpha,\hat{\alpha}}$ from (12.2.4a) We have defined

$$\mathbb{T}(\alpha_\Lambda, \hat{\alpha}_\Sigma) = (\alpha^t_\Lambda\mathbb{C}\partial_\xi)(\hat{\alpha}^t_\Sigma\mathbb{C}\partial_\xi)\,\mathbb{S}. \tag{3.6.9}$$

For the details of the computations, see appendix E.2. It is straightforward to obtain all the other constraints (electric/magnetic and magnetic/magnetic) from the electric/electric ones.

Without hidden vectors it reduces to

$$0 = \mathbb{U}_\Lambda\alpha_\Sigma - \mathbb{U}_\Sigma\alpha_\Lambda + \epsilon_{0\Lambda}\alpha_\Sigma - \epsilon_{0\Sigma}\alpha_\Lambda, \tag{3.6.10a}$$

$$0 = \alpha^t_\Lambda\mathbb{C}\alpha_\Sigma + 2(\epsilon_{+\Sigma}\epsilon_{0\Lambda} - \epsilon_{+\Lambda}\epsilon_{0\Sigma}) \tag{3.6.10b}$$

and for $\epsilon_{0\Lambda} = 0$ furthermore to

$$0 = \mathbb{U}_\Lambda \alpha_\Sigma - \mathbb{U}_\Sigma \alpha_\Lambda, \tag{3.6.11a}$$

$$0 = \alpha_\Lambda^t \mathbb{C} \alpha_\Sigma, \tag{3.6.11b}$$

which can be found in [39, eq. (2.20)].

Part II

Kähler manifolds

Chapter 4

Hermitian manifold

In $N = 1$ supergravity the manifold described by the scalars of the chiral multiplets is Kähler. Hence we first start by describing separately the Kähler manifold and the more generic Hermitian and complex manifolds of which a Kähler manifold is a subcase. On the other hand in $N = 2$ supergravity the manifold described by the vector scalars is special Kähler and we explain in chapter 6 what are the additional conditions for making a Kähler manifold special.

Great references for this section and the next one are [90, chap. 13, 147, chap. 8] (see also [18, sec. 9.A, 174]).

4.1 Definition and properties

Consider a manifold $(\mathcal{M}, g)$ of (real) dimension $2n$ and with metric

$$\mathrm{d}s^2 = g_{ab}\, \mathrm{d}\phi^a \mathrm{d}\phi^b, \qquad a = 1, \ldots, 2n, \tag{4.1.1}$$

endowed with a torsionless Levi–Civita covariant derivative, i.e.

$$\mathrm{D}_k g_{ij} = 0. \tag{4.1.2}$$

Definition 4.1 (Almost-complex manifold) The manifold $\mathcal{M}$ is almost-complex if it admits an almost-complex structure ${J_a}^b(\phi)$ which square to $-{\delta_a}^b$

$${J_a}^c {J_c}^b = -{\delta_a}^b. \tag{4.1.3}$$

An almost-complex manifold is necessarily even-dimensional (in fact it can be shown that any such manifold is almost-complex). The definition (4.1.3) implies that the eigenvalues of J are $\pm i$ (and of equal numbers).

From the almost-complex structure one defines the Nuijenhuis tensor

$${N_{ab}}^c = {J_a}^d \partial_{[c} {J_{b]}}^k - {J_b}^d \partial_{[c} {J_{a]}}^k. \tag{4.1.4}$$

The qualifier "almost" is used to indicate that J may not be defined globally.

Definition 4.2 (Complex manifold) An almost-complex manifold $(\mathcal{M}, J)$ is said to be complex if J is integrable, i.e. if it can be defined globally.

For a complex manifold the Nijenhuis tensor vanishes

$${N_{ab}}^c = 0. \tag{4.1.5}$$

Definition 4.3 (Hermitian manifold) A manifold $(\mathcal{M}, J)$ is said to be hermitian if J is compatible with the metric

$$J_a{}^c g_{cd} J_b{}^d = g_{ab} \Longleftrightarrow J\, g\, J^t = g. \tag{4.1.6}$$

Using the metric to lower an index produces the antisymmetric tensor

$$J_{ab} = J_a{}^c g_{cb}, \qquad J_{ab} = -J_{ba} \tag{4.1.7}$$

as can be seen by multiplying (4.1.6) by $J_e{}^b$

$$g_{ab} J_e{}^b = J_{ea},$$
$$J_a{}^c g_{cd} J_b{}^d J_e{}^b = -J_a{}^c g_{cd} \delta_e{}^d = g_{ab} J_e{}^b = -J_a{}^c g_{ce} = -J_{ae}$$

(in one word, hermicity implies antisymmetry). Thus it defines a 2-form called the fundamental form of $\mathcal{M}$, denoted by Ω

$$\Omega = -J_{ab}\, \mathrm{d}\phi^a \wedge \mathrm{d}\phi^b. \tag{4.1.8}$$

Note that Ω is real.

Since Ω^n is a $(2n)$-form nowhere vanishing it can serves as a volume element on the manifold [147, sec. 8.4.2].

4.2 Complex coordinates

Locally it is possible to introduce complex coordinates

$$\phi^a = (\tau^i, \bar\tau^{\bar\imath}), \qquad i, \bar\imath = 1, \dots, n \tag{4.2.1}$$

such that the metric reads

$$\mathrm{d}s^2 = g_{i\bar\jmath}\, \mathrm{d}\tau^i \mathrm{d}\bar\tau^{\bar\jmath} + g_{\bar\imath j}\, \mathrm{d}\bar\tau^{\bar\imath} \mathrm{d}\tau^j = 2\, g_{i\bar\jmath}\, \mathrm{d}\tau^i \mathrm{d}\bar\tau^{\bar\jmath}. \tag{4.2.2}$$

Note that this metric is real since it was in the original coordinates, and as a consequence

$$g_{i\bar\jmath} = g^*_{j\bar\imath}. \tag{4.2.3}$$

A generic complex manifold that is not hermitian cannot be set in this form [90, sec. 13.1]. Conversely it can be shown that in coordinates where J is diagonal, the definition (4.1.6) implies that g_{ij} and its conjugate vanish. In matrix form one has

$$g_{ab} = \begin{pmatrix} 0 & g_{i\bar\jmath} \\ g_{j\bar\imath} & 0 \end{pmatrix}. \tag{4.2.4}$$

The index i and $\bar\imath$ are called holomorphic and antiholomorphic. The convention is to write the holomorphic index first. Moreover it is always possible to use the metric to convert a (anti)holomorphic index into its counterpart. For example one can use the metric on $A_i{}^{\bar\jmath}$ to get A_{ij}

$$A_{ij} = g_{j\bar\jmath} A_i{}^{\bar\jmath} \tag{4.2.5}$$

or $\partial_{\bar\imath}$ to ∂^i. Vectors of dimension n_v will sometimes be denoted in boldface, for example $\boldsymbol{\tau}$.

In these coordinates the almost-complex structure takes the diagonal form

$$J_a{}^b = i\, \mathrm{diag}(\delta_i{}^j, -\delta_{\bar\imath}{}^{\bar\jmath}). \tag{4.2.6}$$

Inserting this expression into (4.1.8), one obtains the fundamental form in complex coordinates

$$J_{i\bar{\jmath}} = -i\, g_{i\bar{\jmath}}, \tag{4.2.7a}$$

$$\Omega = 2i\, g_{i\bar{\jmath}}\, \mathrm{d}\tau^i \wedge \mathrm{d}\bar{\tau}^{\bar{\jmath}}. \tag{4.2.7b}$$

Due to the hermicity some Christoffel symbols vanish

$$\Gamma^i{}_{\bar{\jmath}\bar{k}} = \Gamma^{\bar{\imath}}{}_{jk} = 0. \tag{4.2.8}$$

The Dobeault operators are defined by

$$\mathrm{d} = \partial + \bar{\partial}, \qquad \partial = \mathrm{d}\tau^i\, \partial_i, \qquad \bar{\partial} = \mathrm{d}\bar{\tau}^{\bar{\imath}}\, \partial_{\bar{\imath}}. \tag{4.2.9}$$

A useful relation is

$$\partial\bar{\partial} = -\frac{1}{2}\, \mathrm{d}(\partial - \bar{\partial}). \tag{4.2.10}$$

Chapter 5

Kähler geometry

5.1 Definition

Definition 5.1 (Kähler manifold) A hermitian manifold $\mathcal{M}$ is said to be Kähler if the fundamental form Ω is closed

$$\mathrm{d}\Omega = 0. \tag{5.1.1}$$

In this case Ω is also called the Kähler 2-form.

This is equivalent to J being covariantly constant[1]

$$\mathrm{D}_k J_{ij} = 0. \tag{5.1.2}$$

A Kähler manifold has a holonomy group $\mathrm{U}(n)$. The Kähler form is a symplectic form, and as such Kähler manifolds also have a symplectic structure [174, p. 20].

Example 5.1 Examples of Kähler manifolds include:

- Calabi–Yau manifolds, for which the holonomy is restricted to $\mathrm{SU}(n)$. They have a vanishing first Chern class c_1 and admit a non-vanishing holomorphic n-form [174, sec. 5].
- All Hermitian manifolds of real dimension 2 due to the fact that any 2-form in 2 dimensions is closed [174, p. 20].
- The complex projective planes $\mathbb{C}P^n$.

In complex coordinates the condition (5.1.1) translates to

$$\mathrm{d}\Omega = -i(\partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}})\mathrm{d}\tau^i \wedge \mathrm{d}\tau^j \wedge \mathrm{d}\bar{\tau}^{\bar{k}} + \text{c.c.} = 0 \tag{5.1.3}$$

where the expression (4.2.7) of J_{ab} was used. Then the Kähler form is closed if

$$\partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}} = 0. \tag{5.1.4}$$

The latter implies the existence of a real function $K(\tau, \bar{\tau})$ called the Kähler potential that determines the metric

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K. \tag{5.1.5}$$

[1] Indeed if a form is closed, then one gets derivatives of the components which can be transformed to covariant ones since the Christoffel symbols will vanish by antisymmetry.

This presents a huge simplification since a single function gives the full metric. The *Kähler cone* is defined as the range of coordinates τ^i for which the metric is positive definite.

This function is not unique as shifts – called Kähler transformations – by holomorphic and antiholomorphic functions $f(\tau)$ and $\bar{f}(\bar{\tau})$

$$K(\tau, \bar{\tau}) \longrightarrow K(\tau, \bar{\tau}) + f(\tau) + \bar{f}(\bar{\tau}) \tag{5.1.6}$$

leave the metric invariant. Moreover K does not need to be defined globally, and the Kähler potentials on various patches are related by Kähler transformations

$$K_j(\tau, \bar{\tau}) = K_i(\tau, \bar{\tau}) + f_{ij}(\tau) + \bar{f}_{ij}(\bar{\tau}). \tag{5.1.7}$$

Using Dobeault operators (4.2.9) one can write the Kähler form as

$$\Omega = 2i\, \partial\bar{\partial} K. \tag{5.1.8}$$

5.2 Riemannian geometry

Recall that

$$\Gamma^i{}_{\bar{j}\bar{k}} = \Gamma^{\bar{i}}{}_{jk} = 0 \tag{5.2.1}$$

because the manifold is hermitian. Additional symbols vanish because of the Kähler condition

$$\Gamma^i{}_{j\bar{k}} = \Gamma^{\bar{i}}{}_{\bar{j}k} = 0. \tag{5.2.2}$$

Then the only non-vanishing symbols are

$$\Gamma^i{}_{jk} = g^{i\bar{\ell}} \partial_j g_{k\bar{\ell}} = g^{i\bar{\ell}} \partial_j \partial_k \partial_{\bar{\ell}} K \tag{5.2.3}$$

and their conjugates. The trace of the Christoffel is particularly simple

$$\Gamma^j{}_{ij} = \partial_i \ln \det g. \tag{5.2.4}$$

Similarly only the component $R_{i\bar{j}k\bar{\ell}}$ of the Riemann tensor and its permutations do not vanish

$$R^i{}_{jk\bar{\ell}} = -\partial_{\bar{\ell}} \Gamma^i{}_{jk}, \tag{5.2.5a}$$

$$R_{i\bar{j}k\bar{\ell}} = \partial_i \partial_{\bar{j}} g_{k\bar{\ell}} - g^{m\bar{n}} \partial_{\bar{j}} g_{m\bar{\ell}}\, \partial_i g_{k\bar{n}} \tag{5.2.5b}$$

$$= \partial_i \partial_{\bar{j}} \partial_k \partial_{\bar{\ell}} K - g^{m\bar{n}} (\partial_{\bar{j}} \partial_{\bar{\ell}} \partial_m K) \partial_i \partial_{\bar{n}} \partial_k K. \tag{5.2.5c}$$

The Ricci tensor

$$R_{i\bar{j}} = R^k{}_{ki\bar{j}} = -g^{k\bar{\ell}} R_{i\bar{\ell}k\bar{j}} \tag{5.2.6}$$

can be obtained directly from

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \ln \det g. \tag{5.2.7}$$

The Ricci 2-form is defined by

$$\mathcal{R} = i\, R_{i\bar{j}} \mathrm{d}\tau^i \wedge \mathrm{d}\bar{\tau}^{\bar{j}} = i\, \partial\bar{\partial} \ln \sqrt{g}. \tag{5.2.8}$$

This form is closed

$$\mathrm{d}\mathcal{R} = 0 \tag{5.2.9}$$

and as a consequence of (4.2.10) is it locally exact (but not globally since the determinant is a density). The first Chern class corresponds to the cohomology class defined by the Ricci form

$$c_1 = \frac{1}{2\pi} [\mathcal{R}]. \tag{5.2.10}$$

5.3 Symmetries

To each symmetry of the manifold preserving both structures g (in order to be an isometry) and J corresponds an holomorphic Killing vector k which generates infinitesimal transformations (or holomorphic isometries) through Lie derivative [8, sec. 7.1, 94, sec. 2]: its Lie derivative acting on g and J should vanish

$$\mathcal{L}_k g_{ij} = \nabla_a k_b + \nabla_b k_a = 0, \tag{5.3.1a}$$

$$\mathcal{L}_k J_a{}^b = J_c{}^b \nabla_a k^c - J_a{}^c \nabla_c k_a = 0. \tag{5.3.1b}$$

Together these implies the invariance of the Kähler form

$$\mathcal{L}_k \Omega = 0. \tag{5.3.2}$$

In fact the last requirement is more fundamental than the vanishing of $\mathcal{L}_k J_i{}^j$, since it means that the volume is invariant (the Lie derivative of the volume element Ω^n vanishes) and we will see that a condition similar to $\mathcal{L}_k \Omega = 0$ is the correct on in the case of quaternionic manifold.

Using the explicit formula (A.2.12) for $\mathcal{L}_k$ and the fact that $\mathrm{d}\Omega = 0$ gives

$$\mathrm{d} i_k \Omega = 0. \tag{5.3.3}$$

Then the Poincaré lemma states that it exists a (real) function P called the moment map (or Killing potential) such that

$$i_k \Omega = -2\, \mathrm{d} P_k. \tag{5.3.4}$$

P_k is not unique as it can be shifted by a constant (note that it depends on k)

$$P_k \longrightarrow P_k + \xi_k. \tag{5.3.5}$$

In the rest of this section we omit the index k.

In complex coordinates the condition (5.3.1b) gives the constraints

$$\partial_{\bar\imath} k^j = 0, \qquad \partial_i k^{\bar\jmath} = 0, \tag{5.3.6}$$

which mean that the Killing vector (with the index up) splits into a holomorphic and an antiholomorphic parts

$$k = k^a(\phi)\partial_a = k^i(\tau)\partial_i + k^{\bar\imath}(\bar\tau)\partial_{\bar\imath}. \tag{5.3.7}$$

Then a variation of the coordinates with parameter θ reads

$$\delta\tau^i = \theta\, k^i(\tau), \qquad \delta\bar\tau^{\bar\imath} = \theta\, k^{\bar\imath}(\bar\tau) \tag{5.3.8}$$

and the transformation preserves the split in holomorphic and antiholomorphic coordinates. On the other hand the Killing equation (5.3.1a) gives two conditions

$$\nabla_i k_j + \nabla_j k_i = 0, \qquad \nabla_i k_{\bar\jmath} + \nabla_{\bar\jmath} k_i = 0. \tag{5.3.9}$$

The first equation is trivial since

$$\nabla_i k_j = g_{j\bar k} \nabla_i k^{\bar k} = g_{j\bar k} \partial_i k^{\bar k} = 0 \tag{5.3.10}$$

In coordinates the definition (5.3.4) of the moment map reads (from now on we remove the index k denoting the vector)

$$k_i = g_{i\bar\jmath} k^{\bar\jmath} = i\, \partial_i P, \qquad k_{\bar\imath} = -i\, \partial_{\bar\imath} P. \tag{5.3.11}$$

Then the second equation of (5.3.9) is immediately satisfied. An equation for P can be obtained from the first condition in (5.3.9)

$$\nabla_i \partial_j P = 0. \tag{5.3.12}$$

Kähler manifolds are simpler than arbitrary manifolds because a Killing vector is fully determined by one unique real function, mirroring the fact that the metric is given by the Kähler potential.

In general the Kähler potential is not invariant under Killing transformation which can induces a Kähler transformation

$$\mathcal{L}_k K = (k^i \partial_i + k^{\bar{\imath}} \partial_{\bar{\imath}}) K = f + \bar{f}, \tag{5.3.13}$$

which leaves the metric invariant. This makes possible to find an explicit expression for P. Indeed using the expression of the metric, (5.3.11) can be rewritten as

$$k_{\bar{\jmath}} = g_{i\bar{\jmath}} k^i = k^i \partial_i \partial_{\bar{\jmath}} K, \tag{5.3.14}$$

and comparing with (5.3.11) gives

$$P = i\big(k^i \partial_i K - r\big) \tag{5.3.15}$$

where $r = r(\tau)$. This last function can be identified by requiring the reality of P

$$P + \bar{P} = 2P \Longrightarrow (k^i \partial_i + k^{\bar{\imath}} \partial_{\bar{\imath}}) K = r + \bar{r}. \tag{5.3.16}$$

Then the equation (5.3.13) implies that $r = f$ and one obtains

$$P = i\big(k^i \partial_i K - f\big) = -i\big(k^{\bar{\imath}} \partial_{\bar{\imath}} K - \bar{f}\big). \tag{5.3.17}$$

In particular any constant shift ξ of the prepotential can be taken into account by shifting f to $f + i\xi$. There will be an ambiguity only for $\mathrm{U}(1)$ factors.

In general a metric admits several Killing vectors k_Λ that generate a non-abelian group with Lie algebra

$$[k_\Lambda, k_\Sigma] = f_{\Lambda\Sigma}{}^{\Omega} k_\Omega. \tag{5.3.18}$$

All quantities then get a Λ index. The bracket does not mix holomorphic and antiholomorphic vectors, and in components they read

$$k_\Lambda^j \partial_j k_\Sigma^i - k_\Sigma^j \partial_j k_\Lambda^i = f_{\Lambda\Sigma}{}^{\Omega} k_\Omega^i \tag{5.3.19}$$

with $\mathcal{L}_\Lambda \equiv \mathcal{L}_{k_\Lambda}$.

For a simple non-abelian group the moment map can be shifted by the constants such that they transform into the adjoint

$$\mathcal{L}_\Lambda P_\Sigma = (k_\Lambda^i \partial_i + k_\Lambda^{\bar{\imath}} \partial_{\bar{\imath}}) P_\Sigma = f_{\Lambda\Sigma}{}^{\Omega} P_\Omega. \tag{5.3.20}$$

This last condition, which is also called the equivariance condition, can be rewritten as

$$k_\Lambda^i g_{i\bar{\jmath}} k_\Sigma^{\bar{\jmath}} - k_\Sigma^i g_{i\bar{\jmath}} k_\Lambda^{\bar{\jmath}} = i f_{\Lambda\Sigma}{}^{\Omega} P_\Omega. \tag{5.3.21}$$

There are four families and two exceptional cases of symmetric Kähler space [90, p. 270]

$$\begin{gathered}\frac{\mathrm{SU}(p,q)}{\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)}, \qquad \frac{\mathrm{SO}^*(2n)}{\mathrm{U}(n)}, \qquad \frac{\mathrm{Sp}(2n)}{\mathrm{U}(n)}, \qquad \frac{\mathrm{SO}(n,2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}, \\ \frac{\mathrm{E}_{6,-14}}{\mathrm{SO}(10)\times\mathrm{U}(1)}, \qquad \frac{\mathrm{E}_{7,-25}}{\mathrm{E}_6\times\mathrm{U}(1)}.\end{gathered} \tag{5.3.22}$$

5.4 Kähler–Hodge manifold

Kähler–Hodge manifolds (or Kähler manifold of restricted type) are discussed in [7, sec. 2, 8, sec. 4.1, 4.2, 67, sec. 4.1, 90, sec. sec. 17.3.6, 17.5.1, app. 17A]. In the context of supergravity, the presence of fermions implies a Dirac-like quantization condition on the Kähler form and this is equivalent to the Hodge condition [67, sec. 4.1].

Definition 5.2 (Kähler–Hodge manifold) A Kähler–Hodge manifold $\mathcal{M}$ is a Kähler manifold for which it exists a line bundle $\mathcal{L} \to \mathcal{M}$ such that the first Chern class is equal to the (de Rham) cohomology class of the Kähler form

$$c_1(\mathcal{L}) = [\Omega]. \tag{5.4.1}$$

Given a metric $h(z^i, \bar{z}^{\bar{\imath}})$ on the fiber, the connection reads[2]

$$\theta = \partial \ln h = h^{-1} \partial h \tag{5.4.2}$$

and similarly for $\bar{\theta}$. Then the cohomology class is

$$c_1(\mathcal{L}) = 2i\,[\bar{\partial}\theta] = 2i\,[\bar{\partial}\partial \ln h]. \tag{5.4.3}$$

Recalling (5.1.8)

$$\Omega = 2i\,\partial\bar{\partial} K, \tag{5.4.4}$$

the definition implies that the metric is given by the exponential of the Kähler potential

$$h = \mathrm{e}^K \Longrightarrow \theta = \partial K. \tag{5.4.5}$$

Note that a Kähler transformation corresponds to a gauge transformation on θ

$$\theta \longrightarrow \theta + \partial f, \tag{5.4.6}$$

since the derivative of the Kähler potential transforms as

$$\partial_i K \longrightarrow \partial_i K + \partial_i f. \tag{5.4.7}$$

Then the transition function between two patches if given by e^f which corresponds to a Kähler transformation. A line bundle can be mapped to a U(1) bundle $\mathcal{U} \to \mathcal{M}$, and the corresponding transition function is $\exp(i \operatorname{Im} f)$. The connection on the line and on the U(1) bundles are related by

$$\mathcal{A} = \operatorname{Im} \theta = \frac{i}{2}\,(\theta - \bar{\theta}). \tag{5.4.8}$$

A way to motivate this result is that $\partial_i f = 2i\partial_i \operatorname{Im} f$, whereas taking the real part would give a total derivative and thus a vanishing curvature [90, p. 379]. Using the expression for θ, one obtains

$$\mathcal{A} = -\frac{i}{2}\big(\partial_i K\, \mathrm{d}\tau^i - \partial_{\bar{\imath}} K\, \mathrm{d}\bar{\tau}^{\bar{\imath}}\big). \tag{5.4.9}$$

In real coordinates this can be written

$$\mathcal{A}_a = -\frac{1}{2}\, J_a{}^b \partial_b K. \tag{5.4.10}$$

A field ψ^i (corresponding to a section of $\mathcal{U}$) is said to be of weight $(p, \bar{p})$ if it transforms as

$$\psi^i \longrightarrow \psi'^i = \mathrm{e}^{-\frac{1}{2}(pf + \bar{p}\bar{f})} \psi^i \tag{5.4.11}$$

[2] h is just a function since the line is 1-dimensional, such that $h^{-1} = 1/h$.

under a Kähler transformation (5.1.6). Then the covariant derivative is

$$\mathrm{D}_i\psi^j = \partial_i\psi^j + \Gamma^j{}_{ik}\psi^k + \frac{p}{2}\,\partial_i K\,\psi^j, \qquad \mathrm{D}_{\bar{\imath}}\psi^j = \partial_{\bar{\imath}}\psi^j + \frac{\bar{p}}{2}\,\partial_{\bar{\imath}} K\,\psi^j. \tag{5.4.12}$$

Moreover the conjugate field $\bar{\psi}^{\bar{i}}$ has weight $(-p, -\bar{p})$. In general one has $\bar{p} = -p$ from the fact that the derivative of a section ϕ on $\mathcal{U}$ is

$$\mathrm{D}\phi = (\mathrm{d} + ip\,\mathcal{A})\phi. \tag{5.4.13}$$

Then one can map the sections of $\mathcal{U}$ into sections of $\mathcal{L}$ through

$$\Psi^i = \mathrm{e}^{-\frac{\bar{p}}{2}K}\psi^i, \tag{5.4.14}$$

such that the covariant derivatives are

$$\mathrm{D}_i\Psi^j = \partial_i\Psi^j + \Gamma^j{}_{ik}\Psi^k + p\,\partial_i K\,\Psi^j, \qquad \mathrm{D}_{\bar{\imath}}\psi^j = \partial_{\bar{\imath}}\Psi^j. \tag{5.4.15}$$

If ψ^i is holomorphic then the field Ψ^i is covariantly holomorphic

$$\partial_{\bar{\imath}}\psi^j = 0 \Longrightarrow \mathrm{D}_{\bar{\imath}}\Psi^j = 0. \tag{5.4.16}$$

Note also that

$$R_{i\bar{\jmath}} = [\mathrm{D}_i, \mathrm{D}_{\bar{\jmath}}] = i\,g_{i\bar{\jmath}} = -J_{i\bar{\jmath}} \tag{5.4.17}$$

meaning that the curvature of the bundle is the Kähler form.

Part III

Special Kähler manifolds

Chapter 6

Special Kähler geometry

Special Kähler (SK) manifolds appear as target spaces of non-linear sigma models of the vector scalars in $N = 2$ supergravity. These spaces correspond to Kähler–Hodge manifolds endowed with a symplectic bundle. The U(1) bundle associated to the Hodge condition has the interpretation of the $\mathrm{U}(1)_R$ R-symmetry of the supersymmetry algebra. The simplest formulation is using projective coordinates which are necessary for using a symplectic covariant formalism, which can then be used to formulate more efficiently the $N = 2$ theory. In particular many analytic results for BPS and non-BPS solutions rely heavily on this formulation, and additionally some quaternionic Kähler (QK) manifolds – and more specifically most of those of interest in $N = 2$ supergravity – can be described as a fibration over a SK manifold (see chapter 11). Finally both for SK and QK manifolds the isometries are more easily understood using symplectic covariant expressions. For these reasons we propose to review these manifolds in some details: we first start by defining the manifold, its projective parametrization and its Riemannian properties. Then in the following chapters we cover in details other important aspects such as the symplectic invariants, the classification of the homogeneous spaces and the most important models (called quadratic and cubic) and at the end the isometries.

The first axiomatic definition was given in [166], and it was refined in [67] (see also [89]). Major references on the topic are the book [90] and the papers [8, 44, 45].

6.1 Definition

Definition 6.1 (Special Kähler manifold) A special Kähler (SK) manifold $(\mathcal{M}_v, g)$ of real dimension $2n_v$ with complex (or special) coordinates $\{\tau^i, \bar{\tau}^{\bar{\imath}}\}$, $i = 1, \ldots, n_v$, is a Kähler–Hodge manifold equipped with a (flat) holomorphic vector bundle with group $\mathrm{Sp}(2n_v + 2, \mathbb{R})$, and for which there exists a section v such that the exponential of the Kähler potential is given by

$$K = -\ln(-i\ \langle v, \bar{v}\rangle) \tag{6.1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the symplectic inner product [8, sec. 4, 67, sec. 4.2.2, 166, sec. 4]. An additional necessary property is[1]

$$\langle v, \partial_i v\rangle = 0. \tag{6.1.2}$$

Other equivalent definitions can be found in [67, sec. 4.2]. Since this manifold is Kähler–Hodge it satisfies all the properties from chapter 5.

[1] This condition was missing in [166].

The line and vector bundles are respectively denoted by $\mathcal{L} \to \mathcal{M}_v$ and $\mathcal{SV} \to \mathcal{M}_v$. The section v is an element of the tensor bundle $\mathcal{L} \otimes \mathcal{SV}$.

The metric is written

$$\mathrm{d}s^2 = 2\, g_{i\bar{\jmath}}\, \mathrm{d}\tau^i \mathrm{d}\tau^{\bar{\jmath}}, \qquad i = 1, \ldots, n_v \tag{6.1.3}$$

6.2 Homogeneous coordinates and symplectic structure

6.2.1 Vectors

Let's denote the components of the section v by

$$v = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}, \qquad \Lambda = 0, \ldots, n_v. \tag{6.2.1}$$

The X^Λ are called homogeneous coordinates (or projective) coordinates and they provide a projective parametrization of the manifold such that

$$\tau^i = \frac{X^i}{X^0}. \tag{6.2.2}$$

The special coordinates are left unchanged by rescaling of the homogeneous coordinates X^Λ. As a consequence the section v are defined up to rescaling

$$v \longrightarrow \mathrm{e}^{-f(\tau)} v. \tag{6.2.3}$$

A convenient gauge choice is[2]

$$X^0 = 1, \qquad X^i = \tau^i. \tag{6.2.4}$$

The transformation properties of this section will be addressed in more details in section 7.1.

We restrict ourselves to the case where the components F_Λ can be derived from a prepotential F which is an homogeneous (holomorphic) function of order 2 in the X^Λ

$$F(\lambda X) = \lambda^2 F(X). \tag{6.2.5}$$

Then one has

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda} \equiv \partial_\Lambda F. \tag{6.2.6}$$

One can write [8, sec. 4.5, 69, sec. 5]

$$F(X^0, \tau) = (X^0)^2 f(\tau) \tag{6.2.7}$$

where $f(\tau)$ is invariant under rescaling of the coordinates due to the property (6.2.5).

More generally a symplectic vector A of dimension $2(n_v + 1)$ is defined by

$$A = \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}, \tag{6.2.8}$$

where the upper and lower components are distinguished only by the positions of the index (and from the vector itself by the presence of the index).

The symplectic 2-form Ω reads explicitly

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.2.9}$$

[2] Sometimes the name "special coordinates" is used to designate explicitly this gauge choice.

It defines a scalar product

$$\langle A, B\rangle \equiv A^t \Omega B = A^\Lambda B_\Lambda - B^\Lambda A_\Lambda. \tag{6.2.10}$$

Sometimes we will need to write explicitly the symplectic indices

$$A^M = \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}, \qquad \Omega_{MN} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M = 1, \ldots, 2(n_v + 1). \tag{6.2.11}$$

With these notations the symplectic product is

$$\langle A, B\rangle = A^M \Omega_{MN} B^N. \tag{6.2.12}$$

6.2.2 Metric and Kähler potential

The Kähler potential is

$$K = -\ln(-i\, \langle v, \bar v\rangle) = -\ln i\big(\bar X^\Lambda F_\Lambda - X^\Lambda \bar F_\Lambda\big). \tag{6.2.13}$$

This definition can be understood from the following fact: the inner product between v and its conjugate transforms as

$$\langle v, \bar v\rangle \longrightarrow \mathrm{e}^{-f-\bar f}\, \langle v, \bar v\rangle\,, \tag{6.2.14}$$

under rescaling of v (6.2.3), and one recognizes in the exponential a possible Kähler transformation [44, p. 4, 166, sec. 2].

The metric is derived from the Kähler potential

$$g_{i\bar\jmath} = \partial_i \partial_{\bar\jmath} K. \tag{6.2.15}$$

An expression in homogeneous coordinates is given by [90, p. 445]

$$g_{i\bar\jmath} = 2\, \mathrm{Im}\, F_{\Lambda\Sigma}\, \partial_i X^\Lambda \partial_{\bar\jmath} X^\Sigma. \tag{6.2.16}$$

The metric is invariant under Kähler transformations

$$K \longrightarrow K' = K + f + \bar f. \tag{6.2.17}$$

Let's come back to the condition (6.1.2): despite that v is a section of the bundle, the covariant derivative is not necessary because

$$\langle v, \mathrm{D}_i v\rangle = \langle v, \partial_i v\rangle \tag{6.2.18}$$

since the symplectic product is antisymmetric [67, sec. 4.2.2].

6.2.3 Covariant holomorphic fields

The manifold is Kähler–Hodge which means that there is a U(1) bundle (see section 5.4 for more details). The section v has weight $p = 1$

$$\mathrm{D}_i v = \partial_i v + \frac{1}{2}\, \partial_i K\, v \tag{6.2.19}$$

and is holomorphic

$$\partial_{\bar\imath} v = 0, \tag{6.2.20}$$

such that one can define the holomorphic section

$$\mathcal{V} = \mathrm{e}^{\frac{K}{2}} v \equiv \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \tag{6.2.21}$$

and its covariant derivative

$$U_i = \mathrm{D}_i \mathcal{V} \equiv \begin{pmatrix} f^i_\Lambda \\ h_{i\Lambda} \end{pmatrix}. \tag{6.2.22}$$

One then has

$$\mathrm{D}_{\bar{\imath}} \mathcal{V} = 0. \tag{6.2.23}$$

Note that the coordinates τ^i can also be written as

$$\tau^i = \frac{L^i}{L^0}. \tag{6.2.24}$$

Moreover the section $\mathcal{V}$ is invariant under Kähler transformations by construction (see the previous section).

Taking the exponential of the Kähler potential (6.2.13) and using the expression of the sections (6.2.21) give the normalizations

$$\langle \mathcal{V}, \bar{\mathcal{V}} \rangle = i, \qquad \langle U_i, \bar{U}_{\bar{\jmath}} \rangle = -i\, g_{i\bar{\jmath}}. \tag{6.2.25}$$

The last relation can be used to obtain the metric if one knows U_i.

Decomposing $\mathcal{V}$ into its real and imaginary part, (6.2.25) implies that

$$\langle \operatorname{Re} \mathcal{V}, \operatorname{Im} \mathcal{V} \rangle = -\frac{1}{2}, \qquad \langle \operatorname{Re} U_i, \operatorname{Im} \bar{U}_{\bar{\jmath}} \rangle = -\frac{1}{2} \operatorname{Re} g_{i\bar{\jmath}}. \tag{6.2.26}$$

The symplectic product of a vector A with $\mathcal{V}$ and U_i are defined by

$$\Gamma(A) = \langle \mathcal{V}, A \rangle, \qquad \Gamma_i(A) = \mathrm{D}_i \Gamma(A) = \langle U_i, A \rangle \tag{6.2.27}$$

and similarly for the complex conjugates $\bar{\Gamma}(A)$ and $\bar{\Gamma}_{\bar{\imath}}(A)$. Note that these operators are linear and $\Gamma_i(A)$ can be defined only if the vector A is independent of τ^i. In particular one has

$$\Gamma(\bar{\mathcal{V}}) = i, \qquad \Gamma(\operatorname{Re} \mathcal{V}) = \frac{i}{2}, \qquad \Gamma(\operatorname{Im} \mathcal{V}) = -\frac{1}{2}, \qquad \Gamma(U_i) = 0. \tag{6.2.28}$$

Note that as a consequence of the previous relations one has

$$\mathrm{D}_{\bar{\imath}} \Gamma(A) = 0, \qquad \mathrm{D}_{\bar{\jmath}} \mathrm{D}_i \Gamma(A) = g_{i\bar{\jmath}}\, \Gamma(A). \tag{6.2.29}$$

6.2.4 Prepotential properties

The nth derivative of the prepotential is

$$F_{\Lambda_1 \cdots \Lambda_n} \equiv \frac{\partial F}{\partial X^{\Lambda_1} \cdots \partial X^{\Lambda_n}}. \tag{6.2.30}$$

The homogeneity of the prepotential implies several identities for its derivatives [69, sec. 2, 90, p. 433]

$$X^{\Lambda_n}\, F_{\Lambda_1 \cdots \Lambda_n} = (3 - n)\, F_{\Lambda_1 \cdots \Lambda_{n-1}} \tag{6.2.31}$$

(for $n = 1$ we define $F_{\Lambda_1 \Lambda_0} \equiv F$) and in particular [180]

$$F = \frac{1}{2} F_\Lambda X^\Lambda, \qquad F_\Lambda = F_{\Lambda\Sigma} X^\Sigma, \qquad F_{\Lambda\Sigma\Delta} X^\Delta = 0. \tag{6.2.32}$$

The special case $n = 3$ implies the following relation

$$\mathrm{d}F_\Lambda = F_{\Lambda\Sigma}\mathrm{d}X^\Sigma \tag{6.2.33}$$

since

$$\mathrm{d}F_\Lambda = \mathrm{d}(F_{\Lambda\Sigma}X^\Sigma) = F_{\Lambda\Sigma}\mathrm{d}X^\Sigma + X^\Sigma \mathrm{d}F_{\Lambda\Sigma} = F_{\Lambda\Sigma}\mathrm{d}X^\Sigma + \cancel{X^\Sigma F_{\Lambda\Sigma\Xi}\mathrm{d}X^\Xi}. \tag{6.2.34}$$

Two prepotentials that differ by a quadratic polynomial in X^Λ with real coefficients are equivalent as they do not contribute to the Kähler potential [23, p. 5, 184, p. 5]. Moreover such terms can be removed/added by a symplectic transformation (see section 7.1).

6.3 Homogeneous matrices

6.3.1 Hessian matrix

The Hessian matrix $\mathcal{F}$ of the prepotential F is written

$$F_{\Lambda\Sigma} = \partial_\Lambda F_\Sigma = \partial_\Sigma F_\Lambda. \tag{6.3.1}$$

In section 6.4 we will prove that $\operatorname{Im}\mathcal{F}$ has n_v positive and one negative eigenvalues.

Note that the quantity

$$T_\Lambda = -i\,\frac{\operatorname{Im} F_{\Lambda\Sigma}X^\Sigma}{\operatorname{Im} F_{\Delta\Xi}\, X^\Delta X^\Xi} \tag{6.3.2}$$

will correspond to the graviphoton projector [44].

6.3.2 Period matrix

The period matrix[3] [90, p. 448]

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i\,\frac{\operatorname{Im} F_{\Lambda\Delta}\operatorname{Im} F_{\Sigma\Xi}\, X^\Delta X^\Xi}{\operatorname{Im} F_{\Delta\Xi}\, X^\Delta X^\Xi} \tag{6.3.3}$$

is symmetric and is an object that allows to lower the index of L^Λ as

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma}L^\Sigma. \tag{6.3.4}$$

On the other hand f^i_Λ and $h_{i\Lambda}$ are related by

$$h_{i\Lambda} = \bar{\mathcal{N}}_{\Lambda\Sigma}f_i^\Sigma. \tag{6.3.5}$$

This means that $\mathcal{N}$ is not a metric for Λ index. Note also that $\mathcal{I}$ is negative definite, which is a consequence of the positivity of the metric. The real and imaginary parts of this matrix are written as $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{R}_{\Lambda\Sigma} + i\,\mathcal{I}_{\Lambda\Sigma}. \tag{6.3.6}$$

The inverse of the matrices is denoted with upper indices.

There are some useful identities

$$L^\Lambda \mathcal{I}_{\Lambda\Sigma}\bar{L}^\Sigma = -\frac{1}{2}, \qquad f_i^\Lambda \mathcal{I}_{\Lambda\Sigma}\bar{f}_{\bar{\jmath}}^\Sigma = -\frac{1}{2}\,g_{i\bar{\jmath}}, \qquad L^\Lambda \mathcal{I}_{\Lambda\Sigma}f_i^\Sigma = 0, \tag{6.3.7a}$$

$$U^{\Lambda\Sigma} = f_i^\Lambda g^{i\bar{\jmath}}\bar{f}_{\bar{\jmath}}^\Sigma = -\frac{1}{2}\,\mathcal{I}^{\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma. \tag{6.3.7b}$$

[3] This expression could also be given in terms of L^Λ because it has weight 0.

6.4 Symplectic matrices

For general references on the symmetric symplectic matrices that can be defined for SK spaces, see [5, sec. 3.2, 42, sec. 1, 88, sec. 1] (see also [76, sec. 2.2, 90, p. 514, 96, app. A, 104, app. A]).

6.4.1 Definition

Let's denote by $\mathcal{T}_{\Lambda\Sigma}$ a symmetric matrix of dimension $(n_v + 1)$, and define its real and imaginary parts[4]

$$\mathcal{T} = \mathcal{R} + i\,\mathcal{I}. \tag{6.4.1}$$

Then the (real) symplectic matrix $\mathcal{M}(\mathcal{T})$ is defined by

$$\mathcal{M}(\mathcal{T}) = \begin{pmatrix} 1 & -\mathcal{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathcal{R} & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}, \tag{6.4.2}$$

of dimension $2(n_v + 1)$, where 1 denotes the identity matrix of dimension $(n_v + 1)$. The matrix is symmetric

$$\mathcal{M}^t = \mathcal{M} \tag{6.4.3}$$

since $\mathcal{R}$ and $\mathcal{I}$ are symmetric. It is also symplectic because it satisfies the relation

$$\mathcal{M}^t \Omega \mathcal{M} = \Omega. \tag{6.4.4}$$

The product[5] of Ω with this matrix $\mathcal{M}$

$$\Omega\mathcal{M} = -\begin{pmatrix} \mathcal{I}^{-1}\mathcal{R} & -\mathcal{I}^{-1} \\ \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \end{pmatrix} \tag{6.4.5}$$

is also symplectic

$$(\Omega\mathcal{M})^t\, \Omega\, (\Omega\mathcal{M}) = \Omega. \tag{6.4.6}$$

This relation can be rewritten by expanding the transpose in order to show

$$(\Omega\mathcal{M})^2 = -1. \tag{6.4.7}$$

As a consequence the product $\Omega\mathcal{M}$ defines a complex structure on the bundle and the eigenvalues of $\Omega\mathcal{M}$ are $\pm i$ ($n_v + 1$ of each)

$$\Omega\mathcal{M}\mathcal{V} = \epsilon_1\, i\mathcal{V}, \quad \Omega\mathcal{M}\bar{\mathcal{V}} = -\epsilon_1\, i\mathcal{V}, \quad \Omega\mathcal{M}U_i = \epsilon_1\, iU_i, \quad \Omega\mathcal{M}\bar{U}_i = -\epsilon_1\, i\bar{U}_i, \tag{6.4.8}$$

where $\epsilon_1, \epsilon_2 = \pm 1$ depends on $\mathcal{T}$.[6]

The expression (6.4.7) gives the inverse of $\Omega\mathcal{M}$ as

$$(\Omega\mathcal{M})^{-1} = -\Omega\mathcal{M} \tag{6.4.9}$$

and this can be rewritten (6.4.6) as

$$(\Omega\mathcal{M})^t\Omega = -\Omega(\Omega\mathcal{M}). \tag{6.4.10}$$

Since $\mathcal{M}$ and $\Omega\mathcal{M}$ are symplectic they preserve the inner product and they can be moved inside

$$\langle \Omega\mathcal{M}A, \Omega\mathcal{M}B \rangle = \langle A, B \rangle\,, \qquad \langle \Omega\mathcal{M}A, B \rangle = \langle A, \Omega\mathcal{M}B \rangle\,. \tag{6.4.11}$$

[4] Later we will use normal letters instead of curly ones for the real and imaginary parts.

[5] Some authors call this product $\mathcal{M}$ [39, sec. 2.2].

[6] Because of this fact four cases are possible and we can predict that only two matrices will be relevant, one with $\epsilon_1 = \epsilon_2 = 1$ and one with $\epsilon_1 = -\epsilon_2 = 1$, the two other cases being equivalent to changing the overall sign.

6.4.2 Matrices $\mathcal{M}(\mathcal{N})$ and $\mathcal{M}(\mathcal{F})$

Two matrices of this type are of interest

$$\mathcal{M}_+ \equiv \mathcal{M}(\mathcal{N}) \equiv \mathcal{M}, \qquad \mathcal{M}_- \equiv \mathcal{M}(\mathcal{F}), \tag{6.4.12}$$

where $\mathcal{F}$ and $\mathcal{N}$ are respectively the period (6.3.3) and Hessian (6.3.1) matrices. Similarly by convention $\mathcal{R}$ and $\mathcal{I}$ without further specifications are the real and imaginary parts of $\mathcal{N}$.

In the case of $\Omega\mathcal{M}_+$ one has

$$\Omega\mathcal{M}\mathcal{V} = i\mathcal{V}, \qquad \Omega\mathcal{M}U_i = -iU_i \tag{6.4.13}$$

while for $\Omega\mathcal{M}_-$ one finds

$$\Omega\mathcal{M}(\mathcal{F})\mathcal{V} = i\mathcal{V}, \qquad \Omega\mathcal{M}(\mathcal{F})U_i = iU_i. \tag{6.4.14}$$

From Ω and $\mathcal{M}$ another matrix[7] can be defined [76, sec. 2.2]

$$\mathcal{C} = \frac{1}{2}(\mathcal{M} - \epsilon_\Omega\, i\Omega) \tag{6.4.15}$$

This matrix is hermitian [50, sec. 3]

$$\mathcal{C}^\dagger = \mathcal{C}. \tag{6.4.16}$$

and from (6.4.13) it satisfies the twisted self-duality

$$\mathcal{C}\mathcal{V} = -\epsilon_\Omega\, i\,\Omega\mathcal{V}. \tag{6.4.17}$$

Using equation (6.4.4) one can show that

$$\mathcal{C}\Omega\mathcal{C} = \epsilon_\Omega\, i\,\mathcal{C}. \tag{6.4.18}$$

6.4.3 Expansions and sum rules

Since the vectors $(\mathcal{V}, \bar{\mathcal{V}}, U_i, \bar{U}_i)$ form a complete basis of $\mathcal{M}_v$ [60, app. A], the identity and $\mathcal{M}$ can be expanded

$$1 = i\,\mathcal{V}\bar{\mathcal{V}}^t\Omega - i\,\bar{\mathcal{V}}\mathcal{V}^t\Omega - i\,g^{i\bar{\jmath}}\,U_i\bar{U}_{\bar{\jmath}}^t\Omega + i\,g^{i\bar{\jmath}}\,\bar{U}_{\bar{\jmath}}U_i^t\Omega, \tag{6.4.19a}$$

$$-\Omega\mathcal{M} = \mathcal{V}\bar{\mathcal{V}}^t\Omega + \bar{\mathcal{V}}\mathcal{V}^t\Omega + g^{i\bar{\jmath}}\,U_i\bar{U}_{\bar{\jmath}}^t\Omega + g^{i\bar{\jmath}}\,\bar{U}_{\bar{\jmath}}U_i^t\Omega, \tag{6.4.19b}$$

$$-\Omega\mathcal{M}(\mathcal{F}) = \mathcal{V}\bar{\mathcal{V}}^t\Omega + \bar{\mathcal{V}}\mathcal{V}^t\Omega - g^{i\bar{\jmath}}\,U_i\bar{U}_{\bar{\jmath}}^t\Omega - g^{i\bar{\jmath}}\,\bar{U}_{\bar{\jmath}}U_i^t\Omega, \tag{6.4.19c}$$

$$\mathcal{C} = \Omega\,\bar{\mathcal{V}}\mathcal{V}^t\,\Omega + g^{i\bar{\jmath}}\,\Omega\,U_i\bar{U}_{\bar{\jmath}}^t\,\Omega. \tag{6.4.19d}$$

The decompositions of Ω and $\mathcal{M}$ are straightforward. These relations can be checked by multiplying them on the right by $\mathcal{V}$ and U_i and their conjugates before using the orthonormality (6.2.25) (implying that only one term of the sum contributes) and the properties of the complex structure (6.4.13); as an example multiply the second one by $\mathcal{V}$

$$\mathcal{M}\mathcal{V} = -i\mathcal{V} = \mathcal{V}\bar{\mathcal{V}}^t\Omega\mathcal{V}. \tag{6.4.20}$$

In particular any (real) vector A on can be expanded on the basis $(\mathcal{V}, \bar{\mathcal{V}}, U_i, \bar{U}_i)$ through (6.4.19a) [104, app. A]

$$A = i\,\langle\bar{\mathcal{V}}, A\rangle\,\mathcal{V} - i\,\langle\mathcal{V}, A\rangle\,\bar{\mathcal{V}} + i\,g^{i\bar{\jmath}}\,\langle U_i, A\rangle\,\bar{U}_{\bar{\jmath}} - i\,g^{\bar{\imath}j}\,\langle\bar{U}_{\bar{\imath}}, A\rangle\,U_j \tag{6.4.21a}$$

$$= i\,\bar{\Gamma}(A)\,\mathcal{V} - i\,\Gamma(A)\,\bar{\mathcal{V}} + i\,g^{i\bar{\jmath}}\,\Gamma_i(A)\,\bar{U}_{\bar{\jmath}} - i\,g^{\bar{\imath}j}\,\bar{\Gamma}_{\bar{\imath}}(A)\,U_j \tag{6.4.21b}$$

$$= 2\,\mathrm{Im}\,\big(\langle\bar{\mathcal{V}}, A\rangle\;\mathcal{V}\big) - 2g^{i\bar{\jmath}}\,\mathrm{Im}\,\big(\langle\bar{U}_{\bar{\jmath}}, A\rangle\,U_i\big) \tag{6.4.21c}$$

$$= 2\,\mathrm{Im}\,\big(\bar{\Gamma}(A)\,\mathcal{V}\big) - 2g^{i\bar{\jmath}}\,\mathrm{Im}\,\big(\bar{\Gamma}_{\bar{\jmath}}(A)\,U_i\big). \tag{6.4.21d}$$

[7] It would be simpler to define $\mathcal{C} = 1/2(\epsilon_\Omega\mathcal{M} - i\Omega)$ since ϵ_Ω would not appear in the following relations, but we wanted to recover the formulas given in other references.

Taking a symplectic vector A, the decomposition (6.4.19b) implies the sum rule [5, sec. 3.2]

$$- \frac{1}{2} A^t \mathcal{M} A = |\Gamma(A)|^2 + |\Gamma_i(A)|^2. \tag{6.4.22}$$

Hence $\mathcal{M}$ defines a quadratic form which is negative definite if the metric is positive definite [90, p. 448], which reflects the fact that $\operatorname{Im} \mathcal{N}$ is negative definite. This is a consequence of the fact that $\mathcal{R}$ does not play any role since, defining the vector

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ -\mathcal{R} & 1 \end{pmatrix} A, \tag{6.4.23}$$

one can rewrite the previous relation as

$$A^t \mathcal{M} A = \tilde{A}^t \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I}^{-1} \end{pmatrix} \tilde{A}. \tag{6.4.24}$$

Similarly $\mathcal{M}(\mathcal{F})$ defines a quadratic form through another sum rule

$$- \frac{1}{2} A^t \mathcal{M}(\mathcal{F}) A = |\Gamma(A)|^2 - |\Gamma_i(A)|^2. \tag{6.4.25}$$

This shows that $\operatorname{Im} \mathcal{F}$ has one negative and n_v positive eigenvalues.

Note also the relation

$$- \frac{1}{2} A^t \mathcal{M}(\mathcal{F}) A = \frac{1}{2} A^t \mathcal{M} A + 2 |\Gamma(A)|^2. \tag{6.4.26}$$

6.5 Structure coefficients

For a summary of this section, see [8, sec. 4.3, 21, sec. 4] (and also [7, sec. 2, 166, sec. 2]).

The structure constant of the SK space is a symmetric 3-tensor defined by

$$C_{ijk} = \langle \mathrm{D}_i U_j, U_k \rangle \tag{6.5.1}$$

and it is covariantly holomorphic of weight 2

$$\mathrm{D}_{\bar{m}} C_{ijk} = 0. \tag{6.5.2}$$

(this covariant derivative does not involve Christoffel symbol). Notice that, as it is a 3-tensor, the covariant derivative reads explicitly

$$\mathrm{D}_i C_{jk\ell} = \partial_i C_{jk\ell} + (\partial_i K) C_{jk\ell} + \Gamma^m{}_{ij} C_{mk\ell} + \Gamma^m{}_{ik} C_{jm\ell} + \Gamma^m{}_{i\ell} C_{jkm} \tag{6.5.3}$$

(this expression is symmetric in ij).

One also has the formula

$$\mathrm{D}_i U_j = i\, C_{ijk} g^{k\bar{k}} \bar{U}_{\bar{k}} \tag{6.5.4}$$

which implies

$$\mathrm{D}_i \mathrm{D}_j \Gamma(A) = i\, C_{ijk} g^{k\bar{k}} \bar{\Gamma}_{\bar{k}}(A). \tag{6.5.5}$$

We will use the abbreviations

$$W_\tau = W_{\tau\tau\tau} = W_{ijk}\, \tau^i \tau^j \tau^k, \qquad W_{\tau,i} = W_{ijk}\, \tau^j \tau^k \tag{6.5.6}$$

and similarly for other quantities like W_y (y being the imaginary part of τ).

From this tensor one defines the rescaled structure constant

$$W_{ijk} = \mathrm{e}^{-K} C_{ijk} \tag{6.5.7}$$

which satisfies

$$\partial_{\bar{m}} W_{ijk} = 0. \tag{6.5.8}$$

It is related to the third derivative of the prepotential

$$W_{ijk} = -F_{\Lambda\Sigma\Delta}\, \partial_i X^\Lambda \partial_j X^\Sigma \partial_k X^\Delta = -(X^0)^3\, F_{ijk} \tag{6.5.9}$$

where the last equality is valid for special coordinates (6.2.2).

The complex conjugate is written $\bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}}$, and the quantities with upper indices are obtained from

$$C^{\bar{\imath}\bar{\jmath}\bar{k}} = g^{i\bar{\imath}} g^{j\bar{\jmath}} g^{k\bar{k}} C_{ijk}, \qquad \bar{C}^{ijk} = g^{i\bar{\imath}} g^{j\bar{\jmath}} g^{k\bar{k}} \bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}}. \tag{6.5.10}$$

The corresponding rescaled quantities are

$$\bar{W}^{ijk} = g^{i\bar{\imath}} g^{j\bar{\jmath}} g^{k\bar{k}} \bar{W}_{\bar{\imath}\bar{\jmath}\bar{k}} = \mathrm{e}^{-K} \bar{C}^{ijk}. \tag{6.5.11}$$

A more convenient quantity[8] is a rescaled $\bar{W}^{ijk}$

$$\bar{\mathcal{W}}^{ijk} = \mathrm{e}^{2K} \bar{W}^{ijk}. \tag{6.5.12}$$

Given a vector A the so-called cubic norm reads [30, sec. 2.1, 42, sec. 5]

$$N(A) = C_{ijk} \bar{\Gamma}^i(A) \bar{\Gamma}^j(A) \bar{\Gamma}^k(A), \qquad \bar{N}(A) = \bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}} \Gamma^{\bar{\imath}}(A) \Gamma^{\bar{\jmath}}(A) \Gamma^{\bar{k}}(A). \tag{6.5.13}$$

Note that

$$N(\mathcal{V}) = 0 \Longrightarrow N(\mathrm{Re}\,\mathcal{V}) = N(\mathrm{Im}\,\mathcal{V}) = 0 \tag{6.5.14}$$

because of the orthogonality conditions (6.2.25).

One defines finally the rank 5 E-tensor

$$E^m{}_{ijk\ell} = g^{m\bar{m}} E_{\bar{m}ijk\ell}, \qquad E_{\bar{m}ijk\ell} = \frac{1}{3}\, \bar{\mathrm{D}}_{\bar{m}} \mathrm{D}_i C_{jk\ell}. \tag{6.5.15}$$

It is symmetric in all covariant indices. An explicit expression can be computed

$$E^m{}_{ijk\ell} + \frac{4}{3}\, C_{(ijk} \delta_{\ell)}{}^m = g^{m\bar{m}} g^{n\bar{n}} g^{p\bar{p}}\, C_{n(ij} C_{k\ell)p} \bar{C}_{\bar{m}\bar{n}\bar{p}}. \tag{6.5.16}$$

6.6 Riemannian geometry

The Riemann geometry of SK manifolds is described in [23, 69, sec. 2, 166, sec. 2], and additional details on symmetric spaces are in [42, sec. 5, 68, 184].

[8] In analogy with the cubic case one could define it as $\bar{W}^{ijk}/W_y^2$ but this would not be define for quadratic prepotentials. Moreover the normalization is simpler with the above definition. For cubic prepotentials both quantities are related by a constant factor.

6.6.1 General properties

Since the space is Kähler, the expressions from section 5.2 can be used. But the additional properties give alternative expressions.

The Riemann tensor read

$$R_{i\bar{j}k\bar{\ell}} = g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}} - g^{m\bar{n}} C_{ikm} \bar{C}_{\bar{j}\bar{\ell}\bar{n}}, \tag{6.6.1}$$

the sign being chosen such that $R < 0$ [21, sec. 4]. In the rigid limit only the last term survives.

Contracting with the metric gives the Ricci tensor

$$R_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{\ell}k\bar{j}} = -(n_v + 1) g_{i\bar{j}} + g^{k\bar{\ell}} g^{m\bar{n}} C_{ikm} \bar{C}_{\bar{j}\bar{\ell}\bar{n}}. \tag{6.6.2}$$

And finally one finds the curvature

$$R = g^{i\bar{j}} R_{i\bar{j}} = -n_v(n_v + 1) + g^{i\bar{j}} g^{k\bar{\ell}} g^{m\bar{n}} C_{ikm} \bar{C}_{\bar{j}\bar{\ell}\bar{n}}. \tag{6.6.3}$$

6.6.2 Symmetric space

The space $\mathcal{M}_v$ is symmetric if the Riemann tensor is covariantly constant

$$\mathrm{D}_m R_{i\bar{j}k\bar{\ell}} = 0. \tag{6.6.4}$$

This implies that[9]

$$\mathrm{D}_\ell C_{ijk} = \mathrm{D}_{(\ell} C_{i)jk} = 0, \tag{6.6.5}$$

and as a consequence the E-tensor (6.5.15) vanishes

$$E^m{}_{ijk\ell} = 0. \tag{6.6.6}$$

From (6.5.16) this implies the relation

$$\frac{4}{3} C_{(ijk} g_{\ell)\bar{m}} = g^{n\bar{n}} g^{p\bar{p}} C_{n(ij} C_{k\ell)p} \bar{C}_{\bar{m}\bar{n}\bar{p}}, \tag{6.6.7}$$

and thus

$$g^{n\bar{n}} R_{(i|\bar{m}|j|\bar{n}} C_{n|k\ell)} = -\frac{2}{3} g_{(i|\bar{m}} C_{|jk\ell)}. \tag{6.6.8}$$

6.7 Formulation without prepotential

It is possible to write expressions for all the quantities defined in this section even if the prepotential does not exist. We refer to [5, 42, 67].

In particular since the matrix $(\bar{f}^\Lambda_{\bar{\imath}}, L^\Lambda)$ is invertible, one can obtain the period matrix through

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \bar{h}_{\Lambda\bar{\imath}} \\ M_\Lambda \end{pmatrix} \begin{pmatrix} \bar{f}^\Lambda_{\bar{\imath}} \\ L^\Lambda \end{pmatrix}^{-1}. \tag{6.7.1}$$

[9] Note that the next two equations are necessary conditions for the manifold to be symmetric, but they are not sufficient [21, sec. 4].

Chapter 7

Symplectic transformations and invariants

The description of SK manifolds in terms of the section and its derivative is symplectic covariant and we are free to change the parametrization of the bundle section $\mathcal{V}$ by performing a $\mathrm{Sp}(2n_v+2,\mathbb{R})$ rotation. This means that the expressions are not *invariant* when written in coordinates (for example the prepotential changes) but they keep the same form when given in terms of symplectic vectors. This can be compared to general relativity where expressions are covariant/invariant with respect to diffeomorphisms/isometries. A given basis is called a (symplectic) *frame*.

The next question is to construct objects that are invariant under isometries. It appears that a quartic symmetric tensor exist for SK symmetric manifolds G/H since the group G is of type E_7. This invariant tensor plays an important role in many places, such as the definition of isometries of special quaternionic manifolds (see chapter 12), in the construction of analytic solutions to the BPS equations or in some important quantities defining the black holes, such as the area of the adS_4 radius. This structure is most clearly seen using a symplectic covariant formalism, which also simplifies the formulation of the equations and of the Lagrangian.

7.1 Symplectic transformations

References include [8, sec. 2, 44, 45, 67, sec. 2, app. A].

7.1.1 Holomorphic section

A matrix $\mathcal{U}$ is symplectic if

$$\mathcal{U}^t \Omega \mathcal{U} = \Omega. \tag{7.1.1}$$

Parametrizing the matrix as

$$\mathcal{U} = \begin{pmatrix} Q & R \\ S & T \end{pmatrix} = \begin{pmatrix} Q^\Lambda{}_\Sigma & R^{\Lambda\Sigma} \\ S_{\Lambda\Sigma} & T_\Lambda{}^\Sigma \end{pmatrix}. \tag{7.1.2}$$

this implies the following constraints

$$Q^t S - S^t Q = 0, \qquad R^t T - T^t R = 0, \qquad Q^t T - S^t R = 1. \tag{7.1.3}$$

From these one can determine the dimension of the group [90, p. 85]

$$\dim \mathrm{Sp}(2n_v+2,\mathbb{R}) = (n_v+1)(2n_v+3). \tag{7.1.4}$$

The matrix $\mathcal{U}$ acts on $\mathcal{V}$ as

$$\mathcal{V}' = \mathcal{U}\mathcal{V} \quad \Longrightarrow \quad \begin{cases} L'^{\Lambda} = Q^{\Lambda}{}_{\Sigma} L^{\Sigma} + R^{\Lambda\Sigma} M_{\Sigma}, \\ M'_{\Lambda} = S_{\Lambda\Sigma} L^{\Sigma} + T_{\Lambda}{}^{\Sigma} M_{\Sigma}. \end{cases} \tag{7.1.5}$$

Since the matrix is constant it acts in the same way on U_i

$$U'_i = \mathcal{U} U_i = \mathrm{D}_i(\mathcal{U}\mathcal{V}). \tag{7.1.6}$$

In order to preserve the relation (6.3.4) in the new frame

$$M_{\Lambda} = \mathcal{N}_{\Lambda\Sigma} L^{\Sigma} \quad \Longrightarrow \quad M'_{\Lambda} = \mathcal{N}'_{\Lambda\Sigma} L'^{\Sigma} \tag{7.1.7}$$

it is necessary for the matrix $\mathcal{N}$ to transform as

$$\mathcal{N}' = (S + T\mathcal{N})(Q + R\mathcal{N})^{-1}. \tag{7.1.8}$$

For this one needs to replace M_{Λ} in (7.1.5)

$$L' = (Q + R\mathcal{N})L, \qquad M' = (S + T\mathcal{N})M = \mathcal{N}' L'. \tag{7.1.9}$$

For some applications it is convenient to consider infinitesimal transformations

$$\mathcal{U} = \mathrm{e}^{\mathfrak{U}} \sim 1 + \mathfrak{U} \tag{7.1.10}$$

where $\mathfrak{U} \in \mathfrak{sp}(2n_v + 2, \mathbb{R})$ and one writes

$$\delta\mathcal{V} = \mathfrak{U}\,\mathcal{V}. \tag{7.1.11}$$

The condition (7.1.1) translates into

$$\mathfrak{U}^t \Omega + \Omega \mathfrak{U} = 0, \tag{7.1.12}$$

or as

$$t = -q^t, \qquad r = r^t, \qquad s = s^t \tag{7.1.13}$$

in terms of the parametrization

$$\mathfrak{U} = \begin{pmatrix} q & r \\ s & t \end{pmatrix}. \tag{7.1.14}$$

7.1.2 Section and coordinates

The variation of the homogeneous coordinates can be written as [69, sec. 6, 179]

$$\delta X^{\Lambda} = q^{\Lambda}{}_{\Sigma} X^{\Sigma} + r^{\Lambda\Sigma} F_{\Sigma} = \Big(q^{\Lambda}{}_{\Sigma} + r^{\Lambda\Xi} F_{\Xi\Sigma} \Big) X^{\Sigma} \tag{7.1.15}$$

using the homogeneity of F. One sees that $\delta X^0 \neq 0$ which implies that the two sets of special coordinates

$$\tau^i = \frac{X^i}{X^0}, \qquad \tau'^i = \frac{X'^i}{X'^0} \tag{7.1.16}$$

are not equivalent anymore, i.e. the transformation does not preserve the gauge choice imposed on X^0 for defining the special coordinates. For this reason one needs to rescale the coordinates X^{Λ} by multiplying by X'^0/X^0. Infinitesimally this implies

$$\delta\tau^i = \Big(q^i{}_{\Sigma} + r^{i\Xi} F_{\Xi\Sigma} \Big) \tau^{\Sigma} - \tau^i \Big(q^0{}_{\Sigma} + r^{0\Xi} F_{\Xi\Sigma} \Big) \tau^{\Sigma} \tag{7.1.17}$$

where $\tau^0 = 1$.

A first condition on these transformations is that [184, app. C]

$$\frac{\partial X'^\Lambda}{\partial X^\Sigma} \neq 0 \tag{7.1.18}$$

is non-singular, which means that the transformation of X'^Λ in terms of X^Λ (with F_Λ taken as a function of X^Λ) is invertible.

If one wants to keep the same class of Lagrangian – derivable from a prepotential – then one also needs that it exists a function F' such that

$$F'_\Lambda = \frac{\partial F'(X')}{\partial X'^\Lambda}. \tag{7.1.19}$$

This is the case when

$$F'_{\Lambda\Sigma} = \frac{\partial F'_\Lambda}{\partial X'^\Sigma} \tag{7.1.20}$$

is symmetric.

The new prepotential F' is obtained by using the relation

$$F' = \frac{1}{2}\, F'_\Lambda X'^\Lambda \tag{7.1.21}$$

and the explicit expression for F'_Λ and X'^Λ.

The expression for the new prepotential is

$$F'(X') = F(X) + X S^t R F + \frac{1}{2}\, X S^t Q X + \frac{1}{2}\, F T^t R F \tag{7.1.22}$$

where all F except in the first term are denoting the vector F_Λ.

It is always possible to find a frame where a prepotential exists [67, sec. 4.2].

7.1.3 Induced Kähler transformation

Under a change of basis, the section v can undergo an additional Kähler transformation (see also section 6.2.2)

$$v = \mathrm{e}^{-K/2}\mathcal{V} = \mathrm{e}^{-f}\mathcal{U} v, \tag{7.1.23}$$

where $f(\tau)$ is the Kähler transformation parameter (6.2.17). The variation of v is then

$$\delta v = \mathfrak{U} v - f v. \tag{7.1.24}$$

7.1.4 Symplectic embedding of diffeomorphisms

We have seen that a symplectic transformation implies a change in the coordinates τ^i. Conversely a diffeomorphism $\xi \in \mathrm{Diff}(\mathcal{M}_v)$ induces a symplectic transformation $\mathcal{U}_\xi \in \mathrm{Sp}(2n_v + 2, \mathbb{R})$ together with a $\mathrm{U}(1)$ Kähler transformation [8, sec. 2]. Then it should exist a homomorphism

$$\begin{array}{cccc} i_\delta: & \mathrm{Diff}(\mathcal{M}_v) & \longrightarrow & \mathrm{Sp}(2n_v+2,\mathbb{R}) \\ & \xi & \longmapsto & \mathcal{U}_\xi \end{array} \tag{7.1.25}$$

Since the diffeomorphism group is infinite while the symplectic group is finite dimensional this cannot be an isomorphism. Defining the Torelli subgroup as

$$\mathrm{Tor}(\mathcal{M}_v) = \ker i_\delta \subset \mathrm{Diff}(\mathcal{M}_v) \tag{7.1.26}$$

one always have

$$\dim \mathrm{Tor}(\mathcal{M}_v) = \infty. \tag{7.1.27}$$

7.2 Symplectic invariants

Any quantity made from symplectic products behaves as a scalar under symplectic transformations – and by an abuse of language we write sometimes "symplectic invariant". This corresponds to H-invariance [47, sec. 1]. In particular this is the case of the structure constant (6.5.1) since it is defined as a symplectic product, and – given a vector A – of the products $\Gamma(A)$ and $\Gamma_i(A)$, and of the cubic norm $N(A)$, given by (6.2.27) and (6.5.13).

If the manifold is a coset $\mathcal{M}_v \equiv G/H$, then symplectic scalars with no free (anti)holomorphic indices are only H-invariant if the coordinates are fixed. Conversely H-invariant expressions are also symplectic covariant.

In the following invariants associated to a vector A are built, and we write $\Gamma \equiv \Gamma(A)$, $\Gamma_i \equiv \Gamma_i(A)$ and $N(A) \equiv N$. The independent invariants were listed in [42, sec. 5] (see also [47, 48, sec. 2]). Two invariants are given by

$$I_\pm(A, \mathcal{V}) = -\frac{1}{2}\, A^t \mathcal{M}_\pm = |\Gamma|^2 \pm |\Gamma_i|^2. \tag{7.2.1}$$

They can be written in terms of the two invariants

$$i_1 = |\Gamma|^2, \tag{7.2.2a}$$

$$i_2 = |\Gamma_i|^2. \tag{7.2.2b}$$

Two others can be introduced

$$i_3 = \frac{1}{3!}\,(\Gamma\, N + \bar\Gamma\, \bar N) = \frac{1}{3!}\,\big(C_{ijk}\Gamma\, \bar\Gamma^i \bar\Gamma^j \bar\Gamma^k + \bar C_{\bar\imath\bar\jmath\bar k} \bar\Gamma\, \Gamma^{\bar\imath}\Gamma^{\bar\jmath}\Gamma^{\bar k}\big), \tag{7.2.2c}$$

$$i_4 = \frac{i}{3!}\,(\Gamma\, N - \bar\Gamma\, \bar N) = \frac{1}{3!}\,\big(C_{ijk}\Gamma\, \bar\Gamma^i \bar\Gamma^j \bar\Gamma^k - \bar C_{\bar\imath\bar\jmath\bar k} \bar\Gamma\, \Gamma^{\bar\imath}\Gamma^{\bar\jmath}\Gamma^{\bar k}\big), \tag{7.2.2d}$$

along with the Poisson bracket

$$i_5 = \big\{N, \bar N\big\} = g^{i\bar\imath}\, \frac{\partial N}{\partial \bar\Gamma^i}\, \frac{\partial \bar N}{\partial \Gamma^{\bar\imath}} = g^{i\bar\imath} C_{ijk} \bar C_{\bar\imath\bar\jmath\bar k}\, \bar\Gamma^j \bar\Gamma^k \Gamma^{\bar\jmath}\Gamma^{\bar k}. \tag{7.2.2e}$$

We recall the definition

$$\Gamma^{\bar\imath} = g^{\bar\imath i}\Gamma_i. \tag{7.2.3}$$

7.3 The I_4 function

7.3.1 Definition

Given a vector A, the following (real) quartic polynomial – called the quartic function – possesses very special properties [42, sec. 5] (for other references, see [30, sec. 2.1, 86, sec. 4, 87, sec. 4.3, 118, app. A])

$$I_4 = (i_1 - i_2)^2 - 4\, i_4 - i_5 \tag{7.3.1a}$$

or using explicit expressions for the invariants

$$\begin{aligned} I_4 &= \big(|\Gamma|^2 - |\Gamma_i|^2\big)^2 - \frac{2i}{3}\,\big(\Gamma\, N - \bar\Gamma\, \bar N\big) - \big\{N, \bar N\big\}\,, &(7.3.1b)\\ &= \big(|\Gamma|^2 - |\Gamma_i|^2\big)^2 - \frac{2i}{3}\,\big(C_{ijk}\Gamma\, \bar\Gamma^i \bar\Gamma^j \bar\Gamma^k - \bar C_{\bar\imath\bar\jmath\bar k} \bar\Gamma\, \Gamma^{\bar\imath}\Gamma^{\bar\jmath}\Gamma^{\bar k}\big) - g^{i\bar\imath} C_{ijk} \bar C_{\bar\imath\bar\jmath\bar k}\, \bar\Gamma^j \bar\Gamma^k \Gamma^{\bar\jmath}\Gamma^{\bar k}. &(7.3.1c) \end{aligned}$$

This expression does not depend of the symplectic frame and it depends both on A and (in general) on the coordinates

$$I_4 = I_4(A, \tau^i). \tag{7.3.2}$$

The above general expression is sometimes said to be given in the complex basis [30] (as opposed to its expression for cubic prepotentials which is real). In [87, sec. 4.3] it is called the "entropy functional".

For an example with cubic prepotential, see section 8.3.4.

7.3.2 Quartic tensor

Then one can define a symmetric[1] 4-tensor [86, sec. 4, 118, app. B]

$$t_{MNPQ} = \frac{\partial^4 I_4(A)}{\partial A^M \partial A^N \partial A^P \partial A^Q}. \tag{7.3.3}$$

The explicit expression for this tensor is [87, sec. 4.3]

$$\begin{aligned} t_{MNPQ} = {} & (\mathcal{V}\Omega)_{(M}(\mathcal{V}\Omega)_N(\bar{\mathcal{V}}\Omega)_P(\bar{\mathcal{V}}\Omega)_{Q)} \\ & + g^{i\bar{\imath}} g^{j\bar{\jmath}} (U_i\Omega)_{(M}(U_j\Omega)_N(\bar{U}_{\bar{\imath}}\Omega)_P(\bar{U}_{\bar{\jmath}}\Omega)_{Q)} \\ & - 2g^{i\bar{\imath}}(\mathcal{V}\Omega)_{(M}(\mathcal{V}\Omega)_N(U_i\Omega)_P(\bar{U}_{\bar{\imath}}\Omega)_{Q)} \\ & - \frac{2i}{3}\Big(C_{ijk}(\mathcal{V}\Omega)_{(M}(\bar{U}^i\Omega)_N(\bar{U}^j\Omega)_P(\bar{U}^k\Omega)_{Q)} \\ & \qquad - \bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}}(\bar{\mathcal{V}}\Omega)_{(M}(U^{\bar{\imath}}\Omega)_N(U^{\bar{\jmath}}\Omega)_P(U^{\bar{k}}\Omega)_{Q)}\Big) \\ & - g_{i\bar{\imath}} C_{ijk}\bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}}\,(U^{\bar{\imath}}\Omega)_{(M}(U^{\bar{k}}\Omega)_N(\bar{U}^j\Omega)_P(\bar{U}^k\Omega)_{Q)} \end{aligned} \tag{7.3.4}$$

and all indices are symmetrized as indicated by the parenthesis. Another way to write it would be to use the notation (see appendix A.3)

$$\mathcal{V}_M = (\mathcal{V}\Omega)_M, \qquad U_{iM} = (U_i\Omega)_M, \tag{7.3.5}$$

but we wanted to be explicit. Using formula (6.5.4) one can replace the C-tensor by a derivative of $\bar{U}_i$

$$\begin{aligned} t_{MNPQ} = {} & (\mathcal{V}\Omega)_{(M}(\mathcal{V}\Omega)_N(\bar{\mathcal{V}}\Omega)_P(\bar{\mathcal{V}}\Omega)_{Q)} \\ & + g^{i\bar{\imath}} g^{j\bar{\jmath}} (U_i\Omega)_{(M}(U_j\Omega)_N(\bar{U}_{\bar{\imath}}\Omega)_P(\bar{U}_{\bar{\jmath}}\Omega)_{Q)} \\ & - 2g^{i\bar{\imath}}(\mathcal{V}\Omega)_{(M}(\mathcal{V}\Omega)_N(U_i\Omega)_P(\bar{U}_{\bar{\imath}}\Omega)_{Q)} \\ & - \frac{2}{3}\Big((\mathcal{V}\Omega)_{(M}(\bar{U}^i\Omega)_N(\bar{U}^j\Omega)_P(\mathrm{D}_i U_j\Omega)_{Q)} \\ & \qquad - (\bar{\mathcal{V}}\Omega)_{(M}(U^{\bar{\imath}}\Omega)_N(U^{\bar{\jmath}}\Omega)_P(\mathrm{D}_{\bar{\imath}}\bar{U}_{\bar{\imath}}\Omega)_{Q)}\Big) \\ & - g^{i\bar{\imath}}(U^{\bar{\jmath}}\Omega)_{(M}(\mathrm{D}_{\bar{\imath}}\bar{U}_{\bar{\imath}}\Omega)_N(\bar{U}^j\Omega)_P(\mathrm{D}_i U_j\Omega)_{Q)}. \end{aligned} \tag{7.3.6}$$

Finally using the definition of the Riemann tensor (6.6.1) a third rewriting is possible

$$\begin{aligned} t_{MNPQ} = {} & (\mathcal{V}\Omega)_{(M}(\mathcal{V}\Omega)_N(\bar{\mathcal{V}}\Omega)_P(\bar{\mathcal{V}}\Omega)_{Q)} \\ & - g^{i\bar{\imath}} g^{j\bar{\jmath}} (U_i\Omega)_{(M}(U_j\Omega)_N(\bar{U}_{\bar{\imath}}\Omega)_P(\bar{U}_{\bar{\jmath}}\Omega)_{Q)} \\ & - 2g^{i\bar{\imath}}(\mathcal{V}\Omega)_{(M}(\mathcal{V}\Omega)_N(U_i\Omega)_P(\bar{U}_{\bar{\imath}}\Omega)_{Q)} \\ & - \frac{2}{3}\Big((\mathcal{V}\Omega)_{(M}(\bar{U}^i\Omega)_N(\bar{U}^j\Omega)_P(\mathrm{D}_i U_j\Omega)_{Q)} \\ & \qquad - (\bar{\mathcal{V}}\Omega)_{(M}(U^{\bar{\imath}}\Omega)_N(U^{\bar{\jmath}}\Omega)_P(\mathrm{D}_{\bar{\imath}}\bar{U}_{\bar{\imath}}\Omega)_{Q)}\Big) \\ & - R_{i\bar{\imath}j\bar{\jmath}}(\bar{U}^i\Omega)_{(M}(U^{\bar{\imath}}\Omega)_N(\bar{U}^j\Omega)_P(U^{\bar{\jmath}}\Omega)_{Q)} \end{aligned} \tag{7.3.7}$$

[1] It would be more convenient to normalize by $1/4!$ which would avoid the factors in the formulas relating $I_4(A)$ and $I_4(A, A, A, A)$ below, but this is not the usual convention.

(note the minus sign on the second line). The last term is called the sectional curvature of matter and corresponds to the symplectic pull-back of the Riemann tensor.

Then one can define a function I_4 that takes four arguments

$$I_4(A,B,C,D) = t_{MNPQ} A^M B^N C^P D^Q \tag{7.3.8}$$

along with its gradient

$$I_4'(A,B,C)^M = \Omega^{MR}\, t_{RNPQ} A^N B^P C^Q \tag{7.3.9}$$

where Ω is used to get a vector and not a form.

Finally one defines the formulas for equal arguments

$$I_4(A) = \frac{1}{4!}\, I_4(A,A,A,A), \qquad I_4'(A) = \frac{1}{3!}\, I_4'(A,A,A). \tag{7.3.10}$$

Note that by definition one has

$$\langle A_1, I_4'(A_2,A_3,A_4)\rangle = I_4(A_1,A_2,A_3,A_4). \tag{7.3.11}$$

7.3.3 Identities

In this section we want to show various identities that hold for any SK space.[2]

A first thing to note is that any I_4 or I_4' evaluated with at least two $\mathcal{V}$ (or its real/imaginary parts) will be independent of C_{ijk} due to the orthogonality of $\mathcal{V}$ and U_i (6.2.25) and the fact that these terms involve only one product without U_i.

The simpler formulas are

$$I_4(\mathcal{V}) = 0, \qquad I_4(\operatorname{Re}\mathcal{V}) = I_4(\operatorname{Im}\mathcal{V}) = \frac{1}{16} \tag{7.3.12}$$

using (6.2.28) and the fact that all other terms vanish.

One of the most useful identity is [118, app. B]

$$I_4'(A, \operatorname{Im}\mathcal{V}, \operatorname{Im}\mathcal{V}) = -4\,\langle \operatorname{Im}\mathcal{V}, A\rangle \operatorname{Im}\mathcal{V} - 8\,\langle \operatorname{Re}\mathcal{V}, A\rangle \operatorname{Re}\mathcal{V} - \Omega\mathcal{M}A. \tag{7.3.13}$$

and from it one deduces the relation

$$\operatorname{Re}\mathcal{V} = 2\, I_4'(\operatorname{Im}\mathcal{V}) = \frac{I_4'(\operatorname{Im}\mathcal{V})}{2\sqrt{I_4(\operatorname{Im}V)}}. \tag{7.3.14}$$

The latter also gives directly

$$I_4(A, \operatorname{Im}\mathcal{V}, \operatorname{Im}\mathcal{V}, \operatorname{Im}\mathcal{V}) = -3\epsilon_\Omega\, \langle \operatorname{Re}\mathcal{V}, A\rangle\,. \tag{7.3.15}$$

Since I_4' defines a vector it is possible to nest expressions, for example $I_4'(A,B,I_4'(C,D,E))$ is again a vector. These expressions can be simplified using identities for the product $t_{MNPQ}\Omega^{MR}t_{RSTU}$, which depend on the type of the manifold under consideration (magical, cubic non-magical and quadratic models) [84, sec. 2] (see also [33]).

It is remarkable that none of these identities changes when $\mathcal{V}$ is multiplied by a phase.

[2] In earlier papers [79, 105, 118] it was believed that these identities hold only for symmetric spaces but this is not true [119].

7.4 Duality invariants

In this section we assume that the SK space is symmetric

$$\mathcal{M}_v = \frac{G}{H}, \tag{7.4.1}$$

where G is called the duality group.

7.4.1 General definition

A duality invariant

$$I_n = I_n(A, \mathcal{V}) = I_n\big(\Gamma(A), \Gamma_i(A)\big) \tag{7.4.2}$$

(where A is any symplectic vector) is a homogeneous polynomial of order n which is invariant under G-transformations (i.e. under the isometries). One consequence is that it does not depend on the manifold coordinates [42, 47, footnote 1]

$$\partial_i I_n = 0 \Longleftrightarrow I_n = I_n(A). \tag{7.4.3}$$

In $d = 4$ duality invariants for all symmetric manifolds G/H are quartic,[3] i.e. $n = 4$. This is a consequence of the fact that the group G is always of type E_7 [49, 84, 86].

Definition 7.1 ($\mathbf{E_7}$-type Lie group) A group of type E_7 is a Lie groups for which there exists a representation $\mathbf{R}$ such that ($A_i \in \mathbf{R}$ in the following) [33, sec. 4, 84, sec. 2.1]:

1. $\mathbf{R}$ is symplectic, which means that the singlet $\mathbf{1}$ sits into the antisymmetric product

$$1 = (\mathbf{R} \times \mathbf{R})_a, \tag{7.4.4}$$

and the associated invariant tensor $\mathbb{C}$ corresponds to the symplectic metric (skew-symmetric 2-form). The latter defines a symplectic product for vectors in $\mathbf{R}$

$$\langle A_1, A_2 \rangle = \mathbb{C}_{MN} A_1^M A_2^N. \tag{7.4.5}$$

2. There exists a unique invariant symmetric 4-tensor t (called a primitive G-invariant structure)

$$1 = (\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R})_s, \tag{7.4.6}$$

and then one can define the map $I_4 : \mathbf{R}^4 \to \mathbb{R}$

$$I_4(A_1, A_2, A_3, A_4) = t_{MNPQ} A_1^M A_2^N A_3^P A_4^Q. \tag{7.4.7}$$

3. The trilinear map $I_4' : \mathbf{R}^3 \to \mathbf{R}$ defined by

$$\langle I_4'(A_1, A_2, A_3), A_4 \rangle = I_4(A_1, A_2, A_3, A_4), \tag{7.4.8}$$

satisfies

$$\langle I_4'(A_1, A_1, A_1), I_4'(A_2, A_2, A_2) \rangle = -2\, I_4(A_1, A_1, A_2, A_2)\, \langle A_1, A_2 \rangle\,. \tag{7.4.9}$$

These properties are linked to the connection between Jordan algebras (and Freudenthal triple system) and special Kähler manifolds. They imply various identities for the quartic invariant.

For symmetric SK manifolds the quartic invariant is given by the quartic function defined in (7.3.1). In this case it is independent of the scalars which is due to the fact that W_{ijk} and $\mathcal{W}^{ijk}$ are constant.

It is invariant under diffeomorphisms of $\mathcal{M}_v$ (detailed in section 9) [86, sec. 4]. Moreover the quartic invariant can be built directly from the generators of the group G [9, sec. 3, 49].

[3] As we will see later, the groups are *degenerate* for quadratic prepotential, and there is a quadratic invariant.

7.4.2 Freudenthal duality

The Freudenthal dual $\mathfrak{f}(A)$ of a vector A is defined by [86]

$$\mathfrak{f}(A)^M = \Omega^{MN} \frac{\partial \left| \sqrt{I_4(A)} \right|}{\partial A^N}. \tag{7.4.10}$$

This operator $\mathfrak{f}$ is an anti-involution and preserves the quartic invariant

$$\mathfrak{f}(\mathfrak{f}(A)) = -A, \qquad I_4(\mathfrak{f}(A)) = I_4(A). \tag{7.4.11}$$

Then $\mathfrak{f}$ is a complex structure.

Chapter 8

Manifold classification. Quadratic and cubic prepotentials

We provide elements concerning the classification of homogeneous symmetric and non-symmetric spaces, and we give more details on quadratic and cubic models. Both these models appear frequently in $N = 2$ supergravity and they contain all the possible symmetric spaces: we will use them frequently in our study of BPS solutions and we will also classify the isometries in these two cases.

8.1 Classification of spaces

Spaces with cubic prepotentials are referred to as very special Kähler spaces. They are obtained from the dimensional reduction of $d = 5$ $N = 2$ supergravity for which the scalar manifold is real; this operation is called the r-map. As a consequence they have real structure constants.

The classification of symmetric spaces have been done in [68, 102], while homogeneous spaces were described in [41, 181] (see also [182–184]). Other useful references include [91, p. 78, tab. 2, 90, p. 443, tab. 20.5].

8.1.1 Symmetric spaces

For all symmetric SK spaces there exists a symplectic basis where the prepotential is quadratic or cubic [42, p. 29]. Properties of the Riemann tensor and the curvature of theses spaces are described in [68].

Spaces with quadratic prepotentials correspond to complex projective spaces (see section 8.2) [68]

$$\mathbb{C}P^{n_v} \equiv \frac{\mathrm{SU}(n_v, 1)}{\mathrm{SU}(n_v) \times \mathrm{U}(1)} \tag{8.1.1}$$

(for $n_v = 1$ there is only one U(1) in the denominator). They originally appeared in [135].

Günaydin, Sierra and Townsend obtained all symmetric spaces with cubic prepotentials by studying the link between Jordan algebra and symmetric real geometries in $d = 5$ $N = 2$ supergravity and reducing to $d = 4$ [102]. It was proven by Cremmer and van Proeyen that this list was indeed complete, using a classification of symmetric Kähler spaces (5.3.22) and imposing the "special" conditions [68] (see also [69, sec. 5, app.]).

There is an infinite family of cubic spaces (sometimes called the *generic Jordan sequence* [21])

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(n_v,2)}{\mathrm{SO}(n_v)\times\mathrm{SO}(2)} \tag{8.1.2}$$

(for $n_v = 1$ there is only the first factor), along with four exceptional cases (sometimes called magical models) [68, 102, sec. 5]

$$\frac{\mathrm{Sp}(6)}{\mathrm{U}(3)}, \quad \frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1)}, \quad \frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6)\times\mathrm{U}(1)}, \quad \frac{\mathrm{E}_{7,-25}}{\mathrm{E}_6\times\mathrm{U}(1)} \tag{8.1.3}$$

for $n_v = 6, 9, 15, 27$ respectively (related to the magic square – they are linked with the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O}$). An interesting point of the generic sequence is that they are the only SK spaces with a direct product structure [8, p. 11].

Note that

$$\mathrm{SU}(1,1) \sim \mathrm{SL}(2,\mathbb{R}). \tag{8.1.4}$$

The cubic case $n_v = 3$ (called the STU model) is very special because [90, p. 452]

$$\mathcal{M}_v = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,2)}{\mathrm{SO}(2)\times\mathrm{SO}(2)} \sim \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)^3. \tag{8.1.5}$$

This implies that the geometry will factorize and this manifold exhibits very interesting properties.

In the case $n_v = 1$, the manifolds are $\mathrm{SU}(1,1)/\mathrm{U}(1)$ for both the quadratic and cubic prepotentials, but they are different since they have different curvature [68, p. 451]

$$R_{\text{quad}} = -2, \qquad R_{\text{cubic}} = -\frac{2}{3}. \tag{8.1.6}$$

Symmetric spaces are also Einstein

$$R_{i\bar{\jmath}} = \Lambda g_{i\bar{\jmath}}, \qquad \Lambda = \frac{R}{n_v}, \tag{8.1.7}$$

where [21, sec. 5]

$$\Lambda_{\text{quad}} = -(n_v + 1), \qquad \Lambda_{\text{cubic}} = -\frac{n_v^2 - 2n_v + 3}{n_v}, \qquad \Lambda_{\text{magic}} = -\frac{2}{3}\,n_v. \tag{8.1.8}$$

8.1.2 Homogeneous spaces

The classification of homogeneous SK spaces with cubic prepotential was started by Cecotti [41] and completed by de Wit and van Proeyen [181]. As reviewed in section 11, QK manifolds can be obtained from SK manifolds through the c-map. Homogeneous quaternionic spaces were classified by Alekseevskii and Cecotti used this fact to obtain homogeneous SK manifolds as the inverse of the c-map. In their paper de Wit and van Proeyen discovered new SK spaces, showing that Alekseevskii's classification was incomplete (since new QK manifolds could be derived from the c-map).

De Wit and van Proeyen found interesting links with Clifford algebras, while Cecotti showed that these spaces were related to T-algebras, which are a generalization of Jordan algebras.

8.2 Quadratic prepotential

For references see [87, sec. 4.2, 90, sec. 13.3].

Quadratic prepotentials

$$F = \frac{i}{2}\, \eta_{\Lambda\Sigma}\, X^\Lambda X^\Sigma \tag{8.2.1}$$

correspond to the complex projective spaces $\mathbb{C}P^{n_v}$

$$\mathcal{M}_v = \frac{\mathrm{SU}(n_v, 1)}{\mathrm{SU}(n_v) \times \mathrm{U}(1)} \tag{8.2.2}$$

which are maximally symmetric. The flat metric on this space is given by

$$\eta_{\Lambda\Sigma} = \mathrm{diag}(-1, 1, \dots, 1). \tag{8.2.3}$$

The coefficients of F are imaginary because real quadratic terms are irrelevant as seen in section 6.2.4.

Because the isotropy group is $\mathrm{SU}(n_v) \times \mathrm{U}(1)$ there is a natural split between the timelike direction $\Lambda = 0$ and the spacelike ones $\Lambda = i$.

8.2.1 General formulas

In special coordinates [78, app. A.1]

$$X^\Lambda = \begin{pmatrix} 1 \\ \tau^i \end{pmatrix} \tag{8.2.4}$$

the F_Λ are given by

$$F_\Lambda = i\, \eta_{\Lambda\Sigma} X^\Sigma = i \begin{pmatrix} -1 \\ \tau^i \end{pmatrix} \tag{8.2.5}$$

The "spatial" indices are raised and lowered with $\delta_{i\bar{\jmath}}$ and $\delta^{i\bar{\jmath}}$.

The Kähler potential is given by

$$\mathrm{e}^{-K} = 2(|\boldsymbol{\tau}|^2 - 1) \tag{8.2.6}$$

where $\boldsymbol{\tau}$ is the vector with components τ^i. The metric reads

$$g_{i\bar{\jmath}} = \frac{\delta_{i\bar{\jmath}}}{1 - |\boldsymbol{\tau}|^2} + \frac{\bar{\tau}_i \tau_{\bar{\jmath}}}{(1 - |\boldsymbol{\tau}|^2)^2}. \tag{8.2.7}$$

The structure constants vanish

$$C_{ijk} = 0 \tag{8.2.8}$$

and for this reason these models in supergravity are called minimally coupled. This implies that three invariants from (7.2.2) are zero [84, sec. 8.4]

$$i_3 = i_4 = i_5 = 0. \tag{8.2.9}$$

The curvature of these spaces is read from (6.6.3)

$$R = -n_v(n_v + 1). \tag{8.2.10}$$

Quadratic spaces can be obtained as a truncation from symmetric cubic spaces since [84, sec. 8.3]

$$\frac{\mathrm{SU}(n_v, 1)}{\mathrm{SU}(n_v) \times \mathrm{U}(1)} \subset \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2n_v, 2)}{\mathrm{SO}(2n_v) \times \mathrm{SO}(2)} \tag{8.2.11}$$

8.2.2 Quartic and quadratic invariants

The groups $\mathrm{SU}(n_v, 1)$ are degenerate groups of type E_7, as is seen in the vanishing of the structure constants [84]. As a consequence the quartic invariant I_4 becomes the square of a quadratic invariant I_2 [180, p. 227, 78, sec. 3.2]

$$I_4(A) = I_2(A)^2. \tag{8.2.12}$$

The quadratic invariant reads [84, sec. 8.4]

$$I_2 = i_1 - i_2 \tag{8.2.13}$$

and one denotes by θ_{MN} the associated tensor

$$I_2(A_1, A_2) = \theta_{MN} A_1^M A_2^N, \qquad \theta_{MN} = \frac{1}{2}\, \frac{\partial^2 I_2(A)}{\partial A^M \partial A^N}. \tag{8.2.14}$$

The quartic tensor (8.2.12) can be derived from this 2-tensor

$$t_{MNPQ} = 4!\, \theta_{MN} \theta_{PQ}. \tag{8.2.15}$$

Using (6.4.25) I_2 can also be written

$$I_2(A) = -\frac{1}{2}\, A^t \mathcal{M}(\mathcal{F}) A, \tag{8.2.16}$$

where $\mathcal{M}(F)$ was defined in section 6.4.

Writing explicitly the components with $\mathcal{Q} = (p^\Lambda q_\Lambda)$, the quadratic invariant is [42, sec. 5, 78, sec. 3.2, 87, sec. 1]

$$I_2(\mathcal{Q}) = \frac{i}{2}\, p^\Lambda \eta_{\Lambda\Sigma}\, p^\Sigma + \frac{i}{2}\, q_\Lambda \eta^{\Lambda\Sigma} q_\Sigma. \tag{8.2.17}$$

Note that I_2 can be rewritten as

$$I_2(\mathcal{Q}) = \frac{1}{2}\, T^{\Lambda\Sigma} T^{\Delta\Xi}\, \eta_{\Lambda\Delta} \eta_{\Sigma\Xi}, \qquad T_{\Lambda\Sigma} = p^\Lambda q^\Sigma - p^\Sigma q^\Lambda. \tag{8.2.18}$$

This implies

$$\theta = \frac{i}{2} \begin{pmatrix} \eta_{\Lambda\Sigma} & 0_\Lambda{}^\Sigma \\ 0^\Lambda{}_\Sigma & \eta^{\Lambda\Sigma} \end{pmatrix}. \tag{8.2.19}$$

The gradient defines a new vector

$$I_2'(A)^M = \Omega^{MN} \theta_{NP} A^P. \tag{8.2.20}$$

Because of the existence of I_2, the Freudenthal operator (see section 7.4.2) becomes [84, sec. 10]

$$\mathfrak{f}(A)^M = \Omega^{MN}\, \frac{\partial I_2(A)}{\partial A^N} \tag{8.2.21}$$

while using the definition of the gradient gives

$$\mathfrak{f}(A) = \frac{1}{2}\, I_2'(A). \tag{8.2.22}$$

It preserves the quadratic invariant

$$I_2(\mathfrak{f}(A)) = I_2(A). \tag{8.2.23}$$

In this context the operator I_2' also defines a complex structure (up to a normalization) since we have seen that $\mathfrak{f}$ defines one.

8.3 Cubic prepotential

8.3.1 General case

Manifolds with cubic prepotential are called very special Kähler manifolds or d-geometries. These manifolds can be obtained by reducing $d = 5$ supergravity to $d = 4$ through the r-map.

For details see [21, 69, sec. 5, 184, p. 7, sec. 4, 78, sec. 3.1, app. A].

For space with cubic prepotential there is a frame where F can be put in the form[1]

$$F = -D_{ijk}\,\frac{X^i X^j X^k}{X^0} \tag{8.3.1}$$

where D_{ijk} is a symmetric 3-tensor. The associated f function is

$$f(\tau) = -D_{ijk}\tau^i\tau^j\tau^k. \tag{8.3.2}$$

Using similar notations to (6.5.6) one defines D_τ, $D_{\tau,i}$, etc.

The D-tensor corresponds to the (rescaled) structure constant, and one often denotes by $\widehat{D}^{ijk}$ its inverse

$$D_{ijk} = W_{ijk}, \qquad \widehat{D}^{ijk} = \bar{W}^{ijk} = . \tag{8.3.3}$$

In current notations one has [78, sec. 3.1]

$$\widehat{D}^{ijk} = \frac{1}{D_y^2}\, g^{i\ell} g^{jm} g^{kn}\, D_{\ell mn}. \tag{8.3.4}$$

The (rescaled) structure constant are given in terms of the D-tensor

$$W_{ijk} = D_{ijk}, \tag{8.3.5}$$

and it is convenient to define the tensor $\widehat{D}^{ijk}$ [78, sec. 3.1]

$$\widehat{D}^{ijk} = \frac{1}{D_y^2}\, g^{i\ell} g^{jm} g^{kn}\, D_{\ell mn} = \frac{1}{W_y^2}\,\bar{W}^{ijk}. \tag{8.3.6}$$

In special coordinates, the conjugates are

$$F_\Lambda = \begin{pmatrix} D_\tau \\ -3\,D_{\tau,i} \end{pmatrix}. \tag{8.3.7}$$

The Kähler potential is

$$\mathrm{e}^{-K} = 2\big(\operatorname{Im} f + 2i \operatorname{Im}\tau^i \operatorname{Re}(\partial_i f)\big) = 8\,D_y \tag{8.3.8}$$

since

$$\begin{aligned} \mathrm{e}^{-K} &= -i\big(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda\big) = i(D_\tau - D_{\bar\tau}) - 3i(D_{\tau\tau\bar\tau} - D_{\bar\tau\bar\tau\tau}) \\ &= -2\operatorname{Im} D_\tau + 6\operatorname{Im} D_{\tau\tau\bar\tau} = 2(D_y - 3\,D_{xxy}) + 6(D_{xxy} + D_y). \end{aligned}$$

The metric is real and not only hermitian [105, app A.1]

$$g_{ij} = -\frac{3}{2}\,\frac{D_{y,ij}}{D_y} + \frac{9}{4}\,\frac{D_{y,i}D_{y,j}}{D_y^2} \tag{8.3.9}$$

[1] The minus sign is conventional, other factors can be found in the literature, such as $\pm 1, \pm i$, along with some different normalization, for example $1/3!$ [48, 69].

The Riemann tensor is

$$R^i{}_{jk}{}^\ell = \delta^i_j \delta^\ell_k + \delta^i_k \delta^\ell_j - \frac{9}{16}\, \widehat{D}^{i\ell m} D_{mjk}. \tag{8.3.10}$$

The E-tensor (6.5.15) reads [78, sec. 3.1]

$$E^i{}_{jk\ell m} = \widehat{D}^{ijk} D_{j(\ell m} D_{np)k} - \frac{64}{27}\, \delta^i{}_{(m} D_{np\ell)}. \tag{8.3.11}$$

If $\mathcal{M}_v$ is symmetric, then $\widehat{D}^{ijk}$ entries are constant and they satisfy

$$\widehat{D}^{ijk} D_{j\ell(m} D_{np)k} = \frac{16}{27} \Big(\delta^i_\ell D_{mnp} + 3\, \delta^i_{(m} D_{np)\ell} \Big) \tag{8.3.12a}$$

$$\widehat{D}^{ijk} D_{j(\ell m} D_{np)k} = \frac{64}{27}\, \delta^i_{(\ell} D_{mnp)}. \tag{8.3.12b}$$

8.3.2 Generic symmetric models

As explained in section 8.1.1, the generic cubic symmetric models are the manifolds

$$\mathcal{M}_v = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(n_v,2)}{\mathrm{SO}(n_v)\times \mathrm{SO}(2)}. \tag{8.3.13}$$

In this case again there is a natural split between the timelike direction $\Lambda = 0$ and spacelike ones $\Lambda = i$ because the isotropy group is $\mathrm{SO}(n_v) \times \mathrm{SO}(2)$.

8.3.3 Jordan algebras and quartic invariant

The existence and the form of the quartic invariant for symmetric very special Kähler manifolds is related to Freudenthal triple systems and the associated Jordan algebra; good references includes [30, 84] (for a mathematical paper, see [33]).

For symmetric cubic spaces the quartic invariant is given by [30, sec. 2.1, 42, sec. 5, 78, sec. 3.1, 184, p. 26] (see also [33, sec. 3])

$$I_4(\mathcal{Q}) = -(q_\Lambda p^\Lambda)^2 + \frac{1}{16}\, p^0\, \widehat{D}^{ijk} q_i q_j q_k - 4\, q_0\, D_{ijk} p^i p^j p^k + \frac{9}{16}\, \widehat{D}^{ijk} D_{k\ell m} q_i q_j\, p^\ell p^m \tag{8.3.14}$$

with $\mathcal{Q} = (p^\Lambda, q_\Lambda)$.

The explicit components of the tensor t_{MNPQ} are [40, app. D, 105, app. A.3]

$$\begin{gathered} t_{00}{}^{00} = -4, \qquad t_{0i}{}^{0j} = -2\delta_i{}^j, \qquad t_{ij}{}^{k\ell} = -4\delta_i{}^{(k}\delta_k{}^{\ell)} + \frac{9}{4}\, D_{ijm}\widehat{D}^{k\ell m}, \\ t_0{}^{ijk} = -\frac{3}{8}\, \widehat{D}^{ijk}, \qquad t_{ijk}{}^0 = 24\, D_{ijk}. \end{gathered} \tag{8.3.15}$$

A fundamental identity is [30, sec. 2.1, 118, app. B]

$$I_4'(I_4'(A), A, A) = -8\, A\, I_4(A) \tag{8.3.16}$$

which is called the Freudenthal identity and is a consequence of the Jordan algebra structure of the space. Some identities that are satisfied by combinations of the invariant evaluated with two vectors are given in the appendix D.1.

8.3.4 Non-symmetric spaces

As shown in [21, 23], spaces with a cubic prepotential have a least the isometry group

$$G = \mathrm{SO}(1,1) \times \mathbb{R}^{n_v} \tag{8.3.17}$$

where the first factor is related to overall rescaling while the second corresponds to n_v shifts of the axions $\mathrm{Re}\,\tau^i$. As we will see in the section 9.2, these isometries correspond to the universal transformations associated to parameters $\{\beta, b^i\}$. As a consequence the quartic function can depend only on the dilatons $\mathrm{Im}\,\tau^i$, and the terms that are scalar-dependent will be proportional to the E-tensor (6.5.15)

$$I_4(A, \tau^i) \sim I_4(A) + (D_y)^{5/3} E^m{}_{ijk\ell} p^j p^k p^\ell q_m q_n \frac{\partial D_p}{\partial p^i \partial p^n} \tag{8.3.18}$$

with $I_4(A)$ is the quartic invariant (8.3.14) (but here $\widehat{D}^{ijk}$ depends on the scalars).

Chapter 9

Special Kähler isometries

The main motivation of this chapter is to understand the isometries of the quadratic and cubic models. This is an important step in order to construct gauged supergravities based on these models as one needs to know the correspoding Killing vectors that appear in the covariant derivatives. Moreover some isometries of the QK manifolds are inherited from its base SK space.

9.1 General case

Special Kähler isometries were worked out in [69, sec. 6, 180, p. 222, 183] (see also [78, sec. 3]).

Isometries (also called duality transformations) on special Kähler manifolds are given by symplectic transformations (see section 7.1) that are consistent with the symplectic vectors [90, p. 450, 179]. In particular this means that the duality transformation of F_Λ agrees with the transformation induced by the fact that F_Λ is a function of X^Λ [180, p. 222]. For homogeneous spaces some isometries are constrained while other are universal and their existence is always guaranteed. In the case of symmetric spaces all isometries are realized [180, p. 222]. These isometries are generated by holomorphic Killing vectors since the manifold is Kähler, and all the properties described in section 5.3 also apply.

The isometry group is denoted by

$$G_v = \mathrm{ISO}(\mathcal{M}_v). \tag{9.1.1}$$

It is embedded into the symplectic group through the map (7.1.25)

$$i_\delta : G_v \longrightarrow \mathrm{Sp}(2n_v + 2) \tag{9.1.2}$$

and it is necessary to know this embedding to derive the induced action on the other fields [8, sec. 3]. In this case since G_v is finite dimensional it is possible to provide an explicit construction (in particular it is unique for $N \geq 3$).

The variation of the section is

$$\delta v = \mathfrak{U}\, v \tag{9.1.3}$$

with

$$\mathfrak{U} = \begin{pmatrix} q & r \\ s & t \end{pmatrix} \in \mathfrak{sp}(2n_v + 2) \tag{9.1.4}$$

and the constraints

$$t = -q^t, \qquad r = r^t, \qquad s = s^t. \tag{9.1.5}$$

Consistency of the transformation of the vector v with the expression $F_\Lambda(X)$ implies that the prepotential keeps the same functional form [184, p. 6, app. C]

$$F'(X') = F(X'). \tag{9.1.6}$$

In supergravity this condition implies that the Lagrangian is invariant. Note that this does not mean that the function itself is invariant, and one finds that [69, sec. 6]

$$\delta F(X) = F(X') - F(X) = i\left(X s X - \frac{1}{4}\, F r F\right). \tag{9.1.7}$$

As said in section 6.2.4 pure imaginary quadratic terms have no effect.

This is equivalent to the chain rule

$$\delta F_\Lambda = \frac{\partial F_\Lambda}{\partial X^\Sigma}\, \delta X^\Sigma = F_{\Lambda\Sigma} \delta X^\Sigma. \tag{9.1.8}$$

Contracting this equation with X^Λ and using the homogeneity of F gives

$$X^\Lambda \delta F_\Lambda = F_\Lambda \delta X^\Lambda. \tag{9.1.9}$$

This last condition is sufficient to classify all the isometries and it reads explicitly [69, sec. 6, 180, p. 223]

$$X^\Lambda s_{\Lambda\Sigma} X^\Sigma - 2 X^\Lambda (q^t)_\Lambda{}^\Sigma F_\Sigma - F_\Lambda r^{\Lambda\Sigma} F_\Sigma = 0. \tag{9.1.10}$$

From the relation

$$F = \frac{1}{2}\, F_\Lambda X^\Lambda \tag{9.1.11}$$

one obtains the variation

$$\delta F = \frac{1}{2}\big(\delta F_\Lambda X^\Lambda + F_\Lambda \delta X^\Lambda\big) = \delta F_\Lambda X^\Lambda = F_\Lambda \delta X^\Lambda, \tag{9.1.12}$$

the last two equalities coming from (9.1.9).

The number of isometries is given by the number of independent parameters ω^m in the matrix $\mathfrak{U}$ and they can be found by expanding (9.1.10) in τ^i. Then the Killing vectors and the symplectic matrix can be written as linear combinations

$$k^i = \omega^m\, k^i_m, \qquad \mathfrak{U} = \omega^m\, \mathfrak{U}_m \tag{9.1.13}$$

where each k^i_m and $\mathfrak{U}_m$ generates an independent isometry.

Also the Kähler potential (6.2.13)

$$\mathrm{e}^{-K} = -i\, \langle v, \bar v\rangle \tag{9.1.14}$$

is obviously invariant under isometries since it is written only in terms of symplectic invariant quantities, but this does not need to be the case in special coordinates: there may be a compensating Kähler transformation

$$\mathcal{L}_k K = 2\,\mathrm{Re}\, f_k \tag{9.1.15}$$

associated to the transformation with Killing vector k. The reason is that a transformation may change $X^0 = 1$ to another value $X'^0 \neq 1$, and one needs to perform a compensating Kähler transformation in order to set $X'^0 = 1$ [67].

9.2 Cubic prepotential

Let's consider the cubic prepotential

$$F = -D_{ijk}\,\frac{X^i X^j X^k}{X^0}. \tag{9.2.1}$$

The isometries were studied in [69, sec. 6, 181, 184] (see also [78, sec. 3.1]).

9.2.1 Parameters

The matrix $\mathfrak{U}$ is parametrized as [78, sec. 3.1, 184, p. 7, sec. 4.2]

$$\begin{gathered} q^{\Lambda}{}_{\Sigma} = -(t^t)^{\Lambda}{}_{\Sigma} = \begin{pmatrix} \beta & a_j \\ b^i & B^i{}_j + \frac{1}{3}\,\beta\,\delta^i{}_j \end{pmatrix}, \\ s_{\Lambda\Sigma} = \begin{pmatrix} 0 & 0 \\ 0 & -6D_{ijk}b^k \end{pmatrix}, \qquad r^{\Lambda\Sigma} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{3}{32}\widehat{D}^{ijk}a_k \end{pmatrix}. \end{gathered} \tag{9.2.2}$$

In special coordinates the variation of τ^i is given by

$$\delta\tau^i = b^i - \frac{2}{3}\,\beta\,\tau^i + B^i{}_j\tau^j - \frac{1}{2}\,R^i{}_{jk}{}^{\ell}\,\tau^j\tau^k a_\ell \tag{9.2.3}$$

and the Killing vector is

$$k = k^i\partial_i = k_\beta + b^i\,k_{b,i} + a_i\,k_a^i + B^i{}_j\,(k_B)_i{}^j. \tag{9.2.4}$$

The unconstrained symmetries associated to β and b^i generate respectively a rescaling and a shift of the axions.

The other rescaling symmetries associated to $B^i{}_j$ are constrained by[1]

$$B_{(i}{}^{\ell}D_{jk)\ell} = 0. \tag{9.2.5}$$

Finally the non-linear symmetries must satisfy

$$a_i E^i{}_{jk\ell m} = 0 \tag{9.2.6}$$

where the E-tensor is given by (6.5.16) or (8.3.11). This condition is necessary and sufficient for having $\widehat{D}^{ijk}a_k = \text{cst}$ (which is needed because the matrix $\mathfrak{U}$ is constant) [184, sec. 4.2].

If $\mathcal{M}_v$ is symmetric, then $\widehat{D}^{ijk}$ is constant and $E^i{}_{jk\ell m} = 0$ such that a_i is unconstrained. Then the symmetry group will be a simple Lie algebra, with b^i and a_i being associated to lowering and raising operators, while $(\beta, B^i{}_j)$ are associated to Cartan elements.

9.2.2 Lie derivative

Transformation associated to β and a_i induce a Kähler transformation of the potential with [78, sec. 3.3, app. A.1]

$$f = \beta + a_i\tau^i. \tag{9.2.7}$$

[1] This constraint is discussed more deeply in [181, 184, sec. 5] in which the authors study which d_{ijk} satisfy it, and this has some link with Clifford algebra.

9.2.3 Algebra

The algebra can be found in [69, sec. 6, 184, sec. 4.2]

$$[k_\beta, k_{b,i}] = \frac{2}{3}\, k_{b,i}, \qquad [k_\beta, k_a^i] = -\frac{2}{3}\, k_a^i, \qquad [k_{b,i}, k_a^j] = \delta^i{}_j k_\beta + \tilde{R}^i{}_{jk}{}^\ell (k_B)_\ell{}^k, \tag{9.2.8a}$$

$$\Big[(k_B)_i{}^j, k_{b,k}\Big] = \tilde{R}^i{}_{jk}{}^\ell k_{b,\ell}, \qquad \Big[(k_B)_i{}^j, k_a^k\Big] = -\tilde{R}^i{}_{j\ell}{}^k k_a^\ell \tag{9.2.8b}$$

where

$$\tilde{R}^i{}_{jk}{}^\ell = R^i{}_{jk}{}^\ell + \frac{2}{3}\, \delta^i{}_j \delta_k{}^\ell. \tag{9.2.9}$$

Due to the form of the algebra the existence of a transformation with parameter a_i imply one of the form $B_i{}^j$.

The algebra $\mathfrak{g}_v$ of G_v can be decomposed in eigenspaces associated to the symmetry β [184, sec. 2.2]

$$\mathfrak{g}_v = \mathfrak{g}_{-2/3} + \mathfrak{g}_0 + \mathfrak{g}_{2/3} \tag{9.2.10}$$

where

$$[k_\beta, \mathfrak{g}_a] = a\, \mathfrak{g}_a. \tag{9.2.11}$$

The space $\mathfrak{g}_0$ contains β and $B^i{}_j$ while $\mathfrak{g}_{2/3}$ contains b^i, and as a result

$$\dim \mathfrak{g}_{2/3} = n_v. \tag{9.2.12}$$

Hidden symmetries a_i are in $\mathfrak{g}_{-2/3}$ and the associated roots are located on the left of the root diagram, while the dimension of the space

$$\dim \mathfrak{g}_{-2/3} \le n_v \tag{9.2.13}$$

According to the denomination of [184, sec. 2.2], symmetries associated to a_i are hidden ones. This bound is saturated – meaning that a_i exist – for symmetric spaces, in which case the curvature and $\widehat{D}^{ijk}$ are constant, or equivalently when $E^i{}_{jk\ell m} = 0$. Otherwise the Lie algebra is not semisimple.

9.3 Quadratic prepotential

Now one considers quadratic prepotentials

$$F = \frac{i}{2}\, \eta_{\Lambda\Sigma}\, X^\Lambda X^\Sigma. \tag{9.3.1}$$

9.3.1 Parameters

The solution to the constraints (9.1.10) is given by [78, sec. 3.2, app. A.1]

$$s_{\Lambda\Sigma} = -\eta_{\Lambda\Xi}\, r^{\Xi\Upsilon}\, \eta_{\Upsilon\Sigma}, \qquad \eta_{\Lambda(\Sigma} q^\Lambda{}_{\Xi)} = 0 \tag{9.3.2}$$

where there is no sum on Λ in the last constraint (i.e. all diagonal elements are vanishing). The second constraint is equivalent to

$$q^0{}_i = q^i{}_0, \qquad q^i{}_j = -q^j{}_i, \qquad q^\Lambda{}_\Lambda = 0. \tag{9.3.3}$$

The variations of the coordinates is given by

$$\delta\tau^i = A^i{}_0 + (A^i{}_j - A^0{}_0 \delta^i{}_j)\tau^j - A^0{}_j \tau^j \tau^i \tag{9.3.4}$$

where

$$A = q + i\, r\, \eta. \tag{9.3.5}$$

Looking at the variation of τ^i, the trace of A and ${A^0}_0$ have the same action and one should be removed, and this is equivalent to removing one of them for r. The number of parameters contained in each matrices is

$$r: \quad \frac{1}{2}\,(n_v+1)(n_v+2) - 1, \qquad q: \quad \frac{1}{2}\,n_v(n_v-1) + n_v, \tag{9.3.6}$$

giving a total number of $n_v(n_v+2)$ which agrees with the number of Killing vectors on $\mathbb{C}P^{n_v}$.

9.3.2 Lie derivative

A Kähler transformation is induced for some of the isometries [78, sec. 3.3, app. A.1]

$$f = 2\,{\bar{A}^0}_i\,\tau^i. \tag{9.3.7}$$

Part IV

Quaternionic Kähler manifolds

Chapter 10

Quaternionic geometry

Quaternionic Kähler manifold (QK) manifolds form the target manifold of hypermultiplets in $N = 2$ supergravity. These manifolds possess a SU(2) bundle which correspond to the $\mathrm{SU}(2)_R$ symmetry of the supersymmetry algebra, and as a consequence there is a triplet of complex structures that obey the quaternionic algebra. After giving the definition of these manifolds we describe their geometrical properties followed by a general description of isometries. In particular we describe the SU(2) compensator which is interpreted as a rotation of the complex structures under a transformation, and it will be an important ingredient in the construction of BPS vacua. Finally we describe the special quaternionic manifolds that are constructed as a fibration over a SK manifolds and which are simpler than generic QK spaces, and in the following chapter we build the isometries of these spaces.

General references include [8, sec. 5, 7, 90, chap. 13 and 20, 93, sec. 2] (see also [71, 174, sec. 5]). Some historical and mathematical references are [94, 99, 116, 126, 157, 168, 169].

10.1 Definitions

Definition 10.1 (Quaternionic manifold) A quaternionic Kähler (QK) manifold $(\mathcal{M}_h, h)$ is a $4n_h$-dimensional real manifold with metric

$$\mathrm{d}s^2 = h_{uv}\, \mathrm{d}q^u \mathrm{d}q^v, \qquad u = 1, \ldots, 4n_h \tag{10.1.1}$$

endowed with three (almost-)complex structures $J^x, x = 1, 2, 3$, satisfying the quaternionic algebra

$$J^x J^y = -\delta^{xy} + \varepsilon^{xyz} J^z. \tag{10.1.2}$$

Alternatively a QK manifold is characterized by its holonomy group [99, 116]

$$\mathrm{Hol}(\mathcal{M}_h) = \mathcal{H} \cdot \mathrm{Sp}(1) \equiv \mathcal{H} \times \mathrm{Sp}(1)/\mathbb{Z}_2, \qquad \mathcal{H} \subset \mathrm{Sp}(n_h). \tag{10.1.3}$$

Locally the coordinates q^u can be gathered into quaternions, but in general this is not possible globally [94, p. 126–127]. Similarly these spaces are not Kähler strictly speaking in general and this is an abuse of language.

We note that $\mathrm{Sp}(n_h) \subset \mathrm{SO}(4n_h)$ and it is the subgroup that leaves invariant the J^x. $\mathrm{Sp}(n_h) \cdot \mathrm{Sp}(1)$ is a maximal subgroup of $\mathrm{SO}(4n)$ [99]. We recall that $\mathrm{Sp}(1) \sim \mathrm{SU}(2)$.

The connection 1-form of the SU(2) factor is denoted by

$$\omega^x = (\omega^x)_u\, \mathrm{d}q^u, \tag{10.1.4}$$

and the associated curvature is

$$\Omega^x = \nabla\omega^x = \mathrm{d}\omega^x + \frac{1}{2}\,\varepsilon^{xyz}\omega^y \wedge \omega^z. \tag{10.1.5}$$

Moreover the metric must be hermitian with respect to the three J^x (denoted collectively as $\boldsymbol{J}$), i.e.

$$\forall x: \qquad J^x\, h\, (J^x)^t = h \tag{10.1.6}$$

(no sum over x) and they should be covariantly constant

$$\nabla_w \boldsymbol{J}_u{}^v = \mathrm{D}_w \boldsymbol{J}_u{}^v + 2\boldsymbol{\omega}_w \times \boldsymbol{J}_u{}^v = 0 \tag{10.1.7a}$$

$$\nabla_w (J^x)_u{}^v = \mathrm{D}_w (J^x)_u{}^v + \varepsilon^{xyz}(\omega^y)_w (J^z)_u{}^v = 0, \tag{10.1.7b}$$

where D_u is the covariant derivative associated to h_{uv}. This relation means that the J^x are covariantly constant with respect to D_u up to an SU(2) rotation with vector $(\omega^x)_u(q)$.

The triplet of hyperkähler 2-forms

$$K^x = J^x_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v, \qquad J^x_{uv} = h_{uw}(J^x)_v{}^w. \tag{10.1.8}$$

have to be closed with respect to Sp(1) connection

$$\nabla K^x = \mathrm{d}K^x + \varepsilon^{xyz}\omega^y \wedge K^z = 0. \tag{10.1.9}$$

For a quaternionic manifold the SU(2) curvature 2-form needs to be proportional to the hyperkähler 2-form

$$\Omega^x = \lambda K^x. \tag{10.1.10}$$

In supergravity $\lambda = -1$ [7, p. 6], but we will keep it general for two reasons:

- some authors use different normalizations;
- the limit $\lambda = 0$ corresponds to hyperkähler manifolds and rigid supersymmetry.

Because of the connection the covariant exterior derivative does not square to zero but to [94, sec. 4, 169, sec. 4]

$$\nabla^2 f^x = \varepsilon^{xyz}\Omega^y f^z \tag{10.1.11}$$

for any p-form f^x.

The fundamental (quaternionic) 4-form is defined as [116, 126, 169]

$$\mathit{\Omega} = K^x \wedge K^x = \frac{1}{\lambda^2}\,\Omega^x \wedge \Omega^x, \tag{10.1.12}$$

it is globally defined, non-vanishing and covariantly closed (i.e. parallel)

$$\nabla \mathit{\Omega} = 0 \tag{10.1.13}$$

since it is invariant under $\mathrm{Sp}(n_h)\cdot\mathrm{Sp}(1)$ [126, 157] (or in the opposite sense, a manifold is quaternionic if $\mathit{\Omega}$ is covariantly closed). This implies that $\mathit{\Omega}$ is closed and harmonic (this is equivalent to $K^x = \lambda\Omega^x$) [94, sec. 4]

$$\mathrm{d}\mathit{\Omega} = 0, \qquad \Delta\mathit{\Omega} = 0. \tag{10.1.14}$$

This is automatic for $n_h = 1$ since $\mathit{\Omega} = 3\varepsilon$ (ε being the volume form of the space, not to be counfounded with ε^{xyz}) [169, sec. 2]. Recall that the laplacian on forms is defined by

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d} \tag{10.1.15}$$

where δ is the codifferential.

We want to prove that Ω is closed. Using the definition (10.1.5) of Ω^x we have

$$\begin{aligned}\lambda^2\,\Omega &= \left(\mathrm{d}\omega^x + \frac{1}{2}\,\varepsilon^{xyz}\omega^y\wedge\omega^z\right)\wedge\left(\mathrm{d}\omega^x + \frac{1}{2}\,\varepsilon^{xuv}\omega^u\wedge\omega^v\right)\\ &= \mathrm{d}\omega^x\wedge\mathrm{d}\omega^x + \varepsilon^{xyz}\mathrm{d}\omega^x\wedge\omega^y\wedge\omega^z + \varepsilon^{xuv}\varepsilon^{xyz}\omega^u\wedge\omega^v\wedge\omega^y\wedge\omega^z.\end{aligned}$$

The last term vanishes because the ε will give a symmetric factor, so we have [3, sec. 3]

$$\lambda^2\,\Omega = \mathrm{d}\left(\omega^x\wedge\mathrm{d}\omega^x + \frac{1}{3}\,\varepsilon^{xyz}\omega^x\wedge\omega^y\wedge\omega^z\right). \tag{10.1.16}$$

This implies that Ω is closed as announced. For $n_h > 2$ this is a necessary and sufficient condition for the manifold to be quaternionic and $\mathrm{d}\Omega$ determines entirely $\nabla\Omega$, while for $n_h = 2$ we need to take some care [169, sec. 2, app. A].

The volume element on $\mathcal{M}_h$ is given by Ω^{n_h}.

Closely related to the quaternionic manifolds are the hyperkähler ones, for which the SU(2) bundle is trivial, and the holonomy group is contained in $\mathrm{Sp}(n_h)$.

10.2 Geometry

10.2.1 Vielbein

Let's introduce the vielbein 1-form $U^{\alpha\mathcal{A}}$

$$U^{\alpha\mathcal{A}} = U_u^{\alpha\mathcal{A}}\,\mathrm{d}q^u \tag{10.2.1}$$

such that

$$h_{uv} = \mathbb{C}_{\mathcal{A}\mathcal{B}}\varepsilon_{\alpha\beta}U_u^{\alpha\mathcal{A}}U_v^{\mathcal{B}\beta}. \tag{10.2.2}$$

The flat coordinates have been split in two indices due to the fact that the holonomy group is $\mathrm{Sp}(n_h)\cdot\mathrm{Sp}(1)$: $\mathcal{A}$ and α runs respectively in the fundamental representations of $\mathrm{Sp}(n_h)$ and $\mathrm{Sp}(1)$

$$\alpha = 1,2, \qquad \mathcal{A} = 1,\dots,2n_h, \tag{10.2.3}$$

where the corresponding symplectic flat metrics are $\mathbb{C}$ and ε (see the appendix A.3 for conventions)

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \qquad \mathbb{C}_{\mathcal{A}\mathcal{B}} = -\mathbb{C}_{\mathcal{B}\mathcal{A}}. \tag{10.2.4}$$

The inverse vielbein $U^u_{\alpha\mathcal{A}}$ is defined such that

$$U_u^{\alpha\mathcal{A}}U^v_{\alpha\mathcal{A}} = \delta_u{}^v, \qquad U_u^{\alpha\mathcal{A}}U^u_{\beta\mathcal{B}} = \delta_\alpha{}^\beta\delta_\mathcal{A}{}^\mathcal{B} \tag{10.2.5}$$

and it obeys the reality condition

$$(U^{\alpha\mathcal{A}})^* = U_{\alpha\mathcal{A}} = \mathbb{C}_{\mathcal{A}\mathcal{B}}\varepsilon_{\alpha\beta}U_v^{\mathcal{B}\beta}. \tag{10.2.6}$$

These conditions imply

$$2\,U_u^{\alpha\mathcal{A}}U^v_{\beta\mathcal{A}} = \delta_u{}^v\delta_\alpha{}^\beta + i\,\sigma^x{}_\beta{}^\alpha(J^x)_u{}^v, \tag{10.2.7a}$$

$$(J^x)_u{}^v = -i\,\sigma^x{}_\alpha{}^\beta\,U_u^{\alpha\mathcal{A}}U^v_{\beta\mathcal{A}}. \tag{10.2.7b}$$

Other relations are satisfied, such as

$$\mathbb{C}_{\mathcal{A}\mathcal{B}}(U_u^{\alpha\mathcal{A}}U_v^{\beta\mathcal{B}} + U_v^{\alpha\mathcal{A}}U_u^{\beta\mathcal{B}}) = \varepsilon^{\alpha\beta}\,h_{uv}, \tag{10.2.8a}$$

$$\varepsilon_{\alpha\beta}(U_u^{\alpha\mathcal{A}}U_v^{\beta\mathcal{B}} + U_v^{\alpha\mathcal{A}}U_u^{\beta\mathcal{B}}) = \frac{1}{n_h}\,\mathbb{C}^{\mathcal{A}\mathcal{B}}\,h_{uv}. \tag{10.2.8b}$$

The vielbein is covariantly constant

$$\nabla_v U_u^{\alpha\mathcal{A}} = \partial_v U_u^{\alpha\mathcal{A}} + \omega_{v\beta}{}^{\alpha}\, U_u^{\beta\mathcal{A}} + \Delta_{v\mathcal{B}}{}^{\mathcal{A}}\, U_u^{\alpha\mathcal{B}} - \Gamma^w{}_{vu}\, U_w^{\alpha\mathcal{A}} = 0, \tag{10.2.9}$$

where ω and Δ are the SU(2) and Sp(n_h) (Lie algebra valued) connections

$$\omega_\beta{}^{\alpha} = \omega_{u\beta}{}^{\alpha}\, \mathrm{d}q^u = i\, \omega^x\, \sigma^x{}_\beta{}^{\alpha}, \qquad \Delta_{\mathcal{B}}{}^{\mathcal{A}} = \Delta_{v\mathcal{B}}{}^{\mathcal{A}} \mathrm{d}q^u, \tag{10.2.10}$$

and ω^x is the connection (10.1.4).

10.2.2 Curvature

Due to the holonomy of the manifold the Riemann tensor factorizes. Its precise form can be found from (10.2.9) and it reads

$$R_{uv}{}^w{}_s = U^w_{\alpha\mathcal{A}} U_s^{\alpha\mathcal{B}}\, \mathcal{R}_{uv\mathcal{A}}{}^{\mathcal{B}} - \boldsymbol{J}_s{}^w \cdot \boldsymbol{\Omega}_{uv}, \tag{10.2.11a}$$

$$\mathcal{R}_{uv\mathcal{A}}{}^{\mathcal{B}} = 2\, \partial_{[u} \Delta_{v]\mathcal{B}}{}^{\mathcal{A}} + 2\, \Delta_{[u|\mathcal{C}|}{}^{\mathcal{A}} \Delta_{v]\mathcal{B}}{}^{\mathcal{C}}, \tag{10.2.11b}$$

$$\boldsymbol{\Omega}_{uv} = 2\, \partial_{[u} \boldsymbol{\omega}_{v]} + 2\, \boldsymbol{\omega}_u \times \boldsymbol{\omega}_v \tag{10.2.11c}$$

where Ω^x is the SU(2) curvature (10.1.5), and we recall that it is proportional to the hyperkähler 2-form (10.1.10).

Quaternionic manifolds are Einstein [169]

$$R_{uv} = \frac{R}{4n_h}\, h_{uv} \tag{10.2.12}$$

and thus have constant curvature. Moreover the latter is related to the coefficient of proportionality between Ω^x and K^x

$$\lambda = \frac{R}{8n_h(n_h+2)}. \tag{10.2.13}$$

Even stronger one can prove that the Riemann tensor decomposes as (we omit the indices) [90, p. 455, 94, sec. 4]

$$R = 2\lambda\, R_{\mathbb{H}P} + R_0 \tag{10.2.14}$$

where $R_{\mathbb{H}P}$ is the curvature on quaternionic projective space, and R_0 is the Ricci-flat curvature part (related to the Weyl tensor) of Sp(n_h) (it behaves as a curvature tensor for a Riemannian manifold whose holonomy is a subgroup of Sp(n_h)).

10.3 Symmetries

As for the case of Kähler manifold a Killing vector k acts with a Lie derivative to generate isometries. It should preserve the metric h_{uv} and the fundamental 4-form $\mathit{\Omega}$ [94, sec. 4], that is

$$\mathcal{L}_k h_{uv} = \mathcal{L}_k \mathit{\Omega} = 0. \tag{10.3.1}$$

We have proved that $\mathrm{d}\mathit{\Omega} = 0$ so we have

$$\mathcal{L}_k \mathit{\Omega} = \mathrm{d} i_k \mathit{\Omega} = 0. \tag{10.3.2}$$

Invoking the Poincaré lemma, it exists a 2-form P_k such that [169, sec. 4]

$$i_k \mathit{\Omega} = \mathrm{d}P_k \tag{10.3.3}$$

generalizing the moment map from the Kähler manifolds. We can decompose it (locally) as

$$P_k = P_k^x \Omega^x. \tag{10.3.4}$$

Instead of continuing on this path, we introduce the definitions as in [8, sec. 7.3]. We assume that the action of the Lie group generates triholomorphic isometries, which means that $\mathcal{L}_k$ acts on Ω^x and ω^x [72]

$$\mathcal{L}_k \Omega^x = \varepsilon^{xyz} W_k^y \Omega^z, \qquad \mathcal{L}_k \omega^x = \nabla W_k^x \tag{10.3.5}$$

where W_k^x is an SU(2) compensator.[1] The reason is that the Sp(1) curvature being nonzero, we cannot trivialize the Sp(1) bundle: then all quantities that transform under this group (such as K^x) are defined on this bundle, and not just on the quaternionic base space, and thus they are subject to local Sp(1) gauge transformations [186, sec. 1] or, said another way, they must transform covariantly.

In the same way we associated a prepotential to a Killing vector of Kähler manifolds, we would like to introduce triholomorphic prepotentials (or moment maps) P_k^x satisfying [169, sec. 4]

$$i_k K^x = \nabla P_k^x. \tag{10.3.6}$$

We can express them in terms of the hyperkähler forms (under certain conditions of regularity) [94, sec. 4]. Introduce first the 1-form

$$\beta^x = i_k K^x = \frac{1}{\lambda}\, i_k \Omega^x = \nabla P_k^x, \tag{10.3.7}$$

and take its covariant derivative

$$\nabla \beta^x = \nabla^2 P_k^x \Longrightarrow \mathrm{d}\beta^x + \varepsilon^{xyz} \omega^y \wedge \beta^z = \varepsilon^{xyz} \Omega^y P_k^z \tag{10.3.8}$$

using (10.1.11). Applying i_k and noting that $i_k \beta^x = 0$ since $i_k^2 = 0$ (and $i_k f = 0$ for f a 0-form) we get

$$i_k \mathrm{d}\beta^x + \varepsilon^{xyz} i_k \omega^y\, \beta^z = \varepsilon^{xyz} i_k \Omega^y\, P_k^z. \tag{10.3.9}$$

We can introduce the Lie derivative in the first term since

$$i_k \mathrm{d}\beta^x = i_k\, \mathrm{d} i_k \Omega^x = i_k \mathcal{L}_k \Omega^x \tag{10.3.10}$$

again because $i_k^2 = 0$. The we use (10.3.5) to replace the Lie derivative

$$i_k \mathrm{d}\beta^x = \varepsilon^{xyz} W_k^y\, i_k \Omega^z = \varepsilon^{xyz} W_k^y\, i_k \beta^z. \tag{10.3.11}$$

Replacing $i_k \Omega^y = \lambda \beta^y$ in the last term and switching y and z, we finally find

$$\varepsilon^{xyz} (W_k^y + i_k \omega^y\, \beta^z + \lambda P_k^y) \beta^z = 0. \tag{10.3.12}$$

Under certain condition on $i_k \Omega^x$ [94, sec. 4] this implies

$$P_k^x = \frac{1}{\lambda} \Big(- i_k \omega^x - W_k^x \Big). \tag{10.3.13}$$

We deduce that any isometry is associated to a triplet of moment maps, and moreover we can rewrite (10.3.5) as [186, sec. 2]

$$\mathcal{L}_k \Omega^x = \varepsilon^{xyz} (i_k \omega^x - \lambda P_k^x) \Omega^z, \tag{10.3.14}$$

[1] With respect to [8, 72] we have $W \to -W$ since they define it by $\mathcal{L}_k \Omega^x = \varepsilon^{xyz} \Omega^y W_k^z$.

In terms of the triplet of complex structures this gives

$$\mathcal{L}_k \boldsymbol{J} = 2\lambda \, \boldsymbol{J} \times \boldsymbol{P}_k. \tag{10.3.15}$$

The statement (10.3.5) that a Killing vector is triholomorphic means that its covariant derivative commutes with all three complex structures (we omit the index k in the rest of the section)

$$\nabla_u k^w \, \boldsymbol{J}_w{}^v = \boldsymbol{J}_u{}^w \, \nabla_v k^w \tag{10.3.16}$$

In coordinates equation (10.3.6) reads

$$\lambda \, \nabla_u \boldsymbol{P}^x = k^v \, \boldsymbol{\Omega}_{uv}. \tag{10.3.17}$$

The moment map can also be found from

$$4\lambda n_h \, \boldsymbol{P} = \boldsymbol{J}_u{}^v \, \nabla_v k^u. \tag{10.3.18}$$

From Killing equation

$$\nabla_u k_v + \nabla_v k_u = 0, \tag{10.3.19}$$

using the commutator

$$[\nabla_u, \nabla_v] \, k^w = R_{uv}{}^w{}_s k^s \tag{10.3.20}$$

and the explicit value of the Ricci, one finds that k^u satisfy a Poisson equation [71, app. A]

$$\nabla_v \nabla^v k^u + 2\lambda (n_h + 2) k^u = 0. \tag{10.3.21}$$

Then using the relation with the prepotentials implies that the latter also satisfy a Poisson equation (but with different eigenvalues) The prepotentials are harmonic functions

$$\nabla_u \nabla^u P^x + 4 n_h \lambda \, P^x = 0. \tag{10.3.22}$$

Note that the commutator on P^x yields

$$[\nabla_u, \nabla_v] \, P^x = 2\varepsilon^{xyz} \Omega^y_{uv} P^z. \tag{10.3.23}$$

Then the Poisson equation can be used to find a direct expression for the Killing vector

$$k^u = -\frac{1}{6\lambda^2} \, h^{uv} \Omega^x_{vw} \nabla^w P^x. \tag{10.3.24}$$

Let's denote by $\{k_\Lambda\}$ the set of Killing vectors generating the isometries on $\mathcal{M}_h$ (we will use an index Λ as a shortcut for k_Λ in the compensator, etc.). Then one has the cocycle identity

$$\mathcal{L}_\Lambda W^x_\Sigma - \mathcal{L}_\Sigma W^x_\Lambda + \varepsilon^{xyz} W^y_\Lambda W^z_\Sigma = f_{\Lambda\Sigma}{}^\Xi W^x_\Xi \tag{10.3.25}$$

where $f_{\Lambda\Sigma}{}^\Xi$ are the structure constants of the algebra. There is also an equivariance condition

$$J^x_{uv} k^u_\Lambda k^v_\Sigma = \frac{1}{2} \, f_{\Lambda\Sigma}{}^\Omega P^x_\Omega + \frac{\lambda}{2} \, \varepsilon^{xyz} P^x_\Lambda P^y_\Sigma. \tag{10.3.26}$$

10.4 Classification of spaces

Homogeneous QK manifolds have been classified by Alekseevsky [2], but it was shown by de Wit and van Proeyen that it was incomplete [66, 181]. The symmetric manifolds (called Wolf spaces) were given by Wolf [176] (see also [16, 157]). Useful references include [8, p. 77, tab. 2, 91, p. 78, tab. 2, 90, p. 443, tab. 20.5].

The symmetric spaces that are special (i.e. which can be obtained from the c-map, see chapter 11) consist in two families

$$\frac{\mathrm{SU}(n_h, 2)}{\mathrm{SU}(n_h) \times \mathrm{SU}(2) \times \mathrm{U}(1)}, \qquad \frac{\mathrm{SO}(n_h, 4)}{\mathrm{SO}(n_h) \times \mathrm{SO}(4)}, \tag{10.4.1}$$

(when $n_h = 1$ the factor $\mathrm{SU}(n_h)$ is not present) given respectively by the quadratic and cubic models (section 8.1.1), and five exceptional cases

$$\begin{gathered} \frac{\mathrm{G}_{2,2}}{\mathrm{SO}(4) \times \mathrm{SO}(2)}, \quad \frac{\mathrm{F}_{4,4}}{\mathrm{USp}(6) \times \mathrm{SU}(2)}, \quad \frac{\mathrm{E}_{6,2}}{\mathrm{SU}(6) \times \mathrm{SU}(2)}, \\ \frac{\mathrm{E}_{7,-5}}{\mathrm{SO}(12) \times \mathrm{SU}(2)}, \quad \frac{\mathrm{E}_{8,-24}}{\mathrm{E}_7 \times \mathrm{SU}(2)} \end{gathered} \tag{10.4.2}$$

for $n_h = 7, 10, 16, 28$ respectively. The first of these exceptional spaces corresponds to the c-map with a cubic model since the spaces of the two families are isomorphic for $n_h = 2$ and it is given by a quadratic model [39, p. 5, tab. 2]. Note that $\mathrm{SU}(2) \subset \mathrm{SO}(4)$.

Finally the only symmetric spaces that cannot be obtained from the c-map are the projective quaternionic manifolds

$$\mathbb{H}P^{n_h} \equiv \frac{\mathrm{Sp}(n_h, 1)}{\mathrm{Sp}(n_h) \times \mathrm{Sp}(1)}, \tag{10.4.3}$$

and recall that $\mathrm{Sp}(1) \sim \mathrm{SU}(2)$.

Chapter 11

Special quaternionic manifolds

Special (or dual) quaternionic manifolds $\mathcal{M}_h$ are a subclass of quaternionic manifolds which fully specified by a special Kähler manifold $\mathcal{M}_z$ [80, 90, 181, 184]. The map $\mathcal{M}_z \to \mathcal{M}_h$ is called the c-map. The latter is useful for determining the isometries of the QK manifold; in particular if $\mathcal{M}_z$ is symmetric then $\mathcal{M}_h$ is also symmetric [180, pp. 222, 224].

11.1 Quaternionic metric from the c-map

We recall that $\dim \mathcal{M}_h = 4n_h$. A special quaternionic manifold is made of a base special Kähler manifold $\mathcal{M}_z$ of dimension $2(n_h - 1)$ with a fibration. Homogeneous coordinates on $\mathcal{M}_z$ are denoted by Z^A, and the fibers are $(\phi, \sigma, \xi^A, \tilde{\xi}_A)$ where

$$A = 0, \ldots, n_h - 1. \tag{11.1.1}$$

Physically ϕ is the dilaton (coming from the metric), σ is the axion (coming from dualization of the B-field) and the $(\xi^A, \tilde{\xi}_A)$ corresponds to the NS scalars (coming from the reduction of the NS forms). If $\mathcal{M}_z$ is symmetric then $\mathcal{M}_h$ is also symmetric [91, p. 23].

The explicit construction can be found in [80, 91, sec. 4].

Sometimes we will parametrize the dilaton as

$$\rho = \mathrm{e}^{-2\phi}. \tag{11.1.2}$$

The special coordinates are

$$z^a = \frac{Z^a}{Z^0}, \qquad a = 1, \ldots, n_h - 1. \tag{11.1.3}$$

Finally we group the Ramond coordinates into a symplectic vector

$$\xi = \begin{pmatrix} \xi^A \\ \tilde{\xi}_A \end{pmatrix} \tag{11.1.4}$$

Before describing the metric and other geometrical objects we set up the notation for the base special Kähler manifold.

11.2 Base special Kähler manifold

The properties of this embedded manifold are exactly the same as the ones described in chapter 6. In this section we are just recalling the main quantities and defining the notations: instead of curly letters $\mathcal{A}$ we will use blackboard bold letter $\mathbb{A}$.

The prepotential is denoted by G and its derivatives together with Z^A form the symplectic vector

$$Z = \begin{pmatrix} Z^A \\ G_A \end{pmatrix}. \tag{11.2.1}$$

The symplectic metric is $\mathbb{C}$.

We obtain the Kähler potential from

$$K_z = -\ln\big(-i\bar{Z}^t \mathbb{C} Z\big) = -\ln i\big(\bar{Z}^A G_A - Z^A \bar{G}_A\big) \tag{11.2.2}$$

from which we obtain the metric

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K_z. \tag{11.2.3}$$

We obtain the period matrix

$$\mathbb{N}_{AB} = \bar{G}_{AB} + 2i\, \frac{\operatorname{Im} G_{AC} \operatorname{Im} G_{BD}\; Z^C Z^D}{\operatorname{Im} G_{CD}\; Z^C Z^D}. \tag{11.2.4}$$

and the complex structure

$$\mathbb{M} = \begin{pmatrix} \operatorname{Im}\mathbb{N} + \operatorname{Re}\mathbb{N} (\operatorname{Im}\mathbb{N})^{-1} \operatorname{Re}\mathbb{N} & -\operatorname{Re}\mathbb{N}(\operatorname{Im}\mathbb{N})^{-1} \\ -(\operatorname{Im}\mathbb{N})^{-1}\operatorname{Re}\mathbb{N} & (\operatorname{Im}\mathbb{N})^{-1} \end{pmatrix}. \tag{11.2.5}$$

Cubic prepotentials will be written as

$$G = -d_{abc}\, \frac{Z^a Z^b Z^c}{Z^0}. \tag{11.2.6}$$

The associated manifolds are called very special quaternionic.

11.3 Geometrical structures

The metric $\mathcal{M}_h$ is given by

$$\mathrm{d}s_h^2 = \mathrm{d}\phi^2 + g_{a\bar{b}}\, \mathrm{d}z^a \mathrm{d}\bar{z}^{\bar{b}} + \frac{1}{4}\, \mathrm{e}^{4\phi} \left(\mathrm{d}\sigma + \frac{1}{2}\, \xi^t \mathbb{C} \mathrm{d}\xi\right)^2 - \frac{1}{4}\, \mathrm{e}^{2\phi}\, \mathrm{d}\xi^t \mathbb{M} \mathrm{d}\xi. \tag{11.3.1}$$

Note that the second term in parenthesis can be rewritten as

$$\xi^t \mathbb{C} \mathrm{d}\xi = \xi^A \mathrm{d}\tilde{\xi}_A - \tilde{\xi}_A \mathrm{d}\xi^A. \tag{11.3.2}$$

The spin connection ω_u^x is given[1] by [39, sec. 4.2, 104, sec. 3.1, 129, sec. 4]

$$\begin{aligned} \omega^+ &= \sqrt{2}\, \mathrm{e}^{\phi + K_z/2}\, Z^t \mathbb{C} \mathrm{d}\xi, \\ \omega^3 &= \frac{\mathrm{e}^{2\phi}}{2} \left(\mathrm{d}a + \frac{1}{2}\, \xi^t \mathbb{C} \mathrm{d}\xi\right) - 2\, \mathrm{e}^{K_z} \operatorname{Im}\left(Z^A \operatorname{Im} G_{AB} \mathrm{d}\bar{Z}^B\right). \end{aligned} \tag{11.3.3}$$

where we defined

$$\omega^\pm = \omega^1 \pm i\, \omega^2 \tag{11.3.4}$$

which are complex conjugate. These expressions are not invariant under SU(2) transformations.

We can also rewrite [78, app. B]

$$\operatorname{Im}\left(Z^A \operatorname{Im} G_{AB} \mathrm{d}\bar{Z}^B\right) = \frac{1}{4}\, Z \mathbb{C} \mathrm{d}\bar{Z} + \text{c.c.} \tag{11.3.5}$$

[1] Note that this involves a choice of SU(2) basis. Other possibilities are also fine.

since

$$\operatorname{Im}\left(Z^A \operatorname{Im} G_{AB} \mathrm{d}\bar{Z}^B\right) = \operatorname{Im}\left(\frac{1}{2i}\, Z^A (G_{AB} - \bar{G}_{AB}) \mathrm{d}\bar{Z}^B\right) = -\frac{1}{2} \operatorname{Re}\left(G_A \mathrm{d}\bar{Z}^A - Z^A \mathrm{d}\bar{G}_A\right)$$
$$= -\frac{1}{4}\left(G_A \mathrm{d}\bar{Z}^A - Z^A \mathrm{d}\bar{Z}_A + \bar{G}_A \mathrm{d}Z^A - \bar{Z}^A \mathrm{d}G_A\right)$$

where we used the homogeneity of G (6.2.31)

$$G_{AB} Z^B = G_A, \qquad G_{AB} \mathrm{d}Z^B = \mathrm{d}G_A. \tag{11.3.6}$$

Chapter 12

Quaternionic isometries

In this chapter we focus on the isometries of special quaternionic manifolds. As reviewed in the chapter 9 on SK isometries, knowing the Killing vectors of the target space of the non-linear sigma models involved in the $N = 2$ supergravity is necessary in order to write the gauged theory. Since there is a base SK space we are able to use symplectic covariant expressions which simplify the construction of the Killing vectors and which provide a nice interpretation of them.

The isometries of special quaternionic manifolds were classified by de Wit and Van Proeyen [180, 182–184]. There are three kinds of isometries [78, 184]:

- duality symmetries, inherited from the base special Kähler manifolds;
- extra symmetries, whose origin is seen directly from the gauge transformations;
- hidden symmetries, which are not generic and whose existence depends on specific properties of the manifold.

12.1 Killing vectors

We will denote the isometry group by

$$G_h = \mathrm{ISO}(\mathcal{M}_h). \tag{12.1.1}$$

In order to simplify the notation, we define

$$\partial_\xi = \begin{pmatrix} \partial_A \\ \partial^A \end{pmatrix}, \qquad \partial_A = \partial_{\xi^A} = \frac{\partial}{\partial \xi^A}, \qquad \partial^A = \partial_{\tilde{\xi}_A} = \frac{\partial}{\partial \tilde{\xi}_A}. \tag{12.1.2}$$

We will also make use of

$$\mathbb{C}\partial_\xi = \begin{pmatrix} \partial^A \\ -\partial_A \end{pmatrix}. \tag{12.1.3}$$

Similarly we write

$$\partial_Z = \begin{pmatrix} \partial_{Z^A} \\ \partial_{G_A} \end{pmatrix}, \qquad \partial_{Z^A} = \frac{\partial}{\partial Z^A}, \qquad \partial_{G_A} = \frac{\partial}{\partial G_A}. \tag{12.1.4}$$

12.1.1 Duality symmetries

Isometries of the base SK space (described in section 9) can be lifted to the full quaternionic space by adding a transformation of the fibers [180, p. 223]. They consist in symplectic (infinitesimal) transformations $\mathbb{U} \in \mathfrak{sp}(2n_H, \mathbb{R})$ that leave invariant the prepotential. Since the metric is made only of symplectic products, it is easy to see that the Killing vector on the full space is [39, sec. 4.2]

$$k_{\mathbb{U}} = (\mathbb{U}Z)^t\, \partial_Z + (\mathbb{U}\bar{Z})^t\, \partial_{\bar{Z}} + (\mathbb{U}\xi)^t \partial_\xi. \tag{12.1.5}$$

Writing explicitly the product gives

$$k_{\mathbb{U}} = (\mathbb{U}Z)^A \partial_{Z^A} + (\mathbb{U}Z)_A \partial_{G_A} + (\mathbb{U}\xi)^A \partial_A + (\mathbb{U}\xi)_A \partial^A + \text{c.c.} \tag{12.1.6}$$

In order to use conventions similar to the other Killing vectors we should write this vector as a linear combination of each Killing vector associated to independent parameters, but this is not the usual approach taken in the literature.

The matrix $\mathbb{U}$ is parametrized by (see section 9)

$$\mathbb{U} = \begin{pmatrix} v^A{}_B & t^{AB} \\ s_{AB} & u_A{}^B \end{pmatrix}, \qquad t^{AB} = t^{BA}, \quad s_{AB} = s_{BA}, \quad v^A{}_B = -u_B{}^A \tag{12.1.7}$$

where the constraint are equivalent to

$$\mathbb{U}^t \mathbb{C} + \mathbb{C}\mathbb{U} = 0. \tag{12.1.8}$$

We refer to section 9 for more details on the classification of duality isometries. Since the parameters are subject to the constraints not all these symmetries are universal.

12.1.2 Extra symmetries

These symmetries act on the Heisenberg fiber: they originate from the gauge symmetry of gauge fields that have been dualized to scalar fields [180, p. 223]. Only the derivative of the scalar fields that have been dualized from vector fields appear, and shift symmetries result from this.

The first symmetry is a translation of the axion [39, sec. 4.2]

$$k_+ = \partial_\sigma. \tag{12.1.9}$$

In general nothing depends on the axion and everything is invariant under shift of this field.

Then there is a scaling symmetry of all the fields

$$k_0 = \partial_\phi - 2\sigma\partial_\sigma - \xi^t \partial_\xi. \tag{12.1.10}$$

Expanding the product gives explicitly

$$k_0 = \partial_\phi - 2\sigma\partial_\sigma - \tilde{\xi}_A \partial^A - \xi^A \partial_A. \tag{12.1.11}$$

Finally there are $2n_h$ translations of the Ramond fields ξ accompanied by a transformation of σ [39, sec. 4.2] (this is really a $2n_h$-dimensional vector)

$$k_\xi = \mathbb{C}\partial_\xi + \frac{1}{2}\,\xi\,\partial_\sigma \tag{12.1.12}$$

or more explicitly[1]

$$k^A = \partial^A + \frac{1}{2}\,\xi^A \partial_\sigma, \tag{12.1.13a}$$

$$k_A = -\partial_A + \frac{1}{2}\,\tilde{\xi}_A \partial_\sigma. \tag{12.1.13b}$$

The shift of the fibers can be written

$$k_\xi = \mathbb{C}\partial_\xi + \frac{1}{2}\,\xi\,\partial_\sigma. \tag{12.1.14}$$

All these symmetries are universal and do not depend on the model.

12.1.3 Hidden vectors

There are several hidden symmetries [180, 184, 182, sec. 3]. In [78, sec. 4] these vectors have been expressed in a symplectic covariant form.[2]

Since the quaternionic metric does not contain linear term in $\mathrm{d}z^a$, any isometry of the full space needs to be an isometry of the base SK space when the vector is restricted to the latter

$$\mathcal{L}_k h_{uv} = 0 \Longrightarrow \mathcal{L}_k|_{\mathcal{M}_z} g_{a\bar{b}} = 0. \tag{12.1.15}$$

In particular this implies that the transformation of the homogeneous SK coordinates are of the form

$$\delta Z = \mathbb{S} Z \tag{12.1.16}$$

where $\mathbb{S} \in \mathfrak{sp}(2n_h)$ and it satisfies the equivalent of (9.1.10). In particular this matrix can depend on all the fields of the fiber

$$\mathbb{S} = \mathbb{S}(\phi, \sigma, \xi^A, \tilde{\xi}_A) \tag{12.1.17}$$

as they are just constant from the point of view of the base SK space, but it appears that $\mathbb{S}$ depends only on ξ.

The first vector is given by

$$k_- = -\sigma\,\partial_\phi + (\sigma^2 - \mathrm{e}^{-4\phi} - W)\partial_\sigma + (\sigma\xi - \mathbb{C}\partial_\xi W)^t \partial_\xi - (\mathbb{S}Z)^t \partial_Z + \text{c.c.} \tag{12.1.18a}$$

Then there are $2n_h$ vectors

$$\begin{aligned}\widehat{k}_\xi = -\frac{1}{2}\,\xi\,\partial_\phi + \left(\frac{\sigma}{2}\,\xi - \frac{1}{2}\,\mathbb{C}\partial_\xi W\right)\partial_\sigma + \sigma\,\mathbb{C}\partial_\xi + \left(\frac{1}{2}\,\xi^t\xi - \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\right)\partial_\xi \\ - (\mathbb{C}\partial_\xi \mathbb{S}\, Z)^t \partial_Z + \text{c.c.}\end{aligned} \tag{12.1.18b}$$

Explicitly they are given by

$$\begin{aligned}\widehat{k}^A = -\frac{1}{2}\,\xi^A\,\partial_\phi + \left(\frac{\sigma}{2}\,\xi^A - \frac{1}{2}\,\partial^A W\right)\partial_\sigma + \sigma\,\partial^A + \left(\frac{1}{2}\,\xi^A\,\xi - \mathbb{C}\partial_\xi\partial^A W\right)^t \partial_\xi \\ - (\partial^A \mathbb{S}\, Z)^t \partial_Z + \text{c.c.}\end{aligned} \tag{12.1.18c}$$

$$\begin{aligned}\widehat{k}_A = -\frac{1}{2}\,\tilde{\xi}_A\,\partial_\phi + \left(\frac{\sigma}{2}\,\tilde{\xi}_A + \frac{1}{2}\,\partial_A W\right)\partial_\sigma - \sigma\,\partial_A + \left(\frac{1}{2}\,\tilde{\xi}_A\,\xi + \mathbb{C}\partial_\xi\partial_A W\right)^t \partial_\xi \\ + (\partial_A \mathbb{S}\, Z)^t \partial_Z + \text{c.c.}\end{aligned} \tag{12.1.18d}$$

[1] Note that k_A gets a minus sign with respect to the definition in [39, sec. 4.2].
[2] Also this paper provides corrections to the expression from [184] that were incorrect.

We have used several quantities

$$W = \frac{1}{4}\, h(\xi) - \frac{1}{2}\, \mathrm{e}^{-2\phi}\, \xi^t \mathbb{C}\mathbb{M}\xi, \tag{12.1.19a}$$

$$\mathbb{S} = \frac{1}{2} \left(\xi\xi^t + \frac{1}{2}\, H \right) \mathbb{C}, \tag{12.1.19b}$$

$$H = \mathbb{C}\partial_\xi (\mathbb{C}\partial_\xi h)^t = \begin{pmatrix} \partial^A\partial^B h & -\partial^A\partial_B h \\ -\partial_A\partial^B h & \partial_A\partial_B h \end{pmatrix}. \tag{12.1.19c}$$

h is a homogeneous quartic polynomial constructed from the quartic invariant [86, sec. 4], while $\mathbb{S}$ is a symplectic matrix

$$\mathbb{S}^t = \mathbb{C}\mathbb{S}\mathbb{C}. \tag{12.1.20}$$

H is a symmetric matrix.

Some of the quantities involved are homogeneous in ξ:

- h: order 4;
- $\mathbb{S}, H$: order 2.

This means that

$$\xi^t \partial_\xi h = 4h, \qquad \xi^t \partial_\xi H = 2H, \qquad \xi^t \partial_\xi \mathbb{S} = 2\mathbb{S}. \tag{12.1.21}$$

When the space is symmetric the quartic invariant h is independent of the fields z^i [184, pp. 13, 17]. In particular it is possible to obtain conditions by taking derivatives. If h depends on z^i then some symmetries of $\mathfrak{g}_{-1/2}$ can still exist if some linear combinations of $\partial_A h$ and $\partial^A h$ are independent of z^i. For this last reason it may be interesting to keep parameters in Killing vectors since the Killing vectors $\widehat{k}^A$ and $\widehat{k}_A$ may not exist by themselves, but only linear combinations.

Some interesting results on possible hidden vectors are proved in [184, sec. 4.3] for $\mathcal{M}_z$ with cubic prepotential. For example $\widehat{\alpha}_0$ always exists, whereas $\widehat{\alpha}^0$ exists only for symmetric spaces, and the others exist if

$$E^a{}_{bcde}\, \widehat{\alpha}^e = 0, \qquad E^a{}_{bcde}\, \widehat{\alpha}_a = 0. \tag{12.1.22}$$

Note that the second constraint coincides with the one for the existence of a_a, such that if the later exist, then there also exist symmetries such that $\widehat{\alpha}_a \propto a_a$.

Cubic prepotential

For cubic prepotential the quartic invariant is given by (8.3.14)

$$h(\xi, \tilde{\xi}) = -(\tilde{\xi}_A \xi^A)^2 + \frac{1}{16}\, \xi^0\, \widehat{d}^{abc} \tilde{\xi}_a \tilde{\xi}_b \tilde{\xi}_c - 4\, \tilde{\xi}_0\, d_{abc} \xi^a \xi^b \xi^c + \frac{9}{16}\, \widehat{d}^{abc} d_{cde} \tilde{\xi}_a \tilde{\xi}_b\, \xi^d \xi^e. \tag{12.1.23}$$

The parameters of the matrix $\mathbb{S}$ as written in section 9.2 are

$$\beta = -\frac{1}{2} \left(3\, \tilde{\xi}_0 \xi^0 + \tilde{\xi}_a \xi^a \right), \tag{12.1.24a}$$

$$b^a = -\frac{1}{2} \left(2\, \tilde{\xi}_0 \xi^a - \frac{3}{32}\, \widehat{d}^{abc} \tilde{\xi}_b \tilde{\xi}_c \right), \tag{12.1.24b}$$

$$a_a = -\frac{1}{2} \left(2\, \xi^0 \xi_a + 6\, d_{abc} \xi^b \xi^c \right), \tag{12.1.24c}$$

$$B^a{}_b = -\frac{1}{2} \left(\frac{2}{3}\, \delta^a{}_b \tilde{\xi}_c \xi^c - \frac{9}{8}\, \widehat{d}^{acd} d_{bde} \tilde{\xi}_d \xi^e \right). \tag{12.1.24d}$$

Quadratic prepotential

For quadratic prepotential the quartic invariant is given by (8.2.12) and (8.2.17)

$$h(\xi,\tilde{\xi}) = I_2(\xi,\tilde{\xi})^2, \qquad I_2(\xi,\tilde{\xi}) = \frac{i}{2}\,\xi^A \eta_{AB}\,\xi^B + \frac{i}{2}\,\tilde{\xi}_A \eta^{AB} \tilde{\xi}_B. \tag{12.1.25}$$

The parameters of the matrix $\mathbb{S}$ as written in section 9.3 are

$$r^{AB} = -\frac{1}{2}\,\left(-\xi^A \xi^B - i\, I_2(\xi,\tilde{\xi})\,\eta^{AB} + (\eta^{-1}\tilde{\xi})^A (\eta^{-1}\tilde{\xi})^B\right), \tag{12.1.26a}$$

$$s_{AB} = -\frac{1}{2}\,\left(\tilde{\xi}^A \tilde{\xi}^B + i\, I_2(\xi,\tilde{\xi})\,\eta_{AB} - (\eta\xi)_A (\eta\xi)_B\right), \tag{12.1.26b}$$

$$q^A{}_B = -\frac{1}{2}\,\left(-\xi^A \tilde{\xi}_B - (\eta^{-1}\tilde{\xi})^A (\eta\xi)_B\right). \tag{12.1.26c}$$

Some relations

For later computations we look at various expressions involving the previous objects.

The ϕ derivative of W is equal to

$$\partial_\phi W = \,\mathrm{e}^{-2\phi}\,\xi^t \mathbb{C}\mathbb{M}\xi. \tag{12.1.27}$$

W is not homogeneous (since it has quadratic and quartic pieces) but using the last equation we have

$$(\xi^t \partial_\xi - \partial_\phi) W = 4W, \tag{12.1.28}$$

or written in various other ways

$$\xi^t \partial_\xi W = 2W + \frac{1}{2}\,h = 4W + \,\mathrm{e}^{-2\phi}\,\xi^t \mathbb{C}\mathbb{M}\xi = 4W + \partial_\phi W. \tag{12.1.29}$$

Similarly for the derivative of W we get

$$(\xi^t \partial_\xi - \partial_\phi)\partial_\xi W = 3\,\partial_\xi W \tag{12.1.30}$$

or differently

$$(\xi^t \partial_\xi)\partial_\xi W = W + \frac{1}{2}\,\partial_\xi h = 3\partial_\xi W + \frac{1}{2}\,\mathrm{e}^{-2\phi}\,\partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi) = 3\partial_\xi W + \,\mathrm{e}^{-2\phi}\,\mathbb{C}\mathbb{M}\xi \tag{12.1.31}$$

using the relation (12.1.32) proved below.

The derivative with respect to ξ of the second term in W reads

$$\mathrm{e}^{2\phi}\,\partial_\xi(\partial_\phi W) = \partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi) = 2\mathbb{C}\mathbb{M}\xi \tag{12.1.32}$$

since

$$\partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi) = \mathbb{C}\mathbb{M}\xi + \xi^t \mathbb{C}\mathbb{M} = \mathbb{C}\mathbb{M}\xi - \mathbb{M}^t \mathbb{C}\xi = 2\mathbb{C}\mathbb{M}\xi.$$

Equivalently

$$(\mathbb{C}\partial_\xi)(\xi^t \mathbb{C}\mathbb{M}\xi) = -2\mathbb{M}\xi. \tag{12.1.33}$$

Taking the derivative a second time gives

$$\partial_\xi[\partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi)]^t = 2\mathbb{C}\mathbb{M}, \qquad \mathbb{C}\partial_\xi[\mathbb{C}\partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi)]^t = -2\mathbb{C}\mathbb{M}, \tag{12.1.34}$$

On the other hand we defined

$$H = \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h)^t \tag{12.1.35}$$

so we get that

$$\mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t = H - 2\,\mathrm{e}^{-2\phi}\mathbb{C}\mathbb{M} = -2\xi\xi^t - 4\mathbb{S}\mathbb{C} - 2\,\mathrm{e}^{-2\phi}\mathbb{C}\mathbb{M}. \tag{12.1.36}$$

12.1.4 Summary

As a summary, the list of all the Killing vectors is

$$k_{\mathbb{U}} = (\mathbb{U}Z)^t\,\partial_Z + (\mathbb{U}\bar{Z})^t\,\partial_{\bar{Z}} + (\mathbb{U}\xi)^t\partial_\xi, \tag{12.1.37a}$$

$$k_\xi = \mathbb{C}\partial_\xi + \frac{1}{2}\,\xi\,\partial_\sigma, \tag{12.1.37b}$$

$$k_0 = \partial_\phi - 2\sigma\,\partial_\sigma - \xi^t\partial_\xi, \tag{12.1.37c}$$

$$k_+ = \partial_\sigma, \tag{12.1.37d}$$

for the normal symmetries and

$$k_- = -\sigma\partial_\phi + (\sigma^2 - \mathrm{e}^{-4\phi} - W)\partial_\sigma + (\sigma\xi - \mathbb{C}\partial_\xi W)^t\partial_\xi - (\mathbb{S}Z)^t\partial_Z + \text{c.c.}, \tag{12.1.37e}$$

$$\widehat{k}_\xi = -\frac{1}{2}\,\xi\,\partial_\phi + \left(\frac{\sigma}{2}\,\xi - \frac{1}{2}\,\mathbb{C}\partial_\xi W\right)\partial_\sigma + \sigma\,\mathbb{C}\partial_\xi + \left(\frac{1}{2}\,\xi\xi^t - \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\right)\partial_\xi \\ - (\mathbb{C}\partial_\xi\mathbb{S}\,Z)^t\partial_Z + \text{c.c.} \tag{12.1.37f}$$

for the hidden symmetries.

We have used several quantities

$$W = \frac{1}{4}\,h(\xi^A, \tilde{\xi}_A) - \frac{1}{2}\,\mathrm{e}^{-2\phi}\,\xi^t\mathbb{C}\mathbb{M}\xi, \tag{12.1.38a}$$

$$\mathbb{S} = \frac{1}{2}\left(\xi\xi^t + \frac{1}{2}\,H\right)\mathbb{C}, \tag{12.1.38b}$$

$$H = \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h)^t = \begin{pmatrix} \partial^A\partial^B h & -\partial^A\partial_B h \\ -\partial_A\partial^B h & \partial_A\partial_B h \end{pmatrix}. \tag{12.1.38c}$$

12.2 Algebra

We define the commutator of two vectors of Killing vectors k_1 and k_2 as

$$\left[k_1, k_2^t\right] = k_1 k_2^t - (k_1 k_2^t)^t. \tag{12.2.1}$$

Another possibility is to introduce one parameter for each Killing vector which turns the previous matrix commutator into a normal scalar commutator

$$\left[\epsilon_1^t k_1, \epsilon_2^t k_2\right] = \epsilon_1^t k_1 k_2^t \epsilon_2 - \epsilon_2^t (k_1 k_2^t)^t \epsilon_1 \tag{12.2.2}$$

and specific commutators can be extracted by taking all parameters to zeros except those we are interested in which are set to one.[3]

The non-vanishing commutators of the algebra are [78, sec. 4.3, 184, sec. 3]

$$\begin{gathered}
[k_0, k_+] = 2k_+, \qquad [k_0, k_\xi] = k_\xi, \qquad \left[k_\xi, k_\xi^t\right] = \mathbb{C}\,k_+, \qquad [k_{\mathbb{U}}, k_\xi] = \mathbb{U}\,k_\xi, \\
[k_0, k_-] = -2\,k_-, \qquad \left[k_0, \widehat{k}_\xi\right] = -\widehat{k}_\xi, \qquad [k_-, k_\xi] = -\widehat{k}_\xi, \\
[k_+, k_-] = -k_0, \qquad \left[k_+, \widehat{k}_\xi\right] = k_\xi, \qquad \left[k_{\mathbb{U}}, \widehat{k}_\xi\right] = \mathbb{U}\,\widehat{k}_\xi, \\
\left[\widehat{k}_\xi, \widehat{k}_\xi^t\right] = \mathbb{C}\,k_-, \qquad \left[\widehat{\alpha}^t\widehat{k}_\xi, \alpha^t k_\xi\right] = \frac{1}{2}\,\widehat{\alpha}\mathbb{C}\alpha\,k_0 + k_{\mathbb{T}_{\alpha,\hat{\alpha}}}
\end{gathered} \tag{12.2.3}$$

[3] The same idea is used for supersymmetry where ϵQ can be used to turn anticommutators into commutators.

with

$$\mathbb{T}_{\alpha,\hat{\alpha}} = (\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)\,\mathbb{S} = -\frac{1}{2}\,\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t) + \frac{1}{4}\,H''_{\alpha,\hat{\alpha}}\mathbb{C}, \tag{12.2.4a}$$

$$H''_{\alpha,\hat{\alpha}} = \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}})^t = (\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)H, \tag{12.2.4b}$$

$$h''_{\alpha,\hat{\alpha}} = (\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)h. \tag{12.2.4c}$$

Some commutators are computed in appendix E.1, others have been checked with Mathematica.

We see that there are two Heisenberg subalgebra, one generated by $\{k_\xi, k_+\}$ [39, sec. 4], the other by $\{\widehat{k}_\xi, k_-\}$.

The algebra $\mathfrak{g}_h$ corresponding to these Killing vectors can be decomposed into eigenspaces of k_0 [180, pp. 222–223, 184, sec. 2.3]

$$\mathfrak{g}_h = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1 \tag{12.2.5}$$

where the Killing vectors contained in $\mathfrak{g}_a$ satisfy

$$[k_0, \mathfrak{g}_a] = a\,\mathfrak{g}_a. \tag{12.2.6}$$

We note that the dimensions of extra symmetry subspaces are

$$\dim \mathfrak{g}_{1/2} = 2n_h, \qquad \dim \mathfrak{g}_1 = 1 \tag{12.2.7}$$

while for hidden symmetries the dimensions are

$$\text{symmetric } \mathcal{M}_h\text{:} \qquad \dim \mathfrak{g}_{-1} = 1, \qquad \dim \mathfrak{g}_{-1/2} = 2n_h, \tag{12.2.8a}$$

$$\text{otherwise:} \qquad \dim \mathfrak{g}_{-1} = 0, \qquad \dim \mathfrak{g}_{-1/2} \leq n_h. \tag{12.2.8b}$$

Note that the algebra of $\mathcal{M}_z$ is contained in $\mathfrak{g}_0$. As a conclusion very special quaternionic manifolds have at least $2n_h + 2$ isometries (k_0, k_ξ and k_+) [80]

$$\dim \mathfrak{g} \geq 2n_h + 2. \tag{12.2.9}$$

Using the algebra we can obtain some information about the number of symmetries that will be realized. For example if for a given A the symmetries $\widehat{k}_A$ and $\widehat{k}^A$ exist, then from the algebra we deduce that k_- exists also and the space is symmetric [180, p. 228]. Similarly the bound on the dimension of $\mathfrak{g}_{-1/2}$ is obtained from the commutators with $k_{\mathbb{U}}$, so if we have one symmetry of this subspace we can build other by taking the commutator.

Projective quaternionic space

$$\mathcal{M}_h = \frac{\mathrm{Sp}(n_h, 1)}{\mathrm{Sp}(n_h) \times \mathrm{Sp}(1)} \tag{12.2.10}$$

are associated to the algebra C^1_1 are not in the image of the c-map since

$$\dim \mathfrak{g}_1 = 3 \tag{12.2.11}$$

which is in contradiction with what we have seen above [184, p. 12].

12.3 Compensators

The expressions for the compensators are not invariant under SU(2) transformations, and they depend on the choice of the spin connection.

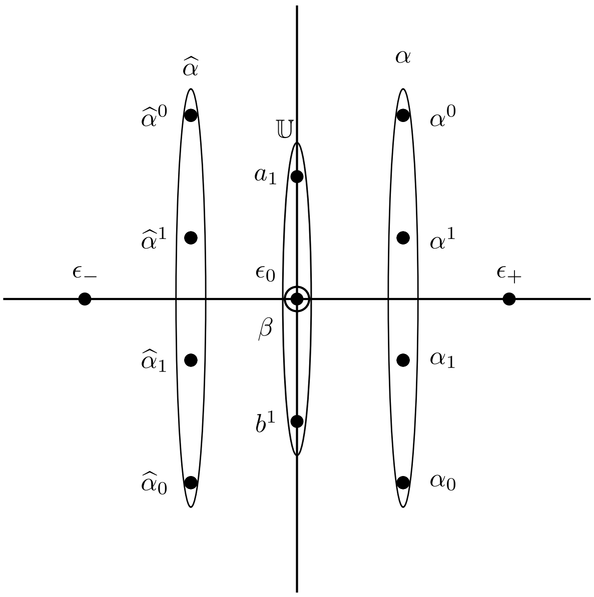

Figure 12.1: G_2 root diagram [184, sec. 2.3], see [64, sec. 3.1] for the construction. This corresponds to $n_h = 2$, and in this case $B^1{}_1 = 0$.

We recall that the compensators are defined by

$$\mathcal{L}_k \omega^+ = \mathrm{d}W_k^+ - i\,\omega^+ W_k^3 + i\,\omega^3 W_k^-, \tag{12.3.1}$$

and also

$$\omega^+ = \sqrt{2}\,\mathrm{e}^{\phi + K_z/2}\, Z^t \mathbb{C} \mathrm{d}\xi. \tag{12.3.2}$$

In homogeneous coordinates, ω^x_u is explicitly invariant and the compensator vanishes

$$W^x = 0. \tag{12.3.3}$$

Then for getting their expressions one needs to compute the Lie derivative in special coordinates.

12.3.1 Duality symmetries

Cubic prepotential

The only non-zero compensator is [78, sec. 5.1.1, app. B.3.1]

$$W^3_{\mathbb{U}} = a_c \operatorname{Im} z^c. \tag{12.3.4}$$

from

$$\mathcal{L}_{\mathbb{U}} \omega^+ = -i\, a_c \operatorname{Im} z^c\, \omega^+. \tag{12.3.5}$$

Quadratic prepotential

The only non-zero compensator is [78, sec. 5.1.1]

$$W^3_{\mathbb{U}} = \operatorname{Im}(A^a{}_0 z^a) = q^a{}_0 \operatorname{Im} z^a + r^{a0} \operatorname{Re} z^a \tag{12.3.6}$$

from

$$\mathcal{L}_{\mathbb{U}} \omega^+ = -i \left(q^a{}_0 \operatorname{Im} z^a + r^{a0} \operatorname{Re} z^a\right) \omega^+. \tag{12.3.7}$$

12.3.2 Hidden symmetries

Compensators for hidden symmetries are [78, sec. 5.1.2, app. B.3.2]

$$W_-^+ = 2i\sqrt{2}\,\mathrm{e}^{K_z-\phi}\,Z\mathbb{C}\xi, \tag{12.3.8a}$$
$$W_-^3 = -W_{\mathbb{S}}^3 - \mathrm{e}^{-2\phi}, \tag{12.3.8b}$$
$$\widehat{W}_\xi^+ = -\mathbb{C}\partial_\xi W_-^+, \tag{12.3.8c}$$
$$\widehat{W}_\xi^3 = -2\,\mathbb{C}\partial_\xi W_-^3. \tag{12.3.8d}$$

12.4 Prepotentials

The expressions for the prepotentials are not invariant under SU(2) transformations, and they depend on the choice of the spin connection.

We recall that Killing prepotentials are given by

$$P_\Lambda^x = k_\Lambda^u \omega_u^x - W_\Lambda^x \tag{12.4.1}$$

and they are real. We will sometimes use

$$P^\pm = P^1 \pm i\,P^2. \tag{12.4.2}$$

The prepotentials for the universal symmetries are

$$P_+^+ = 0, \qquad P_+^3 = \frac{1}{2}\,\mathrm{e}^{2\phi}, \tag{12.4.3a}$$
$$P_0^+ = \sqrt{2}\,\mathrm{e}^{K_z/2+\phi}\,Z\mathbb{C}\xi, \qquad P_0^3 = -\sigma\,\mathrm{e}^{2\phi}, \tag{12.4.3b}$$
$$P_\xi^+ = \sqrt{2}\,\mathrm{e}^{K_z/2+\phi}\,Z, \qquad P_\xi^3 = \frac{1}{2}\,\mathrm{e}^{2\phi}\,Z, \tag{12.4.3c}$$

while those for the base SK isometries are

$$P_{\mathbb{U}}^+ = \frac{1}{2}\,\mathrm{e}^{K_z/2+\phi}\,\xi\mathbb{C}\mathbb{U}Z, \qquad P_{\mathbb{U}}^3 = \frac{1}{4}\,\mathrm{e}^{2\phi}\,\xi\mathbb{C}\mathbb{U}\xi + \frac{1}{2}\,\mathrm{e}^{K_z}\,Z\mathbb{C}\mathbb{U}\bar{Z}, \tag{12.4.3d}$$

and those for the hidden isometries are

$$P_-^+ = -\frac{1}{2}\,\mathrm{e}^{-2\phi} + \frac{\sigma^2}{2}\,\mathrm{e}^{2\phi}(2W - \xi\partial_\xi W) - \frac{1}{2}\,\mathrm{e}^{K_z}\,\bar{Z}\mathbb{C}\mathbb{S}Z, \tag{12.4.3e}$$
$$P_-^3 = \sqrt{2}\,\mathrm{e}^{K_z/2+\phi}(\sigma\,Z\mathbb{C}\xi + Z\partial_\xi W), \tag{12.4.3f}$$
$$\widehat{P}_\xi^+ = \frac{1}{\sqrt{2}}\,\mathrm{e}^{K_z/2+\phi}\,(Z\mathbb{C}\xi)\,\xi + \mathbb{C}\partial_\xi P_-^+, \tag{12.4.3g}$$
$$\widehat{P}_\xi^3 = \frac{\sigma}{2}\,\mathrm{e}^{2\phi}\,\xi + \mathbb{C}\partial_\xi P_-^3. \tag{12.4.3h}$$

Part V

BPS equations for black holes

Chapter 13

Generalities on AdS–NUT black holes

13.1 Ansatz

In this section we consider asymptotically adS and adS–NUT black holes. The goal is to provide an overview of the structure of these solutions [79].

We take the following ansatz for the metric and the gauge fields

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2U}\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big)^2 + \mathrm{e}^{-2U}\mathrm{d}r^2 + \mathrm{e}^{2(V-U)}\,\mathrm{d}\Sigma_g^2, \tag{13.1.1a}$$

$$A^\Lambda = \tilde{q}^\Lambda\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big) + \tilde{p}^\Lambda\,H(\theta)\,\mathrm{d}\phi. \tag{13.1.1b}$$

The functions $U, V, \tilde{q}$ and $\tilde{p}$ depend only on r, and n is the NUT charge. The space Σ_g is defined in section A.7

$$\mathrm{d}\Sigma_g^2 = \mathrm{d}\theta^2 + H'(\theta)^2\,\mathrm{d}\phi^2, \qquad H(\theta) = \begin{cases} -\cos\theta & \kappa = 1, \\ \theta & \kappa = 0, \\ \cosh\theta & \kappa = -1. \end{cases} \tag{13.1.2}$$

We mainly work with $\kappa = \pm 1$, but one can check that key equations are also valid for $\kappa = 0$, possibly with a rescaling of the Maxwell and NUT charges.

13.2 Motivation: constant scalar black holes

13.2.1 Solution

In order to motivate our general analysis let us start with the adS–NUT charged black hole in Einstein–Maxwell theory with a cosmological constant $\Lambda = -3g^2$, which corresponds to minimal gauged supergravity with coupling g ($n_v = n_h = 0$), following [79, sec. 2].

The metric and the gauge field read [4]

$$\mathrm{d}s^2 = -\frac{\mathrm{e}^{2V}}{r^2 + n^2}\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big)^2 + \frac{r^2 + n^2}{\mathrm{e}^{2V}}\,\mathrm{d}r^2 + (r^2 + n^2)\,\mathrm{d}\Sigma_g^2, \tag{13.2.1a}$$

$$A = \frac{Qr - nP}{r^2 + n^2}\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big) + P\,H(\theta)\,\mathrm{d}\phi. \tag{13.2.1b}$$

using the functions

$$\mathrm{e}^{2V} = g^2(r^2+n^2)^2 + (\kappa + 4g^2n^2)(r^2-n^2) - 2mr + P^2 + Q^2, \tag{13.2.2a}$$

$$\mathrm{e}^{2(V-U)} = r^2 + n^2, \tag{13.2.2b}$$

$$\tilde{q} = \frac{Qr - nP}{r^2+n^2}, \tag{13.2.2c}$$

$$\tilde{p} = P. \tag{13.2.2d}$$

The ϕ-component of the gauge field reads

$$A_\phi = \frac{P(r^2-n^2) + 2nQr}{r^2+n^2}\, H(\theta). \tag{13.2.3}$$

The parameters P and Q are the magnetic and electric charges, and m is the mass. The ADM mass and charges depend on the genus of the surface [36, p. 5].

It it well-known that Taub–NUT spacetimes have closed timelike curve (which are present in order to avoid Misner strings), and the periodicity is related to the NUT charge [145, 148, chap. 9]. The only exception to the previous statement is for $\kappa = -1$ where there is a range for n where the solution is free of closed timelike curves [15]

$$0 \le 2g^2n^2 \le 1. \tag{13.2.4}$$

When the NUT charge is set to zero the solution corresponds to the adS Reissner–Nordström.

13.2.2 Root structure and supersymmetry

The supersymmetric properties of adS black holes ($n = 0$) were first studied by Romans in its seminal paper [154]. He found two classes of BPS solutions

$$\frac{1}{2}\text{-BPS}: \qquad m = |Q|, \qquad P = 0, \tag{13.2.5a}$$

$$\frac{1}{4}\text{-BPS}: \qquad m = 0, \qquad P = \pm\frac{1}{2g}, \tag{13.2.5b}$$

and only Q is not constrained. The 1/2-BPS solution has a naked singularity for any κ, while the 1/4-BPS solution also has a naked singularity, except for $\kappa = -1$ and $Q = 0$, in which case it has a horizon $\mathrm{adS}_2 \times H^2$.

This has been generalized in [4] which found again two classes

$$\frac{1}{2}\text{-BPS}: \qquad m = |Q|\sqrt{\kappa + 4g^2n^2}, \qquad P = \pm n\sqrt{\kappa + 4g^2n^2}, \tag{13.2.6a}$$

$$\frac{1}{4}\text{-BPS}: \qquad m = |2gnQ|, \qquad P = \pm\frac{\kappa + 4g^2n^2}{2g}, \tag{13.2.6b}$$

where q and n are not constrained.

On these two BPS branches the root structure corresponds to

$$\mathrm{e}^{2V} = g^2\,(r - r_1^+)(r - r_1^-)(r - r_2^+)(r - r_2^-), \tag{13.2.7}$$

where

$$\frac{1}{2}\text{-BPS}: \qquad r_1^\pm = \frac{i}{2g}\left(\sqrt{\kappa + 4g^2n^2} \pm \sqrt{\kappa + 8g^2n^2 + 4igQ}\right), \tag{13.2.8a}$$

$$\frac{1}{4}\text{-BPS}: \qquad r_1^\pm = i\left(n \pm \frac{1}{\sqrt{2}g}\,\sqrt{\kappa + 4g^2n^2 + 2igQ}\right), \tag{13.2.8b}$$

and in both cases one has $r_2^\pm(Q) = -r_1^\pm(-Q)$.

The 1/4-BPS branch has a real root only if

$$Q^2 = -2n^2(\kappa + 2g^2n^2), \tag{13.2.9}$$

which requires $\kappa = -1$. Then the solution possesses an extremal horizon located at

$$r_1^- = r_2^- = \frac{\sqrt{1-\kappa-4g^2n^2}}{2\sqrt{2}g} > 0. \tag{13.2.10}$$

Note that the squareroot is well defined only if n is situated in the range (13.2.4) where there is no closed timelike curve according to [15]. One can see that if one of the root is real, then another root is automatically real and the black hole is extremal.

On the other hand for the 1/2-BPS solution a real root exists if

$$Q^2 = -n^2(\kappa + 4g^2n^2) \tag{13.2.11}$$

but this is in contraction with the requirement that the magnetic charge is real

$$\kappa + 4g^2n^2 > 0. \tag{13.2.12}$$

In this case the spacetime can reach negative r and there is no horizon. This should be contrasted with the Euclidean analysis where the associated solutions have a single root (corresponding to a bolt). This quantitative difference is due to the fact that one continues also the NUT charge when performing the Wick rotation from Lorentzian to Euclidean signatures.

13.3 Root structure and IR geometry

In general e^{2V} could be any function; nonetheless from known examples it seems that the most general form is a quartic polynomial [79, sec. 4] (see for example [4, 150, 151])

$$\mathrm{e}^{2V} = \sum_{p=0}^{4} v_p\, r^p. \tag{13.3.1}$$

The root structure of this functions is particularly important as it determines the existence and the location of horizons, along which other properties such as extremality. Before proceeding remember that it is possible to shift the radial coordinates. Finally the temperature of the black hole is proportional to $(\mathrm{e}^V)'$.

The various possibilities are:

- Naked singularity: pair of complex conjugate roots, $v_3 = 0$.

  The solution has no horizon.

- Black hole: two real roots, $v_0 = 0$.

  There is at least one horizon and the black hole has a finite temperature.

- Extremal black hole: real double root, $v_0 = v_1 = 0$.

  Two horizons of the previous case coincide, which implies that the first derivative vanishes, and the temperature is zero. We also recall that *static* BPS black holes are extremal.

- Double extremal black hole [35]: pair of real double roots, $v_0 = v_1 = 0$ and $v_3 = \sqrt{v_2 v_4}$.

- Ultracold black hole [154, sec. 3.1]: real triple root, $v_0 = v_1 = v_2 = 0$.

It is implicit that the other roots are different, and they may be real (giving additional horizons) or in complex conjugate pairs. Shifting r has been used to set $v_0 = 0$ – which is equivalent to move one of the root to $r = 0$ – when at least one root is real, or to set $v_3 = 0$. It is possible that for some special values of the v_i the class of a black hole changes, as we have seen in the previous section.

Extremal black holes which have

$$v_0 = v_1 = 0, \qquad v_2 \neq 0 \tag{13.3.2}$$

possess a near-horizon geometry of the form $\mathrm{adS}_2 \times \Sigma_g$ with respective radii R_1 and R_2. They are related to the metric functions by

$$\mathrm{e}^{2V} \sim_0 v_2\, r^2, \qquad \mathrm{e}^{2(V-U)} \sim_0 R_2^2, \qquad v_2 = \frac{R_2}{R_1}. \tag{13.3.3}$$

Plugging these functions into (13.1.1a) gives

$$\mathrm{d}s^2 = -\frac{r^2}{R_1^2} \big(\mathrm{d}t + 2n\, H(\theta)\, \mathrm{d}\phi\big)^2 + \frac{R_1^2}{r^2}\, \mathrm{d}r^2 + R_2^2\, \mathrm{d}\Sigma_g^2 \tag{13.3.4}$$

which approaches $\mathrm{adS}_2 \times \Sigma_g$ after the rescaling

$$r \longrightarrow \epsilon r, \qquad t \longrightarrow t/\epsilon, \tag{13.3.5}$$

followed by $\epsilon \to 0$.

In order to find BPS solutions without NUT charge, Cacciatori and Klemm used an ansatz with two double roots [35]

$$\mathrm{e}^V = \frac{r^2}{R} - v \tag{13.3.6}$$

where R is the radius of the asymptotic adS_4 vacua, and $v > 0$ is fixed by the near-horizon geometry [105]. Hence the function V is completely fixed by the boundary conditions in the IR and in the UV. Solutions in this category include [35, 96]; in the symplectic frame where the gaugings are electric they have magnetic charges.

Chapter 14

Static BPS equations

We are looking for static $\frac{1}{4}$-BPS solutions of $N = 2$ matter-coupled gauged supergravity. As it is well known [117, 143], BPS equations imply the equations of motion for the metric and for the scalar fields, but not Maxwell equations which need to be solved separately.[1]

14.1 Ansatz

The ansatz for the metric and for the gauge fields are

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2U}\mathrm{d}t^2 + \mathrm{e}^{-2U}\mathrm{d}r^2 + \mathrm{e}^{2(V-U)}\,\mathrm{d}\Sigma_g^2, \tag{14.1.1a}$$

$$A^\Lambda = \tilde{q}^\Lambda\,\mathrm{d}t - p^\Lambda F'(\theta)\mathrm{d}\phi. \tag{14.1.1b}$$

The functions $U, V, \tilde{q}$ and p depend only on r, while Σ_g is a Riemann surface of genus g (see appendix A.7) with metric

$$\mathrm{d}\Sigma_g^2 = \mathrm{d}\theta^2 + H'(\theta)^2\,\mathrm{d}\phi^2, \qquad H'(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ 1 & \kappa = 0, \\ \sinh\theta & \kappa = -1. \end{cases} \tag{14.1.2}$$

All scalars are function only on r

$$\tau^i = \tau^i(r), \qquad q^u = q^u(r). \tag{14.1.3}$$

We consider only abelian gaugings.

The magnetic field strength reads

$$G_\Lambda = R_{\Lambda\Sigma}F^\Sigma - I_{\Lambda\Sigma}\star F^\Sigma. \tag{14.1.4}$$

The electric and magnetic charges are given explicitly by

$$p^\Lambda = \frac{1}{4\pi}\int_{\Sigma_g} F^\Lambda, \tag{14.1.5a}$$

$$q_\Lambda = \frac{1}{4\pi}\int_{\Sigma_g} G_\Lambda = -\,\mathrm{e}^{2(V-U)}\,I_{\Lambda\Sigma}\tilde{q}'^\Sigma + \kappa\,R_{\Lambda\Sigma}p^\Sigma. \tag{14.1.5b}$$

The latter can be used for deriving an expression for $\tilde{q}'^\Lambda$

$$\tilde{q}'^\Lambda = \mathrm{e}^{2(U-V)}\,I^{\Lambda\Sigma}\big(R_{\Sigma\Delta}p^\Delta - q_\Sigma\big). \tag{14.1.6}$$

[1] In this section we follow the conventions of [78, 104].

The central and matter charges are[2]

$$\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V} \rangle = p^\Lambda M_\Lambda - q_\Lambda L^\Lambda, \qquad \mathcal{Z}_i = \langle \mathcal{Q}, U_i \rangle \,. \tag{14.1.7}$$

Similarly one defines the prepotential charges

$$\mathcal{L}^x = \langle \mathcal{P}^x, \mathcal{V} \rangle = -P^x_\Lambda L^\Lambda, \qquad \mathcal{L}^x_i = \langle \mathcal{P}^x, U_i \rangle \,. \tag{14.1.8}$$

Another expression for the central charge is

$$\mathcal{Z} = L^\Lambda I_{\Lambda\Sigma} \big(\mathrm{e}^{2(V-U)} \tilde{q}^\Sigma + i\, p^\Sigma \big). \tag{14.1.9}$$

14.2 Equations

BPS equations for $N = 2$ matter-coupled gauged supergravity have been derived in [104, sec. 2.2, app. B] (see also [78, app. D]).

For deriving the equations one choose a frame where the gaugings are purely electric

$$P^{x\Lambda} = 0 \tag{14.2.1}$$

such that

$$\mathcal{L}^x = -P^x_\Lambda L^\Lambda. \tag{14.2.2}$$

The Killing spinor reads

$$\varepsilon_\alpha = \mathrm{e}^{U/2}\, \mathrm{e}^{i\psi/2}\, \varepsilon_{0\alpha} \tag{14.2.3a}$$

where $\varepsilon_{0\alpha}$ is a constant spinor satisfying the two projection conditions

$$\varepsilon_{0\alpha} = i\, \gamma^0 \varepsilon_{\alpha\beta} \varepsilon_0^\beta, \tag{14.2.3b}$$

$$\varepsilon_{0\alpha} = -p^\Lambda P^x_\Lambda\, \gamma^{01} {\sigma^x}_\alpha{}^\beta \varepsilon_{0\beta}. \tag{14.2.3c}$$

Each projection halves the number of independent components. If $p^\Lambda = 0$ then the second projection is removed and one obtains 1/2-BPS solutions.

There are algebraic equations

$$(p^\Lambda P^x_\Lambda)^2 = \kappa^2, \tag{14.2.4a}$$

$$p^\Lambda k^u_\Lambda = 0, \tag{14.2.4b}$$

$$\mathrm{Re}(\mathrm{e}^{-i\psi} \mathcal{L}^x)\, p^\Lambda P^x_\Lambda = -\, \mathrm{e}^{2(U-V)}\, \mathrm{Im}(\mathrm{e}^{-i\psi} \mathcal{Z}) \tag{14.2.4c}$$

and differential equations

$$p'^\Lambda = 0, \tag{14.2.4d}$$

$$\psi' = -\mathcal{A}_r + 2\, p^\Lambda P^x_\Lambda\, \mathrm{e}^{-U}\, \mathrm{Re}(\mathrm{e}^{-i\psi} \mathcal{L}^x), \tag{14.2.4e}$$

$$(\mathrm{e}^U)' = -p^\Lambda P^x_\Lambda\, \mathrm{Im}(\mathrm{e}^{-i\psi} \mathcal{L}^x) + \mathrm{e}^{2(U-V)}\, \mathrm{Re}(\mathrm{e}^{-i\psi} \mathcal{Z}), \tag{14.2.4f}$$

$$(\mathrm{e}^V)' = -2\, \mathrm{e}^{V-U} p^\Lambda P^x_\Lambda\, \mathrm{Im}(\mathrm{e}^{-i\psi} \mathcal{L}^x), \tag{14.2.4g}$$

$$\tau'^i = \mathrm{e}^{-U}\, \mathrm{e}^{i\psi} g^{i\bar{\jmath}} \big(\mathrm{e}^{2(U-V)} \mathrm{D}_{\bar{\jmath}} \mathcal{Z} - i\, p^\Lambda P^x_\Lambda\, \mathrm{D}_{\bar{\jmath}} \mathcal{L}^x \big), \tag{14.2.4h}$$

$$q'^u = -2\, \mathrm{e}^{-U}\, h^{uv} \partial_v \big(p^\Lambda P^x_\Lambda\, \mathrm{Im}(\mathrm{e}^{-i\psi} \mathcal{L}^x) \big), \tag{14.2.4i}$$

$$q'_\Lambda = 2\, \mathrm{e}^{-U}\, \mathrm{e}^{2(V-U)}\, h_{uv} k^u_\Lambda k^v_\Sigma\, \mathrm{Re}(\mathrm{e}^{-i\psi} L^\Sigma), \tag{14.2.4j}$$

[2] There is a minus sign with respect to the notations of appendix A.6.

the primes denoting the radial derivative, and $\mathcal{A}_r$ is the composite U(1) connection. The equation (14.2.4a) corresponds to Dirac quantization condition (3.2.18) for the particular cases where the integer of the RHS is ± 1. The last equation (14.2.4j) corresponds to Maxwell equation: the fact that its RHS is non-trivial implies that some electric charges will not be conserved (they correspond to massive vector fields).

The equations for the vector scalars can also be written in terms of $\mathcal{L}_i^x$ and $\mathcal{Z}_i$.

Combining the equations n_v (complex) equations for τ^i, the one for U and the one for ψ, one can obtain $n_v + 1$ complex equations for the sections [105]

$$2\,\mathrm{e}^{2U}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}L^\Lambda)\big) = -\,\mathrm{e}^{2(U-V)}p^\Lambda + p^\Delta P^x_\Delta\, I^{\Lambda\Sigma} P^x_\Sigma - 8\,p^\Sigma P^x_\Sigma\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}^x)\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda), \tag{14.2.5a}$$

$$2\,\partial_r\big(\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda)\big) = \mathrm{e}^{2(U-V)} I^{\Lambda\Sigma} R_{\Sigma\Delta} p^\Delta - I^{\Lambda\Sigma} q_\Sigma. \tag{14.2.5b}$$

One can also derive equations for M_Λ

$$2\,\mathrm{e}^{2U}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}M_\Lambda)\big) = -\,\mathrm{e}^{2(U-V)}q_\Lambda + p^\Delta P^x_\Delta\, R_{\Lambda\Sigma} I^{\Sigma\Xi} P^x_\Xi - 8\,p^\Sigma P^x_\Sigma\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}^x)\,\mathrm{Re}(\mathrm{e}^{-i\psi}M_\Lambda), \tag{14.2.6a}$$

$$2\,\partial_r\big(\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}M_\Lambda)\big) = \mathrm{e}^{2(U-V)}\Big(R_{\Lambda\Sigma} I^{\Sigma\Delta}\big(R_{\Delta\Xi}p^\Xi - q_\Sigma\big) + I_{\Lambda\Sigma}p^\Sigma\Big) + p^\Sigma P^x_\Sigma\, P^x_\Lambda \tag{14.2.6b}$$

which are not independent.

One finds that

$$\tilde{q}^\Lambda = 2\,\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda). \tag{14.2.7}$$

Let's define

$$P^x_p = p^\Lambda P^x_\Lambda. \tag{14.2.8}$$

Then if $p^\Lambda \neq 0$ one can use a local SU(2) transformation in order to set [78, app. D]

$$P^1_p = P^2_p = 0, \tag{14.2.9}$$

which is a weaker condition than setting $P^1_\Lambda = P^2_\Lambda = 0$ as was done in [104]. This is possible only because p^Λ is constant. Then all remaining P^1_Λ and P^2_Λ in the BPS equations disappear, and the above equations can be rewritten uniquely in terms of $P^\Lambda \equiv P^3_\Lambda$ (this should not be confound with the momentum map of the SK gauged symmetries), and similarly we write $\mathcal{L} \equiv \mathcal{L}^3$.

Then the Dirac condition can be rewritten as

$$p^\Lambda P_\Lambda = \epsilon_D\,\kappa \tag{14.2.10}$$

with $\epsilon_D = \pm 1$ (a common choice is $\epsilon_D = -1$ [76, 78]). Replacing this in all equations one sees that κ only appears in the Dirac condition, meaning that solutions are independent of the curvature of the horizon, but regularity does depend on it [96, p. 6].

If $p^\Lambda = 0$ then the Dirac condition should not be imposed.

14.3 Symplectic extension

In this section we introduce magnetic gaugings by performing a symplectic transformation (see section 3.5). Most parts of the equations (14.2.4) are already written in a symplectic form.

14.3.1 Equations

One can see that $\tilde{q}'^\Lambda$ from (14.1.6) corresponds to the first row of the $-\Omega\mathcal{M}\mathcal{Q}$, where $\mathcal{M}$ was defined in (6.4.2). Then symplectic equations can be obtained from the replacement

$$\tilde{q}'^\Lambda = -\,\mathrm{e}^{2(U-V)}(\Omega\mathcal{M}\mathcal{Q})^\Lambda. \tag{14.3.1}$$

Similarly terms involving the electromagnetic charges and the gaugings, such as $I^{-1}\mathcal{P}$, can be replaced (missing terms due to the fact we had $P^\Lambda = 0$ can be guessed).

We now list the symplectic algebraic equations

$$\langle \mathcal{Q}, \mathcal{P}\rangle = \epsilon_D\,\kappa, \tag{14.3.2a}$$
$$\langle \mathcal{Q}, \mathcal{K}^u\rangle = 0, \tag{14.3.2b}$$
$$\epsilon_D\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) = -\,\mathrm{e}^{2(U-V)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) \tag{14.3.2c}$$

and differential equations

$$(\,\mathrm{e}^U)' = -\epsilon_D\ \mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}) + \,\mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z}), \tag{14.3.2d}$$
$$(\,\mathrm{e}^V)' = -2\epsilon_D\ \mathrm{e}^{V-U}\ \mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}), \tag{14.3.2e}$$
$$\tau'^i = \,\mathrm{e}^{-U}\,\mathrm{e}^{i\psi}g^{i\bar{\jmath}}\big(\,\mathrm{e}^{2(U-V)}\mathrm{D}_{\bar{\jmath}}\mathcal{Z} - i\,\epsilon_D\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\big), \tag{14.3.2f}$$
$$q'^u = -2\epsilon_D\ \mathrm{e}^{-U}\ h^{uv}\partial_v\big(\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L})\big), \tag{14.3.2g}$$
$$\mathcal{Q}' = 2\,\mathrm{e}^{-U}\ \mathrm{e}^{2(V-U)}\ h_{uv}\mathcal{K}^u\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\ \langle\mathcal{V}, \mathcal{K}^u\rangle), \tag{14.3.2h}$$
$$\psi' = -\mathcal{A}_r + 2\epsilon_D\ \mathrm{e}^{-U}\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}). \tag{14.3.2i}$$

We note that the symplectic Maxwell equations correctly reduce to (14.2.4d) and (14.2.4j) in a symplectic frame since $k^{u\Lambda} = 0$.

Instead of working with the real and imaginary parts of $\mathrm{e}^{-i\psi}\,\mathrm{e}^{-U}\mathcal{V}$ as independent equations as in (14.2.5), one can combine (14.2.5a) and (14.2.6a) in the symplectic equation

$$2\,\mathrm{e}^{2U}\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{-U}\mathcal{V}) = -\,\mathrm{e}^{2(U-V)}\mathcal{Q} + \epsilon_D\,\Omega\mathcal{M}\mathcal{P} - 8\,\epsilon_D\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}). \tag{14.3.3a}$$

We stress that this equation is totally equivalent to (14.3.2d), (14.3.2f) and (14.3.2i). Then the remaining equations are combined as

$$2\,\partial_r\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{U}\mathcal{V}) = -\,\mathrm{e}^{2(U-V)}\,\Omega\mathcal{M}\mathcal{Q} + \epsilon_D\,\mathcal{P} \tag{14.3.3b}$$

and they are redundant since $\mathrm{Im}\,\mathcal{V}$ already exhausts the $2n_v+2$ variables τ^i, ψ and U. Here it is useful to have the equations (14.2.6b) for M_Λ because the second term is not visible in (14.2.5b).

For a future purpose we want to obtain another form of (14.3.3a). Multiplying by $\mathrm{e}^{2(V-U)}$, we want to rewrite the LHS with a factor e^V inside the derivative

$$\begin{aligned}\mathrm{e}^{2V}\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{-U}\mathcal{V}) &= \,\mathrm{e}^{V}\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{V-U}\mathcal{V}) - \,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\,\partial_r\,\mathrm{e}^V \\ &= \,\mathrm{e}^{V}\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{V-U}\mathcal{V}) + 2\,\epsilon_D\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V}),\end{aligned}$$

and this combines with the RHS as

$$\begin{aligned}2\,\mathrm{e}^V\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{V-U}\mathcal{V}) = -\mathcal{Q} + \epsilon_D\ \mathrm{e}^{2(V-U)}\Big(&\Omega\mathcal{M}\mathcal{P} - 8\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ &- 4\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V})\Big).\end{aligned} \tag{14.3.4}$$

Equation (14.3.2c) can be directly integrated to get the phase in terms of $\mathcal{L}$ and $\mathcal{Z}$ [76, eq. (2.39)]

$$e^{2i\psi} = \frac{\mathcal{Z} - i\,\epsilon_D\; e^{2(V-U)}\mathcal{L}}{\bar{\mathcal{Z}} + i\,\epsilon_D\; e^{2(V-U)}\bar{\mathcal{L}}}. \tag{14.3.5}$$

This is obtained by writing explicitly the real and imaginary parts in order to get a second order equation for $e^{i\psi}$, which then can be solved.

14.3.2 Simplified form (FI gaugings)

In this section we consider only FI gaugings such that $\mathcal{P} = \text{cst}$. A seminal approach developed in [118] allows to greatly simplify the equations and this lead to complete analytical solution of a full 1/4-BPS black hole in [105]. The idea is to rewrite the equations in terms of the quartic function (7.3.1) (and its gradient) and to exploit the power of special geometry. The identities used in this section do not require the manifold to be symmetric [119] as was first believed in [105, 118].

First let's define a rescaled section

$$\widetilde{\mathcal{V}} = e^{V-U}\, e^{-i\psi}\, \mathcal{V}. \tag{14.3.6}$$

The equation (14.3.4) can be simplified using relation (D.1.2c)

$$2\, e^V \partial_r \operatorname{Im}\widetilde{\mathcal{V}} = -\mathcal{Q} + \epsilon_D\, I_4'(\mathcal{P}, \operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}). \tag{14.3.7}$$

In these terms equation (14.3.2e) reads

$$(e^V)' = -2\epsilon_D \left\langle \widetilde{\mathcal{V}}, \mathcal{P} \right\rangle, \tag{14.3.8}$$

while the constraint (14.3.2c) becomes

$$2\epsilon_D\, I_4(\operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{P}) = \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{Q} \right\rangle \tag{14.3.9}$$

using (D.1.2b) to replace $\operatorname{Re}\widetilde{\mathcal{V}}$

$$\operatorname{Re}\widetilde{\mathcal{V}} = 2\; e^{2(U-V)}\, I_4'(\operatorname{Im}\widetilde{\mathcal{V}}) = \frac{I_4'(\operatorname{Im}\widetilde{\mathcal{V}})}{2\sqrt{I_4(\operatorname{Im}\widetilde{\mathcal{V}})}}. \tag{14.3.10}$$

A more convenient form for this equation can be achieved by writing

$$I_4(\operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{P}) = \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, I_4'(\operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{P}) \right\rangle \tag{14.3.11}$$

and by inserting (14.3.7)

$$e^V \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, \partial_r \operatorname{Im}\widetilde{\mathcal{V}} \right\rangle = \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{Q} \right\rangle. \tag{14.3.12}$$

Let's summarize the equations that have been obtained

$$2\, e^V \partial_r \operatorname{Im}\widetilde{\mathcal{V}} = -\mathcal{Q} + \epsilon_D\, I_4'(\mathcal{P}, \operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}), \tag{14.3.13a}$$

$$(e^V)' = -2\epsilon_D \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{P} \right\rangle, \tag{14.3.13b}$$

$$e^V \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, \partial_r \operatorname{Im}\widetilde{\mathcal{V}} \right\rangle = \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{Q} \right\rangle, \tag{14.3.13c}$$

$$\langle \mathcal{Q}, \mathcal{P} \rangle = \epsilon_D\, \kappa. \tag{14.3.13d}$$

The main advantage of these equations is that they do not involve $\operatorname{Re}\widetilde{\mathcal{V}}$, U or ψ, they only contain $\operatorname{Im}\widetilde{\mathcal{V}}$ and V (as dynamical objects). Another useful point is the removal of the matrix $\mathcal{M}$ whose explicit form is involved in the general case. All other objects can be deduced from them, for example one can obtain $\operatorname{Re}\widetilde{\mathcal{V}}$ from (14.3.10).

It has been shown in [119] that these equations can be obtained from a $d = 11$ supergravity truncation where the role of I_4 is played by the Hitchin functional.

Chapter 15

Static BPS solutions

We will focus on solutions that are black holes interpolating between a (magnetic) adS_4 (of radius R) for $r \to \infty$ and a topological horizon of Bertotti–Robinson type $\mathrm{adS}_2 \times \Sigma_g$ (with respective radius R_1 and R_2) for $r \to 0$. Both these spacetimes are also BPS solutions and can be studied separately, and for this reason the full black hole can be seen as a soliton (or a domain wall) [105].

15.1 $N = 2$ adS_4

An anti-de Sitter vacua is characterized by constant scalars and vanishing charges

$$\tau^i(r) = \tau_0^i, \qquad q^u(r) = q_0^u, \qquad \mathcal{Q} = 0, \tag{15.1.1}$$

which implies in particular $\mathcal{Z} = 0$. The metric functions are

$$\mathrm{e}^U = \frac{r}{R}, \qquad \mathrm{e}^V = \frac{r^2}{R^2} \tag{15.1.2}$$

giving the metric

$$\mathrm{d}s^2 = -\frac{r^2}{R^2}\,\mathrm{d}t^2 + \frac{R^2}{r^2}\,\mathrm{d}r^2 + \frac{r^2}{R^2}\,\mathrm{d}\Sigma_g^2 \tag{15.1.3}$$

As discussed in the previous section vanishing charges imply that the solution is 1/2-BPS. Moreover in the case of adS_4 vacua there is a special enhancement of supersymmetry which increases it to a full BPS solution. Moreover one cannot use the trick of the $\mathrm{SU}(2)$ rotation to set $\mathcal{P}^1 = \mathcal{P}^2 = 0$.

Typically the asymptotic geometry of a 1/4-BPS black hole will be a madS vacua. There is a one-to-one relationship between adS and madS vacua.

From (14.3.2f) one gets the equation

$$\mathcal{L}_i^x = \langle U_i, \mathcal{P}^x \rangle = 0. \tag{15.1.4}$$

In a frame where the gaugings are purely electric, this equation is equivalent to

$$P_\Lambda^x f_i^\Lambda = 0. \tag{15.1.5}$$

In the space spanned by the $n_v + 1$ directions of Λ, f_i^Λ represents n_v vectors indexed by i. Then the previous equation implies that, for fixed x, P_Λ^x is orthogonal to these n_v vectors and thus

$$P_\Lambda^x = c^x(q^u)\, P_\Lambda. \tag{15.1.6}$$

Then a local SU(2) rotation can be used to set

$$c^1 = c^2 = 0. \tag{15.1.7}$$

Note that the latter equations must be enforced as they are not a generic consequence of the theory. We then denote $\mathcal{P} \equiv \mathcal{P}^3$ and $\mathcal{L} \equiv \mathcal{L}^3$ as usual.

The BPS equations are

$$\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) = 0, \tag{15.1.8a}$$

$$\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}) = \frac{1}{R}, \tag{15.1.8b}$$

$$\mathcal{L}_i = 0, \tag{15.1.8c}$$

$$\psi' = 0, \tag{15.1.8d}$$

$$\langle \mathcal{V}, \mathcal{K}^u \rangle = 0. \tag{15.1.8e}$$

From (14.3.2h) one obtains

$$\mathrm{Re}(\,\mathrm{e}^{-i\psi}\,\langle \mathcal{V}, \mathcal{K}^u \rangle) = 0, \tag{15.1.9}$$

while the derivative in (14.3.2g) can be used to replace the prepotential by the Killing vector

$$\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\langle \mathcal{V}, \mathcal{K}^u \rangle) = 0. \tag{15.1.10}$$

Combining both equations gives (15.1.8e).

The equations for the sections are

$$2\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) = R\,\mathcal{P}, \qquad 2\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V}) = R\,\Omega\mathcal{M}\mathcal{P}. \tag{15.1.11}$$

Using the matrix $\mathcal{C}$ defined in (6.4.15) this can be rewritten as

$$\mathrm{e}^{-i\psi}\mathcal{V} = i\,R\,\Omega\mathcal{C}\mathcal{P}. \tag{15.1.12}$$

All the equations but the last one in (15.1.8) do not involve the Killing vectors. Hence a strategy to solve these equations is to consider $\mathcal{P}$ as a constant (which is the case for the FI gaugings $\mathcal{P} = \mathrm{cst}$ and $n_h = 0$) and to solve for the vector scalars in terms of $\mathcal{P}$. Then the remaining equation (15.1.8e) can be used to solve for the hyperscalars which can be replaced at the end in the vector scalars.

Following this strategy we first analyse the equations for the vector scalar sector [96, sec. 3]. Equation (15.1.8d) means that the phase is constant

$$\psi(r) = \psi_0. \tag{15.1.13}$$

We rewrite (15.1.8b) as

$$\mathcal{L} = \frac{i}{R}\,\mathrm{e}^{i\psi_0}. \tag{15.1.14}$$

Because of (15.1.8c) the prepotentials have components only in the direction of $\mathcal{V}$ and its conjugate

$$\mathcal{P} = -2\,\mathrm{Im}(\bar{\mathcal{L}}\mathcal{V}). \tag{15.1.15}$$

Note that these equations are identical to those of the $\mathrm{adS}_2 \times S^2$ near-horizon in ungauged $N = 2$ supergravity, with the replacement $\mathcal{P} \to \mathcal{Q}$, which can be solved explicitly in some cases (such as symmetric cubic $\mathcal{M}_v$) [104, 162]. The value for the phase is taken to be

$$\psi_0 = -\frac{\pi}{2} \tag{15.1.16}$$

which implies

$$\mathcal{L} = \frac{1}{R}, \tag{15.1.17}$$

and also

$$\mathcal{P} = -\frac{2}{R}\, \mathrm{Im}\, \mathcal{V}. \tag{15.1.18}$$

These equations are consistent with (15.1.11). Evaluating this equation on I_4 provides the formula for the radius

$$(-1)^4 \frac{2^4}{R^4} I_4(\mathrm{Im}\, \mathcal{V}) = I_4(\mathcal{P}) \tag{15.1.19}$$

thanks to (7.3.12) (recall that I_4 is homogeneous of order 4) which simplifies to

$$\frac{1}{R^2} = \sqrt{I_4(\mathcal{P})}. \tag{15.1.20}$$

Moreover for FI gaugings $\mathcal{P}$ is constant and for symmetric SK manifold I_4 does not depend on the scalars, which implies that the above formula is given just in terms of the gaugings. Otherwise this gives an implicit equation for the scalar fields that one needs to solve, or one can replace the scalar fields at the end if their solutions have been obtained through another equation.

Let's turn to the last equation (15.1.8e)

$$\langle \mathcal{V}, \mathcal{K}^u \rangle = 0 \tag{15.1.21}$$

following the analysis of [78, sec. 2.2].

First we want to clarify this equation. Using the results of section 10.3, the spin connection ω^x is invariant under symmetry transformation generated by k only up to an SU(2) transformation (we consider only the electric frame here)

$$\mathcal{L}_k \omega^x = \nabla W^x_k \tag{15.1.22}$$

where W^x_k is an SU(2) vector called the compensator. This allows to relate directly the Killing vector and prepotential

$$P^x = k^u \omega^x_u + W^x. \tag{15.1.23}$$

Contracting (15.1.8e) with ω^x_u and plugging this last result gives

$$\mathrm{e}^{-i\psi} \mathcal{L} - \mathrm{e}^{-i\psi} \langle \mathcal{V}, \mathcal{W} \rangle = 0. \tag{15.1.24}$$

If the compensator vanishes $\mathcal{W} = 0$ one obtains a singular solution since $\mathcal{L} = 0$ implies $R \to \infty$. Then a necessary condition for having a $N = 2$ adS_4 vacua is that at least one isometry with a non-trivial compensator is gauged [40, 130]. In the case of special quaternionic manifold, isometries with compensators are not generic as only the isometries inherited from the base special Kähler space and the hidden symmetries have compensators (see section 12).

It may seem that (15.1.8e) are too many equations since there are $2n_h$ equations ($\mathcal{V}$ being complex) for the n_h variables q^u. But in fact the imaginary part is already implied by (15.1.18)

$$\langle \mathrm{Im}\, \mathcal{V}, \mathcal{K}^u \rangle \sim \langle \mathcal{P}, \mathcal{K}^u \rangle = 0 \tag{15.1.25}$$

where the last equality follows from the locality constraints (3.5.9). Then the only equations that we need to solve are

$$\langle \mathrm{Re}\, \mathcal{V}, \mathcal{K}^u \rangle = 0. \tag{15.1.26}$$

We restrict ourselves to the case of symmetric very special Kähler manifold (section 8.3). Using the relation (7.3.14)

$$\operatorname{Re}\mathcal{V} = 2\, I_4'(\operatorname{Im}\mathcal{V}) \tag{15.1.27}$$

the previous equation can be rewritten as

$$\langle I_4'(\operatorname{Im}\mathcal{V}), \mathcal{K}^u \rangle = I_4(\mathcal{K}^u, \operatorname{Im}\mathcal{V}, \operatorname{Im}\mathcal{V}, \operatorname{Im}\mathcal{V}) = 0 \tag{15.1.28}$$

and then as

$$I_4(\mathcal{K}^u, \mathcal{P}, \mathcal{P}, \mathcal{P}) \sim \nabla^u I_4(\mathcal{P}) = 0 \tag{15.1.29}$$

thanks to (15.1.15).

As a summary the equations to solve for are

$$\mathcal{P} = -2\operatorname{Im}(\bar{\mathcal{L}}\mathcal{V}), \tag{15.1.30a}$$

$$\mathcal{L} = \frac{1}{R}, \tag{15.1.30b}$$

$$0 = \nabla^u I_4(\mathcal{P}). \tag{15.1.30c}$$

The first two equations in the case of FI gaugings were explicitly solved in some cases in [96].

15.2 Near-horizon $\mathrm{adS}_2 \times \Sigma_g$

These equations have been studied with $n_h = 0$ and FI gaugings in [35, sec. 4, 76, sec. 3], and further in [104] (see also [96, 118, sec. 5]). For $n_h \neq 0$ they were studied in the electric frame in [104, sec. 2.3] and in general in [78, sec. 2.3].

There is a supersymmetry enhancement at the horizon because there are two extra superconformal charges [78, p. 6].

Denoting the horizon radius by r_h and by r_Λ the radius where the scalars τ^i vanish, the solution is regular only if $r_h > r_\Lambda$ for all Λ [96, p. 15].

Scalars and charges are constant for near-horizon geometries

$$\tau^i(r) = \tau^i_0, \qquad q^u(r) = q^u_0, \qquad \mathcal{Q} = \mathrm{cst}. \tag{15.2.1}$$

The metric functions are

$$\mathrm{e}^U = \frac{r}{R_1}, \qquad \mathrm{e}^V = \frac{R_2}{R_1}\, r \tag{15.2.2}$$

giving the metric

$$\mathrm{d}s^2 = -\frac{r^2}{R_1^2}\,\mathrm{d}t^2 + \frac{R_1^2}{r^2}\,\mathrm{d}r^2 + R_2^2\,\mathrm{d}\Sigma_g^2 \tag{15.2.3}$$

The BPS equations are

$$\langle \mathcal{Q}, \mathcal{P} \rangle = \epsilon_D\, \kappa, \tag{15.2.4a}$$

$$\operatorname{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) = \epsilon_D\, R_2^2\, \operatorname{Re}(\mathrm{e}^{-i\psi}\mathcal{L}), \tag{15.2.4b}$$

$$\operatorname{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}) = \frac{R_2^2}{2R_1}, \tag{15.2.4c}$$

$$\epsilon_D\, \operatorname{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) = -\frac{1}{2R_1}, \tag{15.2.4d}$$

$$\mathcal{Z}_i = i\,\epsilon_D\, R_2^2\, \mathcal{L}_i, \tag{15.2.4e}$$

$$\psi' = 2\epsilon_D\, \frac{R_1}{r}\, \operatorname{Re}(\mathrm{e}^{-i\psi}\mathcal{L}), \tag{15.2.4f}$$

$$\langle \mathcal{Q}, \mathcal{K}^u \rangle = 0, \tag{15.2.4g}$$

$$\langle \mathcal{V}, \mathcal{K}^u \rangle = 0. \tag{15.2.4h}$$

We can adopt the same strategy as in the previous section: all equations except the last two do not contain the Killing vectors, such that they can be solved as if $\mathcal{P}$ was constant, giving a solution for the vector scalars in terms of the charges, the gaugings and the hyperscalars

$$\tau^i = \tau^i(\mathcal{P}, \mathcal{Q}, q^u). \tag{15.2.5}$$

Then the remaining equations can be used to solve for the hyperscalars in terms of the charges and the gaugings

$$q^u = q^u(\mathcal{P}, \mathcal{Q}), \quad \Longrightarrow \quad \tau^i = \tau^i(\mathcal{P}, \mathcal{Q}). \tag{15.2.6}$$

From the equations one can also write

$$\mathrm{Re}(\,\mathrm{e}^{-i\psi} \mathcal{Z}) = -\epsilon_D\, R_2^2\, \mathrm{Im}(\,\mathrm{e}^{-i\psi} \mathcal{L}). \tag{15.2.7}$$

Combining this with (15.2.4b) gives

$$\mathcal{Z} = i\, \epsilon_D\, R_2^2\, \mathcal{L}. \tag{15.2.8}$$

Since R_2^2 is real this means that the phases of $\mathcal{Z}$ and $\mathcal{L}$ differ by $\pi/2$ [76, p. 12]. Plugging the relation (15.2.8) into (14.3.5) implies that ψ is a multiple of π

$$\psi(r) = \pi. \tag{15.2.9}$$

Another way to see this is by taking the imaginary part of (15.2.8): this is consistent with (15.2.4b) only if $\psi = \pi$. Then inserting this result into (15.2.4f) gives

$$\mathrm{Re}(\,\mathrm{e}^{-i\psi} \mathcal{L}) = 0 \Longrightarrow \mathrm{Im}(\,\mathrm{e}^{-i\psi} \mathcal{Z}) = 0, \tag{15.2.10}$$

and as a consequence

$$\mathcal{Z} = \frac{R_2^2}{2R_1}, \qquad \mathcal{L} = -\epsilon_D\, \frac{i}{2R_1}. \tag{15.2.11}$$

Instead of working with (15.2.4e) it is easier to work with the sections. Using the previous elements one has

$$\frac{2R_2^2}{R_1}\, \mathrm{Im}\, \mathcal{V} = \mathcal{Q} - \epsilon_D\, R_2^2\, \Omega\mathcal{M}\mathcal{P}, \tag{15.2.12a}$$

$$\frac{2R_2^2}{R_1}\, \mathrm{Re}\, \mathcal{V} = \Omega\mathcal{M}\mathcal{Q} + \epsilon_D\, R_2^2\, \mathcal{P}. \tag{15.2.12b}$$

Adding the two equations gives

$$\mathcal{V} = i\, \frac{R_1}{2R_2^2}\, \Omega\mathcal{C}(\mathcal{Q} + \epsilon_D\, R_2^2\, \Omega\mathcal{M}\mathcal{P}) \tag{15.2.13}$$

where $\mathcal{C}$ was defined in (6.4.15). Note the similarity with (15.1.12).

Another way to derive the equation for the section is to contract (15.2.4e) with $\Omega\mathcal{M}$. Using the relation (6.4.13)

$$\Omega\mathcal{M} U_i = -i U_i \tag{15.2.14}$$

one obtains

$$\begin{aligned} 0 &= \langle U_i, \mathcal{Q}\rangle - i\, \epsilon_D\, R_2^2\, \langle U_i, \mathcal{P}\rangle = \langle U_i, \mathcal{Q}\rangle + \epsilon_D\, R_2^2\, \langle \Omega\mathcal{M} U_i, \mathcal{P}\rangle \\ &= \langle U_i, \mathcal{Q}\rangle + \epsilon_D\, R_2^2\, \langle U_i, \Omega\mathcal{M}\mathcal{P}\rangle = \big\langle U_i, \mathcal{Q} + \epsilon_D\, R_2^2\, \Omega\mathcal{M}\mathcal{P}\big\rangle \end{aligned}$$

because of (6.4.11). As a consequence the quantity $\mathcal{Q} + \epsilon_D\, R_2^2\, \Omega\mathcal{M}\mathcal{P}$ has no components along the direction U_i in the basis $(\mathcal{V}, U_i)$ such that

$$\mathcal{Q} + \epsilon_D\, R_2^2\, \Omega\mathcal{M}\mathcal{P} = -2\,\mathrm{Im}(\langle \bar{\mathcal{V}}, \mathcal{Q} + \epsilon_D\, R_2^2\, \Omega\mathcal{M}\mathcal{P}\rangle\, \mathcal{V}). \tag{15.2.15}$$

Now we can introduce the central charge and after using the relation (15.2.8) one obtains

$$\mathcal{Q} + \epsilon_D\, R_2^2\, \Omega\mathcal{M}\mathcal{P} = -4\,\mathrm{Im}(\bar{\mathcal{Z}}\, \mathcal{V}). \tag{15.2.16}$$

This is equivalent to (15.2.12a) once $\mathcal{Z}$ is replaced by its value.

Contracting (15.2.16) with $\mathcal{P}$ gives

$$\langle \mathcal{Q}, \mathcal{P}\rangle + \epsilon_D\, R_2^2\, \langle \Omega\mathcal{M}\mathcal{P}, \mathcal{P}\rangle = -4\,\mathrm{Im}(\bar{\mathcal{Z}}\, \mathcal{L}), \tag{15.2.17}$$

while with $\mathcal{Q}$ one gets

$$\langle \Omega\mathcal{M}\mathcal{P}, \mathcal{Q}\rangle = 0. \tag{15.2.18}$$

Then using the relation (15.2.8) modifies the first equation to

$$\langle \mathcal{Q}, \mathcal{P}\rangle - \epsilon_D\, R_2^2\, \mathcal{P}\mathcal{M}\mathcal{P} = 4\epsilon_D\, R_2^2\, |\mathcal{L}|^2, \tag{15.2.19}$$

and using (6.4.26) one obtains [76, p. 13]

$$\frac{\epsilon_D}{R_2^2}\, \langle \mathcal{Q}, \mathcal{P}\rangle = -\mathcal{P}\mathcal{M}(\mathcal{F})\mathcal{P} = 2(|\mathcal{L}|^2 - |\mathcal{L}_i|^2). \tag{15.2.20}$$

A similar relation for $\mathcal{Z}$ follows directly

$$\epsilon_D\, R_2^2\, \langle \mathcal{Q}, \mathcal{P}\rangle = -\mathcal{Q}\mathcal{M}(\mathcal{F})\mathcal{Q} = 2(|\mathcal{Z}|^2 - |\mathcal{Z}_i|^2). \tag{15.2.21}$$

These formulas are helpful for understanding why it is not possible to find asymptotically adS_4 solutions with spherical horizon and constant scalars: the adS_4 vacua has $\mathcal{L}_i = 0$ from (15.1.8b), and the previous equations give

$$R_2^2 = -\frac{\epsilon_D}{2|\mathcal{L}|^2}\, \langle \mathcal{Q}, \mathcal{P}\rangle = -\frac{\kappa}{2|\mathcal{L}|^2}. \tag{15.2.22}$$

The latter is positive only for $\kappa = -1$.

Due to the relation (15.2.8) between $\mathcal{L}$ and $\mathcal{Z}$ one finds the formula

$$\mathcal{Q} + \epsilon_D\, i R_2^2\, \mathcal{P} = 2\bar{\mathcal{Z}}\, \mathcal{V} - 2g^{i\bar{\jmath}}\bar{\mathcal{Z}}_{\bar{\jmath}} U_i \tag{15.2.23}$$

using the expansion (6.4.21) (comparing with previous expressions, the complex structure $\Omega\mathcal{M}$ has been replaced by i). This expression is holomorphic in $\mathcal{V}$ and its derivative, and as a consequence evaluating it on I_4 gives the complex equation

$$I_4(\mathcal{Q} + \epsilon_D\, i R_2^2\, \mathcal{P}) = 0. \tag{15.2.24}$$

Splitting it in its real and imaginary parts gives the radius of Σ_g (and hence the entropy) and a constraint.

The radius of Σ_g reads

$$R_2^4 = I_4(\mathrm{Im}\, \widetilde{\mathcal{V}}) = \frac{1}{I_4(\mathcal{P})}\left(I_4(\mathcal{Q}, \mathcal{Q}, \mathcal{P}, \mathcal{P}) \pm \sqrt{I_4(\mathcal{Q}, \mathcal{Q}, \mathcal{P}, \mathcal{P})^2 - I_4(\mathcal{Q}) I_4(\mathcal{P})}\right). \tag{15.2.25}$$

At this point $\mathcal{P}$ depends on q^u, which needs to be solved for using the other equations, while for FI gaugings this is the final result. The entropy is

$$S = \pi R_2^2. \tag{15.2.26}$$

The aforementioned constraint is

$$\begin{aligned} 0 = 4\, I_4(\mathcal{P}) I_4(\mathcal{P}, \mathcal{Q}, \mathcal{Q}, \mathcal{Q})^2 + 4\, I_4(\mathcal{Q}) I_4(\mathcal{Q}, \mathcal{P}, \mathcal{P}, \mathcal{P})^2 \\ - I_4(\mathcal{P}, \mathcal{Q}, \mathcal{Q}, \mathcal{Q}) I_4(\mathcal{P}, \mathcal{P}, \mathcal{Q}, \mathcal{Q}) I_4(\mathcal{Q}, \mathcal{P}, \mathcal{P}, \mathcal{P}). \end{aligned} \tag{15.2.27}$$

Since it does not depend on r it means that it is valid for any value of the radial direction and it should be enforced on the full solution.

As a summary the equations to solve are

$$\mathcal{Q} + \epsilon_D\, R_2^2\, \Omega \mathcal{M} \mathcal{P} - = -4\, \mathrm{Im}(\bar{\mathcal{Z}}\, \mathcal{V}), \tag{15.2.28a}$$

$$\mathcal{Z} = \frac{R_2^2}{2 R_1}, \tag{15.2.28b}$$

$$\langle \mathcal{Q}, \mathcal{P} \rangle = \epsilon_D\, \kappa, \tag{15.2.28c}$$

$$\langle \mathcal{Q}, \mathcal{K}^u \rangle = 0, \tag{15.2.28d}$$

$$\langle \mathcal{V}, \mathcal{K}^u \rangle = 0. \tag{15.2.28e}$$

The first two equations were solved for FI gaugings with cubic $\mathcal{M}_v$ explicitly in the case of symmetric spaces and implicitly otherwise in [104]. Note that for $\mathcal{P} = 0$ it reduces to the attractor equations of ungauged supergravity.

It is also possible to use the reformulation (14.3.13) of the equations to find other properties more easily. For the near-horizon geometry ansatz they read (we focus on the vector sector)

$$I_4'(\mathcal{P}, \mathrm{Im}\, \widetilde{\mathcal{V}}, \mathrm{Im}\, \widetilde{\mathcal{V}}) = \epsilon_D\, \mathcal{Q}, \tag{15.2.29a}$$

$$2\epsilon_D \left\langle \mathrm{Im}\, \widetilde{\mathcal{V}}, \mathcal{P} \right\rangle = -\frac{R_2}{R_1}, \tag{15.2.29b}$$

$$\left\langle \mathrm{Im}\, \widetilde{\mathcal{V}}, \mathcal{Q} \right\rangle = 0, \tag{15.2.29c}$$

since $\partial_r\, \mathrm{Im}\, \widetilde{\mathcal{V}} = 0$. Moreover contracting the first equation with $\mathrm{Im}\, \widetilde{\mathcal{V}}$ and using the last one gives

$$I_4(\mathcal{P}, \mathrm{Im}\, \widetilde{\mathcal{V}}, \mathrm{Im}\, \widetilde{\mathcal{V}}, \mathrm{Im}\, \widetilde{\mathcal{V}}) = 0. \tag{15.2.30}$$

The entropy can be rewritten

$$S = \pi \sqrt{I_4(\mathrm{Im}\, \widetilde{\mathcal{V}})} \tag{15.2.31}$$

using the definition of $\mathrm{Im}\, \widetilde{\mathcal{V}}$. Note the similarity with ungauged supergravity where $S = \pi\sqrt{\mathcal{Q}}$.

15.3 General solution

A general solution to the set of BPS equations for FI gauged supergravity (14.3.13) was provided in [105]. We will only give the most important details of the analysis.

As explained in section 13.3, BPS static black holes are extremal and we are considering near-horizon geometry $\mathrm{adS}_2 \times \Sigma_g$. As a consequence the ansatz for e^V is

$$\mathrm{e}^{2V} = r^2 (v_4 r^2 + v_3 r + v_2). \tag{15.3.1}$$

This root structure and the degenerate double extremal case are the only ones allowed for this type of black holes [79, p. 11].

The ansatz for $\mathrm{Im}\, \widetilde{\mathcal{V}}$ is more involved

$$\mathrm{Im}\, \widetilde{\mathcal{V}} = \mathrm{e}^{-V} (A_3 r^3 + A_2 r^2 + A_1 r) \tag{15.3.2}$$

where the A_i are symplectic vectors.

The next steps is to expand each of the equations (14.3.13) in powers of r and to identify the coefficients. In principle one should be able to find the constraint (15.2.27) from the analysis, but this did not appear feasible, and for this reason it is used as an input for simplifying the equations, using it for replacing $I_4(\mathcal{P}, \mathcal{P}, \mathcal{Q}, \mathcal{Q})$.

Note also that the system contains much more equations than variables, and there is a lot of redundancy. In particular (14.3.13b) implies the following relations

$$v_{i+1} = \frac{4}{i+1}\, \langle \mathcal{P}, A_i \rangle\,. \tag{15.3.3}$$

The UV boundary condition can be read from (14.3.13a) and gives

$$A_3 = \frac{I_4'(\mathcal{P})}{4\sqrt{I_4(\mathcal{P})}}, \qquad v_4 = \frac{1}{R_{\text{adS}}^2} = \sqrt{I_4(\mathcal{P})}. \tag{15.3.4}$$

The overall normalization was not fixed and it was determined by comparison with [96].

The solution for A_2 and A_3 is found by expanding these vectors on the basis (D.1.1), and it can be found that only third order terms are non-vanishing

$$A_i = a_{i1}\, I_4'(\mathcal{P}) + a_{i2}\, I_4'(\mathcal{P}, \mathcal{P}, \mathcal{Q}) + a_{i3}\, I_4'(\mathcal{P}, \mathcal{Q}, \mathcal{Q}) + a_{i4}\, I_4'(\mathcal{Q}). \tag{15.3.5}$$

Explicit formulas can be found in [105, sec. 3], and one needs to use the identities of appendix D.1.

The real part of $\widetilde{\mathcal{V}}$ can be found from

$$\operatorname{Re} \widetilde{\mathcal{V}} = 2\, \mathrm{e}^{2(U-V)}\, I_4'(\operatorname{Im} \widetilde{\mathcal{V}}), \tag{15.3.6}$$

then the function U from

$$I_4(\operatorname{Im} \widetilde{\mathcal{V}}) = \frac{1}{16}\, \mathrm{e}^{4(V-U)}, \tag{15.3.7}$$

and finally the physical scalars from

$$\tau^i = \frac{\widetilde{L}^i}{\widetilde{L}^0} \tag{15.3.8}$$

(the overall rescaling are cancelling).

The solution has $2n_v$ charges since $\mathcal{Q}$ has $2n_v + 2$ components and there are two constraints, the Dirac condition (14.3.13d) and the constraint (15.2.27). This is the maximum number from the near-horizon analysis from [104].

As a conclusion, it is much easier to find a general solution using a symplectic formalism where the underlying structure simplifies the computations rather than choosing a particular model with electric gaugings.

15.4 Examples

Chapter 16

BPS AdS–NUT black holes

We focus on 1/4-BPS adS–NUT black holes. BPS equations for $N = 2$ FI gauged supergravity and several classes of analytical solutions were derived in [79].[1]

16.1 Ansatz

We consider the ansatz from section 13 where the metric and for the gauge fields are

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2U}\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big)^2 + \mathrm{e}^{-2U}\mathrm{d}r^2 + \mathrm{e}^{2(V-U)}\,\mathrm{d}\Sigma_g^2, \tag{16.1.1a}$$

$$A^\Lambda = \tilde{q}^\Lambda\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big) + \tilde{p}^\Lambda\,H(\theta)\,\mathrm{d}\phi. \tag{16.1.1b}$$

The functions $U, V, \tilde{q}$ and $\tilde{p}$ depend only on r, while Σ_g is a Riemann surface of genus g (see appendix A.7) with metric

$$\mathrm{d}\Sigma_g^2 = \mathrm{d}\theta^2 + H'(\theta)^2\,\mathrm{d}\phi^2, \qquad H'(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ 1 & \kappa = 0, \\ \sinh\theta & \kappa = -1. \end{cases} \tag{16.1.2}$$

All scalars are function only on r

$$\tau^i = \tau^i(r), \qquad q^u = q^u(r). \tag{16.1.3}$$

We consider only abelian gaugings.

The magnetic field strength reads

$$G_\Lambda = R_{\Lambda\Sigma}F^\Sigma - I_{\Lambda\Sigma}\star F^\Sigma. \tag{16.1.4}$$

The electric and magnetic charges are given explicitly by

$$p^\Lambda = \frac{1}{4\pi}\int_{\Sigma_g} F^\Lambda = \tilde{p}^\Lambda - 2n\tilde{q}^\Lambda, \tag{16.1.5a}$$

$$q_\Lambda = \frac{1}{4\pi}\int_{\Sigma_g} G_\Lambda = -\,\mathrm{e}^{2(V-U)}\,I_{\Lambda\Sigma}\tilde{q}'^\Sigma + \kappa\,R_{\Lambda\Sigma}p^\Sigma. \tag{16.1.5b}$$

Using these expressions one can rewrite the gauge field as

$$A^\Lambda = \tilde{q}^\Lambda\,\mathrm{d}t + p^\Lambda\,H(\theta)\,\mathrm{d}\phi, \tag{16.1.6}$$

[1] In this section we follow the conventions of [79]. The main difference is the replacement of Ω by $-\Omega$.

and finds again an expression for $\tilde{q}'^\Lambda$

$$\tilde{q}'^\Lambda = \mathrm{e}^{2(U-V)} I^{\Lambda\Sigma}\big(R_{\Sigma\Delta}p^\Delta - q_\Sigma\big). \tag{16.1.7}$$

The central and matter charges are[2]

$$\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V}\rangle\,, \qquad \mathcal{Z}_i = \langle \mathcal{Q}, U_i\rangle\,. \tag{16.1.8}$$

Similarly one defines the prepotential charges

$$\mathcal{L}^x = \langle \mathcal{P}^x, \mathcal{V}\rangle\,, \qquad \mathcal{L}^x_i = \langle \mathcal{P}^x, U_i\rangle\,. \tag{16.1.9}$$

16.2 BPS equations

For the following we consider FI gaugings and $n_h = 0$.

The Killing spinor has the same form (14.2.3) as for $n = 0$

$$\varepsilon_\alpha = \mathrm{e}^{U/2}\,\mathrm{e}^{i\psi/2}\,\varepsilon_{0\alpha}, \tag{16.2.1a}$$

$$\varepsilon_{0\alpha} = i\,\gamma^0\varepsilon_{\alpha\beta}\varepsilon_0^\beta, \tag{16.2.1b}$$

$$\varepsilon_{0\alpha} = -p^\Lambda P^x_\Lambda\,\gamma^{01}\sigma^x{}_\alpha{}^\beta\varepsilon_{0\beta}, \tag{16.2.1c}$$

$\varepsilon_{0\alpha}$ being a constant spinor.

The symplectic covariant equations are

$$\langle \mathcal{Q}, \mathcal{G}\rangle + 4n\,\mathrm{e}^U\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) = \varepsilon_D\,\kappa, \tag{16.2.2a}$$

$$\varepsilon_D\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) = \mathrm{e}^{2(U-V)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{3U-2V} \tag{16.2.2b}$$

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) &= \big(4n\,\mathrm{e}^U - 8\varepsilon_D\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &\quad - \mathcal{Q} - \varepsilon_D\,\mathrm{e}^{2(V-U)}\Omega\mathcal{M}\mathcal{G}, \end{aligned} \tag{16.2.2c}$$

$$(\mathrm{e}^V)' = -2\varepsilon_D\,\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}), \tag{16.2.2d}$$

$$\mathcal{Q}' = -2n\,\mathrm{e}^{2(U-V)}\,\Omega\mathcal{M}\mathcal{Q}. \tag{16.2.2e}$$

At the end one finds Maxwell equations, while the first one is a generalization of the Dirac condition.

We also have the equation for the real part of $\mathcal{V}$

$$2\,\partial_r\big(\mathrm{e}^U\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = -\mathcal{G} - \mathrm{e}^{2(U-V)}\Omega\mathcal{M}\mathcal{Q}. \tag{16.2.3}$$

Finally we recall the equations for ψ', U' and z'^i

$$\psi' = -A_r - 2\,\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) - n\,\mathrm{e}^{2(U-V)}, \tag{16.2.4a}$$

$$(\mathrm{e}^U)' = -\varepsilon_D\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}), \tag{16.2.4b}$$

$$(z^i)' = \mathrm{e}^{-U}\,\mathrm{e}^{i\psi}g^{i\bar{\jmath}}\big(\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\mathcal{Z} + i\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\big). \tag{16.2.4c}$$

The equation (16.2.2c) can be modified using (E.3.30e) to include one factor e^V inside the derivative

$$\begin{aligned} 2\,\mathrm{e}^V\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) &= 4\,\big(n\,\mathrm{e}^U - 2\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &\quad - 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &\quad - \mathcal{Q} - \mathrm{e}^{2(V-U)}\Omega\mathcal{M}\mathcal{G}. \end{aligned} \tag{16.2.5}$$

[2] There is a minus sign with respect to the notations of appendix A.6.

One can also use Maxwell equation (16.2.2e) to rewrite (16.2.3) as

$$2\,\partial_r\big(\,\mathrm{e}^{U}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = \frac{1}{2n}\,\mathcal{Q}' - \mathcal{G}. \tag{16.2.6}$$

It is then straightforward to integrate this equation

$$4n\,\mathrm{e}^{U}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) = \mathcal{Q} - 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} \tag{16.2.7}$$

where $\widehat{\mathcal{Q}}$ is the integration constant

$$\widehat{\mathcal{Q}} = \begin{pmatrix} P^{\Lambda} \\ Q_{\Lambda} \end{pmatrix}. \tag{16.2.8}$$

In turn one can use this to get the expression for $\mathcal{Q}$ if one knows the other quantities. Moreover plugging this result into Dirac quantization equation (E.3.30a) gives

$$\Big\langle \widehat{\mathcal{Q}}, \mathcal{G} \Big\rangle = \varepsilon_D\,\kappa \tag{16.2.9}$$

where the LHS is constant and $\widehat{\mathcal{Q}}$ corresponds to the conserved charges.

Finally one can use this expression for $\mathcal{Q}$ in order to rewrite the equations for $\mathrm{Im}\,\mathcal{V}$ (16.2.2c)

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\,\mathrm{e}^{-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = 8\big(n\,\mathrm{e}^{U} - \varepsilon_D\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ - 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} - \varepsilon_D\,\mathrm{e}^{2(V-U)}\Omega\mathcal{M}\mathcal{G}. \end{aligned} \tag{16.2.10}$$

and (16.2.5)

$$\begin{aligned} 2\,\mathrm{e}^{V}\partial_r\big(\,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = 8\,\big(n\,\mathrm{e}^{U} - \mathrm{e}^{2(V-U)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ - 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ - 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} - \mathrm{e}^{2(V-U)}\Omega\mathcal{M}\mathcal{G}. \end{aligned} \tag{16.2.11}$$

The main advantage is that $\mathcal{Q}$ has been replaced by the constant $\widehat{\mathcal{Q}}$, while the extra term $\mathcal{G}\,r$ is not a big problem.

Note that we can use (16.2.2b) in order to get an expression for $\mathrm{e}^{i\psi}$. This last expression will not help to solve the equation since it is complicated, but it means that we can always integrate the differential equation for the phase (16.2.4a), and we can obtain the expression if we know all other quantities. The result is[3]

$$\mathrm{e}^{i\psi} = -\frac{n\,\mathrm{e}^{3U-2V}}{\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\,\bar{\mathcal{Z}}} \pm 2\sqrt{\left(\frac{n\,\mathrm{e}^{3U-2V}}{\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\,\bar{\mathcal{Z}}}\right)^2 - \frac{\mathcal{L} + i\,\mathrm{e}^{2(U-V)}\,\mathcal{Z}}{\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\,\bar{\mathcal{Z}}}}. \tag{16.2.12}$$

which is a consequence of the second order equation

$$\mathrm{e}^{2i\psi}\big(\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\,\bar{\mathcal{Z}}\big) - 2n\,\mathrm{e}^{3U-2V}\,\mathrm{e}^{i\psi} + \big(\mathcal{L} + i\,\mathrm{e}^{2(U-V)}\,\mathcal{Z}\big) = 0 \tag{16.2.13}$$

obtained by writing explicitly the real and imaginary parts. For $n = 0$ it reduces to (14.3.5).

[3] To lighten notations we take $g_\Lambda \tilde{p}^\Lambda = \kappa$.

16.3 Symmetric $\mathcal{M}_v$ with FI gaugings

Using techniques similar to section 14.3.2 one obtains the following equations for symmetric cubic $\mathcal{M}_v$

$$2\,\mathrm{e}^V \partial_r \operatorname{Im}\widetilde{\mathcal{V}} = -\widehat{\mathcal{Q}} + \epsilon_D\, I_4'(\mathcal{P}, \operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}) + 2n\mathcal{P}r, \tag{16.3.1a}$$

$$(\,\mathrm{e}^V)' = -2\epsilon_D \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{P} \right\rangle, \tag{16.3.1b}$$

$$\mathrm{e}^V \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, \partial_r \operatorname{Im}\widetilde{\mathcal{V}} \right\rangle = \left\langle \operatorname{Im}\widetilde{\mathcal{V}}, \widehat{\mathcal{Q}} \right\rangle + 3n\,\mathrm{e}^V + 4nr \left\langle \mathcal{P}, \operatorname{Im}\widetilde{\mathcal{V}} \right\rangle, \tag{16.3.1c}$$

$$\left\langle \widehat{\mathcal{Q}}, \mathcal{P} \right\rangle = \varepsilon_D\, \kappa \tag{16.3.1d}$$

where we defined

$$\widetilde{\mathcal{V}} = \,\mathrm{e}^{V-U}\,\mathrm{e}^{-i\psi}\,\mathcal{V}. \tag{16.3.2}$$

16.4 Solutions

In this section we are looking for solutions of the previous equations. Following section 13.3 and the example of section 13.2, we will consider first extremal black holes (of general and CK types), and then solutions with complex roots. Indeed other cases do not seem to appear.

The derivation uses techniques that are similar to those described in section 15.3. In particular one imposes the near-horizon constraint (15.2.27), and the identities from appendix D.1 are used.

16.4.1 Pair of double roots

When there is a pair of double roots our ansatz is

$$\mathrm{e}^{2V} = r^2(v_4 r^2 + 2\sqrt{v_2 v_4}\, r + v_2), \tag{16.4.1}$$

$$\operatorname{Im}\widetilde{\mathcal{V}} = \frac{1}{\epsilon\sqrt{2\,\langle\mathcal{G}, A_1\rangle}}\, A_1 + A_3\, r \tag{16.4.2}$$

where (A_1, A_3) are symplectic vectors which we must determine and we include a sign $\epsilon = \pm 1$ to keep track of both branches of the square root. We have introduced this particular normalization of A_1 to make contact with expressions elsewhere. The IR and UV asymptotics completely fix the solution, the BPS equations then over-constrain this ansatz and for a solution to exist there must be significant cancellations.

We first solve the second equation of (16.3.1b) to get

$$\sqrt{v_2} = \epsilon\sqrt{2\,\langle\mathcal{G}, A_1\rangle}, \qquad \sqrt{v_4} = \langle\mathcal{G}, A_3\rangle\,, \tag{16.4.3}$$

and then expand the BPS equations (16.3.1a) in r to get

$$0 = I_4'(\mathcal{G}, A_3, A_3) - 2\,\langle\mathcal{G}, A_3\rangle\, A_3, \tag{16.4.4a}$$

$$0 = I_4'(\mathcal{G}, A_1, A_3) - 2\,\langle\mathcal{G}, A_1\rangle\, A_3 + n\kappa\epsilon\sqrt{2\,\langle\mathcal{G}, A_1\rangle}\,\mathcal{G}, \tag{16.4.4b}$$

$$0 = I_4'(\mathcal{G}, A_1, A_1) - 2\,\langle\mathcal{G}, A_1\rangle\,\mathcal{Q}. \tag{16.4.4c}$$

The constraint (16.3.1c) is also expanded and we get

$$0 = \sqrt{2}\,\langle A_1, A_3\rangle - n\kappa\epsilon\sqrt{\langle\mathcal{G}, A_1\rangle}, \tag{16.4.5a}$$

$$0 = \langle\mathcal{Q}, A_1\rangle + 2\,\langle A_1, A_3\rangle\,, \tag{16.4.5b}$$

$$0 = \sqrt{2}n\kappa\epsilon\,\langle\mathcal{G}, A_1\rangle^{3/2} + \langle\mathcal{G}, A_3\rangle\,\langle\mathcal{Q}, A_1\rangle + 2\,\langle\mathcal{G}, A_1\rangle\,\big(\,\langle\mathcal{Q}, A_3\rangle + \langle A_1, A_3\rangle\,\big), \tag{16.4.5c}$$

$$0 = \langle\mathcal{Q}, A_1\rangle\,. \tag{16.4.5d}$$

All the free parameters are fixed by the UV and IR asymptotics. From the UV we get

$$A_3 = \frac{I_4'(\mathcal{G})}{4I_4(\mathcal{G})^{1/4}}, \qquad v_4 = \sqrt{I_4(\mathcal{G})} \tag{16.4.6}$$

where we have appealed to [104] to fix the normalization of A_3. The solution for A_1, found from the IR equation (16.4.4c), is the same as in [105]

$$A_1 = a_1\, I_4'(\mathcal{G},\mathcal{G},\mathcal{G}) + a_2\, I_4'(\mathcal{G},\mathcal{G},\mathcal{Q}) + a_3\, I_4'(\mathcal{G},\mathcal{Q},\mathcal{Q}) + a_4\, I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q}) \tag{16.4.7}$$

with

$$a_1 = -\frac{a_3}{3}\,\frac{I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})}{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})}, \tag{16.4.8a}$$

$$a_2 = \frac{a_3}{6}\,\frac{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2}{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 I_4(\mathcal{Q}) - I_4(\mathcal{G})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2}, \tag{16.4.8b}$$

$$a_3 = \frac{9\big(I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})I_4(\mathcal{G}) - I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})I_4(\mathcal{Q})\big)}{I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})(\langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}), I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\rangle + \kappa I_4(\mathcal{G},\mathcal{G},\mathcal{Q},\mathcal{Q}))}, \tag{16.4.8c}$$

$$a_4 = -\frac{a_2}{3}\,\frac{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})}{I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})}. \tag{16.4.8d}$$

The effect of the NUT charge is through (16.4.4b) as well as the constraints (16.4.5a) and (16.4.5c). We find that these three equations are redundant and there is a single non-trivial constraint on the system

$$\begin{aligned} n\kappa\epsilon = &-\frac{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})}{144\sqrt{2}\,I_4(\mathcal{G})^{1/4}}\times \\ &\times\sqrt{\frac{18\,\langle\mathcal{G},\mathcal{Q}\rangle\, I_4(\mathcal{G},\mathcal{G},\mathcal{Q},\mathcal{Q}) - \langle I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q}), I_4'(\mathcal{G},\mathcal{G},\mathcal{G})\rangle}{\big(I_4(\mathcal{G})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2 - I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 I_4(\mathcal{Q})\big)^2 + 16 I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^3 I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3}}. \end{aligned} \tag{16.4.9}$$

When $n = 0$ then (16.4.9) is solved by $I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) = I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) = 0$ and the solutions reduce to those in [35, 96, 118].

16.4.2 Single double root

Only a single double root is required in e^{2V} in order to have an $\mathrm{adS}_2 \times \Sigma_g$ vacuum in the IR but this more general solution is somewhat more complicated. We found that in order to have a pair of double roots, there is a relation between the NUT charge and the electromagnetic charges (16.4.9), whereas there is no such constraint when requiring a single double root. The only constraint is that for $\mathrm{adS}_2 \times \Sigma_g$ vacua (15.2.27).

We take the same ansatz as in section 15.3

$$\mathrm{e}^{2V} = r^2(v_2 + v_3 r + v_4 r^2) \tag{16.4.10}$$

$$\mathrm{Im}\,\widetilde{\mathcal{V}} = \mathrm{e}^{-V}\widehat{A} \tag{16.4.11}$$

$$\widehat{A} = A_1 r + A_2 r^2 + A_3 r^3 \tag{16.4.12}$$

where A_i are constant symplectic vectors whose dependence on $\mathcal{G}$ and $\mathcal{Q}$ we seek to determine.

We first solve (16.3.1b) with

$$v_{i+1} = \frac{4}{i+1}\,\langle\mathcal{G}, A_i\rangle\,, \qquad i = 2,3,4. \tag{16.4.13}$$

The symplectic vector of BPS equations (16.3.1a) is then

$$2\,\mathrm{e}^{2V}\,\widehat{A}' - (\,\mathrm{e}^{2V})'\,\widehat{A} = I_4'(\mathcal{G},\widehat{A},\widehat{A}) + \,\mathrm{e}^{2V}\,(2n\mathcal{G}r - \mathcal{Q}) \tag{16.4.14}$$

which breaks up into five components from different powers of r

$$\begin{aligned}
0 &= I_4'(\mathcal{G},A_3,A_3) - 2\,\langle\mathcal{G},A_3\rangle\,A_3, && (16.4.15)\\
0 &= I_4'(\mathcal{G},A_2,A_3) + n\kappa\,\langle\mathcal{G},A_3\rangle\,\mathcal{G} - 2\,\langle G,A_2\rangle\,A_3, && (16.4.16)\\
0 &= 2I_4'(\mathcal{G},A_1,A_3) + I_4'(\mathcal{G},A_2,A_2) - 8\,\langle\mathcal{G},A_1\rangle\,A_3 - \langle\mathcal{G},A_3\rangle\,\mathcal{Q} && (16.4.17)\\
&\qquad + 2\,\langle\mathcal{G},A_3\rangle\,A_1 + \frac{4}{3}\,\langle\mathcal{G},A_2\rangle\,\big(2\mathcal{G} - A_2\big), &&\\
0 &= I_4'(\mathcal{G},A_1,A_2) + 2\,\langle\mathcal{G},A_1\rangle\,\big(n\kappa\mathcal{G} - A_2\big) + \langle\mathcal{G},A_2\rangle\,\big(A_1 - \mathcal{Q}\big), && (16.4.18)\\
0 &= I_4'(\mathcal{G},A_1,A_1) - 2\,\langle\mathcal{G},A_1\rangle\,\mathcal{Q}. && (16.4.19)
\end{aligned}$$

We also need to the expansion of the single real constraint (16.3.1c)

$$\begin{aligned}
O(r^4):&\quad 0 = 2\,\langle A_2,A_3\rangle - n\kappa\,\langle\mathcal{G},A_3\rangle\,, && (16.4.20)\\
O(r^3):&\quad 0 = 2\,\langle A_1,A_3\rangle + \langle\mathcal{Q},A_3\rangle\,, && (16.4.21)\\
O(r^2):&\quad 0 = \langle A_1,A_2\rangle + n\kappa\,\langle\mathcal{G},A_1\rangle + \langle\mathcal{Q},A_2\rangle\,, && (16.4.22)\\
O(r^1):&\quad 0 = 2\,\langle\mathcal{Q},A_1\rangle\,. && (16.4.23)
\end{aligned}$$

Note that once again, the highest order in r components of (16.4.14) and (16.3.1c) are independent of the NUT charge and therefore the solution for A_3 can be taken from [105]

$$A_3 = \frac{1}{4}\frac{I_4'(\mathcal{G})}{\sqrt{I_4(\mathcal{G})}}, \qquad v_4 = \sqrt{I_4(\mathcal{G})}. \tag{16.4.24}$$

We solve these equations with the ansatz

$$\begin{aligned}
A_1 &= a_1\,I_4'(\mathcal{G},\mathcal{G},\mathcal{G}) + a_2\,I_4'(\mathcal{G},\mathcal{Q},\mathcal{Q}) + a_3\,I_4'(\mathcal{G},\mathcal{Q},\mathcal{Q}) + a_4\,I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q}), && (16.4.25)\\
A_2 &= b_1\,I_4'(\mathcal{G},\mathcal{G},\mathcal{G}) + b_2\,I_4'(\mathcal{G},\mathcal{Q},\mathcal{Q}) + b_3\,I_4'(\mathcal{G},\mathcal{Q},\mathcal{Q}) + b_4\,I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q}), && (16.4.26)
\end{aligned}$$

where $\{a_i, b_j\}$ are real constants with a non-trivial dependence on $(\mathcal{G},\mathcal{Q})$. The IR conditions which give a_i in terms of $(\mathcal{G},\mathcal{Q})$ are the same we obtained for the case when e^{2V} had a pair of double roots and are thus given by (16.4.8a)-(16.4.8d).

Then from (16.4.18) we find the solution for $\{b_1, b_2, b_4\}$ in terms of b_3

$$\begin{aligned}
b_1 &= \frac{b_3 I_4(\mathcal{Q}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})}{3I_4(\mathcal{G}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})} - \frac{2b_3 I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})}{3I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})} + \frac{n\kappa I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2}{18\Pi_3} && (16.4.27a)\\
&\quad + \frac{b_3\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2}{54 I_4(\mathcal{G})\Pi_3}, &&\\
b_2 &= \frac{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\big(6nI_4(\mathcal{G})I_4(\mathcal{Q}) - b_3\Pi_2\big)}{6I_4(\mathcal{G})\Pi_3}, && (16.4.27b)\\
b_4 &= -\frac{I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\big(3nI_4(\mathcal{G}) + b_3 I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\kappa\big)}{9\Pi_3}. && (16.4.27c)
\end{aligned}$$

Finally from (16.4.17) we solve for b_3 and find the rather lengthy expression

$$b_3 = \frac{b_n}{b_d}$$

where the numerator and denominator are given by

$$
\begin{aligned}
b_n = {}& 6n\kappa I_4(\mathcal{G}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2 \left\langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}), I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\right\rangle \Pi_7 \\
& + 3\Big[- I_4(\mathcal{G})^{3/2} I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \Pi_3^2 \Pi_8 \Big[- 18 I_4(\mathcal{G}) \Pi_3^2 \\
& \quad + (\kappa + 4n^2 I_4(\mathcal{G})^{1/2}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^{1/2} I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \Pi_5 \\
& \quad - 8n^2 I_4(\mathcal{G})^{3/2} \big[144\kappa I_4(\mathcal{Q})^2 I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 - \kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3 \\
& \quad + 72 I_4(\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \Pi_6 \big] \Big] \Big]^{1/2}
\end{aligned}
\tag{16.4.28}
$$

and

$$
\begin{aligned}
b_d = {}& 8 I_4(\mathcal{G}) \Big[I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) \Big[2\kappa I_4(\mathcal{Q}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 (144 I_4(\mathcal{Q})^2 I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) \\
& - I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3) + I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) (288 I_4(\mathcal{Q})^2 I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) \\
& - I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3) \left\langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}), I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\right\rangle \\
& + 90\kappa I_4(\mathcal{Q}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2 \left\langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}), I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\right\rangle^2 \\
& + 9 I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3 \left\langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}), I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\right\rangle^3 \Big] + 18\kappa I_4(\mathcal{G}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2 \Pi_3 \Big] \\
& - 4\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^3 I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \Pi_5 .
\end{aligned}
\tag{16.4.29}
$$

We have used the notation

$$
\begin{aligned}
\Pi_1 &= I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \left\langle I_4'(\mathcal{G}), I_4'(\mathcal{Q})\right\rangle + 2\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{Q}), && \text{(16.4.30a)} \\
\Pi_2 &= I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) \left\langle I_4'(\mathcal{G}), I_4'(\mathcal{Q})\right\rangle + 2\kappa I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) I_4(\mathcal{Q}), && \text{(16.4.30b)} \\
\Pi_3 &= I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \left\langle I_4'(\mathcal{G}), I_4'(\mathcal{Q})\right\rangle + 4\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{Q}), && \text{(16.4.30c)} \\
\Pi_4 &= 2\kappa I_4(\mathcal{Q}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 + I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \Pi_1, && \text{(16.4.30d)} \\
\Pi_5 &= I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \left\langle I_4'(\mathcal{G}), I_4'(\mathcal{Q})\right\rangle + 2\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{Q}), && \text{(16.4.30e)} \\
\Pi_6 &= I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) \left\langle I_4'(\mathcal{G}), I_4'(\mathcal{Q})\right\rangle + 2\kappa I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) I_4(\mathcal{G}), && \text{(16.4.30f)} \\
\Pi_7 &= 2\kappa I_4(\mathcal{G}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2 + I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) \Pi_5, && \text{(16.4.30g)} \\
\Pi_8 &= 2\kappa I_4(\mathcal{G}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 + I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \Pi_6 . && \text{(16.4.30h)}
\end{aligned}
$$

These expression are fairly lengthy but in fact their derivation in Mathematica starting from (16.4.15)-(16.4.23) is quite straightforward when using the identities in appendix D.1. The $n \to 0$ limit of these expressions agrees with those found in [105].

16.4.3 Four independent roots

While extremal black holes necessarily have a double real root in e^{2V}, more general configurations are possible. For example we could have one or two pairs of complex conjugate roots. A natural ansatz for such solutions is

$$
\begin{aligned}
\mathrm{e}^{2V} &= v_0 + v_1 r + v_2 r^2 + v_4 r^4, && \text{(16.4.31a)} \\
\operatorname{Im} \widetilde{\mathcal{V}} &= \mathrm{e}^{-V} \widehat{A}, && \text{(16.4.31b)} \\
\widehat{A} &= A_0 + A_1 r + A_2 r^2 + A_3 r^3 . && \text{(16.4.31c)}
\end{aligned}
$$

We have used a shift symmetry in r to set $v_3 = 0$ but one cannot in general use a real shift in r to set $v_0 = 0$.

An example of such solutions is the constant scalar asymptotically adS_4 solution of section 13, corresponding to the STU-model with

$$P^0 = Q_i = P, \qquad Q_0 = -P^i = Q. \tag{16.4.32}$$

In our formalism we find this constant scalar example to be given by the following data

$$A_0 = \frac{n\kappa(P-1)}{2g}\,\mathcal{G} + \frac{n\kappa}{8g^3}\,I_4'(\mathcal{G}), \tag{16.4.33a}$$

$$A_1 = \frac{Q}{2g}\,\mathcal{G} + \frac{P-3gn^2}{8g^3}\,I_4'(\mathcal{G}), \tag{16.4.33b}$$

$$A_2 = \frac{n\kappa}{2}\,\mathcal{G}, \tag{16.4.33c}$$

$$A_3 = \frac{I_4'(\mathcal{G})}{4\sqrt{I_4(\mathcal{G})}}, \tag{16.4.33d}$$

and the metric is given by

$$\mathrm{e}^{2(V-U)} = r^2 + n^2 \tag{16.4.34a}$$

$$\mathrm{e}^{2V} = 2\Big(P^2 + Q^2 + g^2n^4 - 2gn^2P + 4gn\kappa Qr + 2(3gn^2 - gP)r^2 + gr^4\Big). \tag{16.4.34b}$$

The phase of the spinor is given by

$$\sin\psi = \mathrm{e}^{U-2V}\big(gr^3 + (-P + 3gn^2)r + n\kappa Q\big). \tag{16.4.35}$$

We have tried to obtain generalizations of this solution using the ansatz (16.4.31a)-(16.4.31c) but have not managed to decouple the set of algebraic equations. However this should not be seen as evidence that such solutions do not exist. Such solutions would not necessarily correspond to black holes since that requires the existence of a horizon. Since we expect BPS black holes to have extremal horizons, these solutions are covered by our analysis in section 16.4.2. Nonetheless looking ahead to possible extensions to Euclidean solutions, it is of some interest to have more general solutions with single real roots of e^{2V}.

Part VI

Extended supergravity

Chapter 17

Extended supergravity: introduction

17.1 General properties

Extended supergravity with $N > 2$ are very similar to $N = 2$ [5, sec. 3, 4, 42, sec. 2]. The description we are going to give will also (mostly) apply to the case $N = 2$ and will provide a broader view.

The scalar fields ϕ of extended supergravity describe a non-linear sigma model. For $N > 2$ the manifold is symmetric and can be written as the coset

$$\mathcal{M} = \frac{G}{H} \tag{17.1.1}$$

where G is the (non-compact) U-duality group and H its isotropy subgroup, which is a maximal compact subgroup.

Despite that some manifolds $\mathcal{M}$ for $N = 2$ are not coset (which is related to quantum corrections), the fact that the symplectic structure is still present allows to use the same formalism [5, sec. 3.1]. Other exceptions are $d = 4 N = 1$ and $d = 5$ $N = 2$ theories.

The isotropy group H is of the form

$$H = H_{\text{aut}} \times H_{\text{matter}} \tag{17.1.2}$$

where the first part is related to automorphism of the supersymmetry algebra, while the second is linked to the presence of matter multiplets [1] [5, sec. 3.1].

Tables with coset corresponding to $N \geq 3$ supergravities are given in tables 17.1 and 17.2, for references see [6, 10, tab. 4 p. 18, 68, 90, tab. 12.3 p. 250]. There are no scalars for pure $N = 2, 3$ supergravity, while for $N \geq 4$ the gravity multiplets possess scalars and has its own manifold (which is the only one for $N > 4$ when there are no matter multiplets). The number of scalar and vector fields is given in table 17.3 [10, tab. 4 p. 18].

N	4	5	6	7, 8
$\mathcal{M}_v$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$	$\frac{\mathrm{SU}(5,1)}{\mathrm{U}(5)}$	$\frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)}$	$\frac{\mathrm{E}_{7(7)}}{\mathrm{SU}(8)}$

Table 17.1: Scalar manifolds for pure $N \geq 4$ supergravity.

[1] By matter multiplets we also mean vector multiplets. This includes $N \leq 4$ for $d = 4, 5$.

N	3	4
$\mathcal{M}_v$	$\frac{\mathrm{SU}(3,n)}{\mathrm{U}(3)\times\mathrm{SU}(n)}$	$\frac{\mathrm{SO}(6,n)}{\mathrm{SU}(4)\times\mathrm{SO}(n)}$

Table 17.2: Scalar manifolds for n vector multiplets in $N = 3, 4$ supergravity.

N	3	4	5	6	7, 8
n_v	$3 + n$	$6 + n$	10	16	28
m	$6n$	$6n + 2$	10	30	70

Table 17.3: Number of scalar and vector fields in extended supergravity. n is the number of vector multiplets.

The U-duality group is non-compact and contains as a subgroup the product of the T- and S-duality groups of the higher-dimensional string theory [70]. At the quantum level the U-duality group becomes discrete

$$G(\mathbb{R}) \longrightarrow G(\mathbb{Z}). \tag{17.1.3}$$

In general the scalar potential has exponentials of fields associated to the Cartan elements, and polynomials in fields of the nilpotent ones [74, sec. 3].

17.2 Lagrangian and supersymmetric variations

A general construction of the Lagrangian and of the supersymmetric variations can be found in [71, 84, app. A, 158] (see also [10, sec. 3, 88]). General techniques for constructing the Lagrangian and studying the U-duality can be found in [92] (see also [10, sec. 3, 95, chap. 2]).

The field strengths transform in a linear representation of G.

The scalar potential is related to the shift of the supersymmetry through the (schematic) formula [90, p. 290]

$$V = (\delta_{\text{susy}}\psi)(\text{metric})(\delta_{\text{susy}}\psi). \tag{17.2.1}$$

17.3 Symplectic and duality invariants

For general definitions see [42, sec. 2, 47, pp. 2–3, 84].

Symplectic invariants are defined as quantities that do not change under the action of a symplectic transformation of the various symplectic vectors (such as the sections and the charges). When the scalars are described by a symmetric space G/H this corresponds to H-invariance. A duality invariant is an object that is G-invariant (and thus H-invariant). As a consequence it does not depend on the moduli. They are built from combination of H-invariant objects. In $d = 4$ extended supergravity all groups G are of type E_7 and admit a quartic invariant (except for $N = 2$ quadratic model and $N = 3$).

The quartic duality invariant of $N = 8$ is discussed in [70] (which also provide a discussion of invariants in maximal supergravities in higher dimensions: in particular there is a cubic invariant for $d = 5$, and no invariant for $d > 5$).

The invariants for all $N > 2$ are constructed in [6].

17.4 Extremal black holes

For reviews see [43, 88].

Extremal black holes can be classified according to orbits of the charges under the group G [42].

An important quantity is the black hole effective potential [5, sec. 6, 42, 76]

$$V_{\mathrm{BH}} = -\frac{1}{2}\,\mathcal{Q}^t \mathcal{M} \mathcal{Q} \tag{17.4.1}$$

($\mathcal{Q}$ being the symplectic charges) which arises for example by reduction to $3d$. For $N = 2$ one has

$$V_{\mathrm{BH}} = |\mathcal{Z}|^2 + |\mathcal{Z}_i|^2. \tag{17.4.2}$$

Critical points $\phi^* = \phi^*(\mathcal{Q})$ minimize the potential

$$\frac{\partial V_{\mathrm{BH}}}{\partial \phi}(\phi^*) = 0, \tag{17.4.3}$$

and the entropy reads

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{BH}}}{4} = \pi\, V_{\mathrm{BH}}(\phi^*) = \pi\,\sqrt{|I_4(\mathcal{Q})|} \tag{17.4.4}$$

where $I_4(\mathcal{Q})$ is the duality invariant (for symmetric $\mathcal{M}_v$).

Multicenter solutions are studied in [87].

At the attractor point the potential is Freudenthal invariant (see section 7.4.2) [86]. This duality can be extended to non-symmetric (and even non-homogeneous) manifold by replacing $\sqrt{|I_4(\mathcal{Q})|}$ by the entropy $S(\mathcal{Q})$ in (7.4.10)

$$\mathfrak{f}(\mathcal{Q})^M = \Omega^{MN}\frac{\partial S(\mathcal{Q})}{\partial \mathcal{Q}^N}. \tag{17.4.5}$$

In particular this implies the relation

$$S(\mathcal{Q}) = S\left(\Omega^{-1}\,\frac{\partial S}{\partial \mathcal{Q}}\right). \tag{17.4.6}$$

In [86] the authors propose an interpolating Freudenthal duality which depends on scalar fields when $\phi \neq \phi^*$.

17.5 Entropy relations

Entropy sum formulas in gauged supergravity is commented in [177, sec. 5].

Let's consider the $d = 4$ solution

$$\begin{aligned} \mathrm{d}s^2 = &- \frac{\Delta_r}{W}\left(\mathrm{d}t - \frac{a\sin^2\theta}{\Xi}\,\mathrm{d}\phi\right)^2 + W\left(\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}\theta^2}{\Delta_\theta}\right) \\ &+ \frac{\Delta_\theta \sin^2\theta}{W}\left(a\,\mathrm{d}t - \frac{r_1 r_2 + a^2}{\Xi}\,\mathrm{d}\phi\right)^2 \end{aligned} \tag{17.5.1}$$

where

$$\begin{aligned} r_\alpha &= r + 2m\sinh^2\delta_\alpha, && (17.5.2a)\\ \Delta_r &= r^2 + a^2 - 2mr + g^2\, r_1 r_2 (r_1 r_+ a^2), && (17.5.2b)\\ \Delta_\theta &= 1 - g^2 a^2 \cos^2\theta, && (17.5.2c)\\ W &= r_1 r_2 + a^2\cos^2\theta. && (17.5.2d) \end{aligned}$$

Horizons are given by $\Delta_r(r_i) = 0$ and the entropy reads

$$S_i = \frac{A_i}{4}, \qquad A_i = \frac{4\pi}{\Xi}(r_{1i}r_{2i} + a^2). \tag{17.5.3}$$

Then one obtains

$$\sum_{i=1}^{4} S_i = -\frac{2\pi}{g^2}. \tag{17.5.4}$$

This result is independent of the conserved charges m, a and δ_α.

A similar result can be obtained for $d = 6$, and it is possible to see that examples in $d = 5, 7$ have vanishing sum.

17.6 $N = 8$ supergravity

This theory is nicely summarized in [185, sec. 2] (see also [74]).

We have

$$G = \mathrm{E}_{7(7)}, \qquad H = \mathrm{SU}(8), \tag{17.6.1}$$

and

$$\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6, 6) \subset \mathrm{E}_{7(7)} \tag{17.6.2}$$

is a maximal compact subgroup, where $\mathrm{SL}(2, \mathbb{R})$ is the S-duality group, and $\mathrm{SO}(6, 6)$ is the T-duality group. Note that the maximal compact subgroup of the T-duality group is

$$\mathrm{SO}(6) \times \mathrm{SO}(6) \sim \mathrm{SU}(4) \times \mathrm{SU}(4) \subset \mathrm{SU}(8), \tag{17.6.3}$$

while the maximal subgroup of the S-duality is

$$\mathrm{SO}(2) \subset \mathrm{SL}(2, \mathbb{R}). \tag{17.6.4}$$

Transformations of G do not preserve the Lagrangians but only the equations of motion [75].

The symplectic group is $\mathrm{Sp}(56, \mathbb{R})$ and one has

$$\mathrm{E}_{7(7)} \subset \mathrm{Sp}(56, \mathbb{R}). \tag{17.6.5}$$

The classification of the possible gaugings embedded in $\mathrm{SL}(8, \mathbb{R})$ has been proved in [65].

Finally it was shown in [75] that the $N = 8$ $\mathrm{SO}(8)$ gauged supergravity admits a one-parameter extension $\mathrm{SO}(8)_\omega$ where

$$\omega \in [0, \pi/8]. \tag{17.6.6}$$

Additional studies can be found in [29, 185].

This theory can be obtained by compactification of $d = 11$ supergravity on S^7. It is a challenging problem to understanding the role of the ω-parameter in this higher-dimensional setting.

The most general non-extremal black hole in $N = 8$ supergravity can be generated using U-duality from a seed which corresponds to the $N = 2$ STU black hole [55, 57]. The entropy of this black hole was written in terms of $\mathrm{E}_{7(7)}$ invariants in [63].

17.7 $N = 6$ supergravity

One has

$$G = \mathrm{SO}^*(12), \qquad H = \mathrm{U}(6). \tag{17.7.1}$$

Seen as a truncation of $N = 8$ supergravity, it exists a ω-deformation of $N = 6$ gauged supergravity with gauge group $\mathrm{SO}(6) \times \mathrm{U}(1)$ [29] (this possibility was already suggested in [75]).

17.8 $N = 4$ supergravity

This theory is described in [77, 131].

The ω-deformation of $N = 4$ SO(4) gauged supergravity is studied in [132, sec. 4], where it is shown that this parameter is not relevant as it can be absorbed by an isometry.

Part VII

Appendices

Appendix A

Conventions

A.1 Generalities

We mostly follow the conventions of [90] (see also [8, app. C]).

Greek indices are curved, roman indices are flat (local Lorentz). Specific names for curved indices are given, such as (t, r, θ, ϕ), and numbers are reserved for flat indices, such as $(0, 1, 2, 3)$. In most of the text we use Planck units

$$8\pi G = \hbar = c = k = 1. \tag{A.1.1}$$

The signature of spacetime metric

$$\eta_{ab} = \epsilon_\eta \ \mathrm{diag}(-1, 1, 1, 1) \tag{A.1.2}$$

is taken to be mostly plus $\epsilon_\eta = 1$. The Levi–Civita symbol ε_{abcd} (in flat indices) is

$$\varepsilon_{0123} = \epsilon_\varepsilon, \qquad \varepsilon^{0123} = -\epsilon_\varepsilon \tag{A.1.3}$$

and we will use $\epsilon_\varepsilon = 1$.

Given a Lagrangian $\mathcal{L}$ the action reads

$$S = \int \mathrm{d}^d x \sqrt{-g}\, \mathcal{L}. \tag{A.1.4}$$

Partial derivatives are abbreviated as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{A.1.5}$$

The (anti)symmetrization is done with unit weight

$$A_{[ab]} = \frac{1}{2}\,(A_{ab} - A_{ba}), \qquad A_{(ab)} = \frac{1}{2}\,(A_{ab} + A_{ba}). \tag{A.1.6}$$

We summarize the number of degrees of freedom in tables A.1 and A.2.

field	spin	off-shell	on-shell
ϕ	0	1	1
λ	1/2	$2^{\lfloor d/2 \rfloor}$	$2^{\lfloor d/2 \rfloor - 1}$
A_μ	1	$d-1$	$d-2$
ψ_μ	3/2	$(d-1)\, 2^{\lfloor d/2 \rfloor}$	$(d-3)\, 2^{\lfloor d/2 \rfloor - 1}$
$g_{\mu\nu}$	2	$\frac{1}{2}\, d(d-1)$	$\frac{1}{2}\, d(d-3)$

Table A.1: Degrees of freedom off-shell and on-shell for the fields with spin ≤ 2 [90, tab. 6.2].

field	spin	off-shell	on-shell
ϕ	0	1	1
λ	1/2	4	2
A_μ	1	3	2
ψ_μ	3/2	12	2
$g_{\mu\nu}$	2	6	2

Table A.2: Degrees of freedom off-shell and on-shell for the fields with spin ≤ 2 for $d = 4$.

fields	$\psi^\alpha_\mu, \lambda^{\alpha i}$	X^Λ, A^Λ_μ	$A^i_\mu, \lambda^{\alpha i}, \tau^i$	$\zeta^{\mathcal{A}}$	q^u	Z^A, ξ^A	z^a	σ^x
here	α	Λ	i	$\mathcal{A}$	u	A	a	x
[90]	i	I	α	A	u			
[7, 8]	A	Λ	i	α	u			
[40]		A	a			I	i	
range	$1, 2$	$0, \dots, n_v$	$1, \dots, n_v$	$1, \dots, 2n_h$	$1, \dots, 4n_h$	$0, \dots, n_h$	$1, \dots, n_h$	$1, 2, 3$

Table A.3: Indices of the $N = 2$ fields in various conventions. n_v and n_h are the numbers of vector and hypermultiplets. The last column x corresponds to SU(2) index (σ^x are the Pauli matrices).

signs	ϵ_η	ϵ_ε	ϵ_Ω	$\epsilon_\mathbb{C}$
here	+1	+1	+1	+1
[90]	+1	+1	+1	
[7, 8]	−1			
[40]			+1	+1

Table A.4: Sign conventions. For other comparisons of conventions see [28, problem C.1, p. 449–453, 90, app. A].

A.2 Differential geometry

Given a metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu, \tag{A.2.1}$$

the Christoffel symbol and the Riemann tensor are

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2}\, g^{\mu\sigma}\big(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\big), \tag{A.2.2}$$

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\rho\tau}\Gamma^\tau{}_{\nu\sigma} - \Gamma^\mu{}_{\sigma\tau}\Gamma^\tau{}_{\nu\rho}. \tag{A.2.3}$$

The Ricci tensor and the curvature are

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}. \tag{A.2.4}$$

A manifold is said to be Einstein if

$$R_{\mu\nu} = \Lambda\, g_{\mu\nu}, \qquad \Lambda = \frac{R}{d}, \tag{A.2.5}$$

d being the spacetime dimension. In the case $\Lambda = 0$ it is said to be Ricci flat

$$R_{\mu\nu} = 0. \tag{A.2.6}$$

A manifold is symmetric if the Riemann tensor is covariantly constant

$$\mathrm{D}_\tau R_{\mu\nu\rho\sigma} = 0. \tag{A.2.7}$$

A Killing vector k_μ generates an isometry of the corresponding manifold and it is defined by the equation

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0. \tag{A.2.8}$$

A p-form A_p with components $A_{\mu_1\cdots\mu_p}$ is defined by

$$A_p = \frac{1}{p!}\, A_{\mu_1\cdots\mu_p}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{A.2.9}$$

The exterior derivative d is nilpotent and maps a p-form into a $(p+1)$-form (example with a 1-form)

$$F = \mathrm{d}A = \partial_\mu A_\nu\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{A.2.10a}$$

$$F_{\mu\nu} = 2\,\partial_{[\mu} A_{\nu]}. \tag{A.2.10b}$$

The interior derivative i_k by a vector k maps a p-form into a $(p-1)$-form (example with a 1-form)

$$i_k A = k \lrcorner A = k^\mu A_\mu. \tag{A.2.11}$$

The Lie derivative $\mathcal{L}_k$ acting on forms is defined as

$$\mathcal{L}_k = i_k \mathrm{d} + \mathrm{d} i_k \tag{A.2.12}$$

and it commutes with the differential [159, sec. 4.21]

$$[\mathcal{L}_k, \mathrm{d}] = 0. \tag{A.2.13}$$

The integration of a d-form A reads

$$\int A = \frac{1}{d!} \int A_{\mu_1\cdots\mu_d}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_d} = \int A_{0\cdots D-1}\, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{d-1}. \tag{A.2.14}$$

Levi–Civita tensor is given in curved coordinates by

$$\varepsilon_{\mu_1\cdots\mu_d} = e^{-1}\, e^{a_1}_{\mu_1} \cdots e^{a_d}_{\mu_d}\, \varepsilon_{a_1\cdots a_d}, \qquad \varepsilon^{\mu_1\cdots\mu_d} = e\, e^{\mu_1}_{a_1} \cdots e^{\mu_d}_{a_d}\, \varepsilon^{a_1\cdots a_d}, \tag{A.2.15}$$

where e^a_μ is the vielbein. Contraction of two symbols is

$$\varepsilon_{\mu_1\cdots\mu_n \nu_1\cdots\nu_p} \varepsilon^{\mu_1\cdots\mu_n \rho_1\cdots\rho_p} = -n!p!\, \delta_{\nu_1}{}^{[\rho_1} \cdots \delta_{\nu_p}{}^{\rho_p]}. \tag{A.2.16}$$

Using this tensor one can define the Hodge operation

$$\star(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) = \frac{\sqrt{-g}}{(d-p)!}\, \varepsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_d}\, \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_d}, \tag{A.2.17a}$$

$$\star(e^{a_1} \wedge \cdots \wedge e^{a_p}) = \frac{1}{(d-p)!}\, \varepsilon^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_d}\, e^{a_{p+1}} \wedge \cdots \wedge e^{a_d}, \tag{A.2.17b}$$

and the dual of a p-form will produce a $(d-p)$-form. This operation squares to -1

$$\star\star F = -F. \tag{A.2.18}$$

One has the formula

$$\int \star F^{(p)} \wedge F^{(p)} = \frac{1}{p!} \int \mathrm{d}^d x\, \sqrt{-g}\, F^{\mu_1\cdots\mu_p} F_{\mu_1\cdots\mu_p}. \tag{A.2.19}$$

In particular the dual of a 2-form for $d = 4$ is denoted by [90, sec. 4.2.1]

$$\tilde{H}^{ab} = -\frac{i}{2}\,\varepsilon^{abcd} H_{cd} = -i \star F^{\mu\nu}. \tag{A.2.20}$$

Them one can define the self-dual and anti-self-dual of this tensor as

$$H^{\pm}_{ab} = \frac{1}{2}(H_{ab} \pm \tilde{H}_{ab}) \tag{A.2.21}$$

with the properties

$$H^{\pm}_{ab} = \pm \tilde{H}^{\pm}_{ab}, \qquad H^{\pm}_{ab} = (H^{\mp}_{ab})^*. \tag{A.2.22}$$

Moreover the dual operation is an involution (thanks to the factor i). In the curved frame one has

$$\star F_{\mu\nu} = \frac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \qquad \star F^{\mu\nu} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{A.2.23}$$

Given two tensors F and G one has the following identities

$$\tilde{F}^{\mu\nu}\tilde{G}_{\mu\nu} = F_{\mu\nu} G_{\mu\nu}, \qquad \tilde{F}^{\mu\nu} G_{\mu\nu} = F_{\mu\nu}\tilde{G}_{\mu\nu}, \tag{A.2.24a}$$

$$F^{+}_{\mu\nu} G^{-\mu\nu} = 0, \qquad g^{\mu\nu} F^{+}_{\mu[\rho} G^{-}_{\sigma]\nu} = 0, \qquad g^{\mu\nu} F^{+}_{\mu(\rho} G^{+}_{\sigma)\nu} = -\frac{1}{4}\, g_{\rho\sigma}\, F^{+}_{\mu\nu} G^{+\mu\nu}. \tag{A.2.24b}$$

A.3 Symplectic geometry

Let's consider a space of dimension $2n$. We use indices $M, N = 1, \dots, 2n$.

Define the 2-dimensional antisymmetric matrix

$$\varepsilon = \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.3.1}$$

where $\epsilon = \pm 1$. Then the (flat) symplectic metric is defined by

$$\omega = \varepsilon \otimes 1_n = \epsilon \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \tag{A.3.2}$$

1_n denoting the n-dimensional identity. An alternative representation is the block-diagonal form

$$\omega' = 1_n \otimes \varepsilon = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \tag{A.3.3}$$

The symplectic metric squares to -1

$$\omega^2 = -1 \tag{A.3.4}$$

and the inverse is simply $-\omega$

$$\omega^{-1} = -\omega. \tag{A.3.5}$$

Let's consider a vector with contravariant components A^M. Three different conventions exist:

1. The NW–SE convention [8, app. C, 90, p. 421, 471]

$$\omega_{MN}\omega^{NP} = -\delta_M{}^P, \qquad A_M = -\epsilon\,\omega_{MN} A^N, \qquad A^M = \epsilon\,\omega^{MN} A_N. \tag{A.3.6}$$

This implies that $\omega^{MN} = \omega_{MN}$ (in components) and ω^{MN} does *not* correspond to the components of ω^{-1}.

2. The susy convention [28]

$$\omega_{MN}\omega^{NP} = \delta_M{}^P, \qquad A_M = \omega_{MN}A^N, \qquad A^M = \omega^{MN}A_N. \tag{A.3.7}$$

3. The SE–NW convention

$$\omega_{MN}\omega^{NP} = \delta_M{}^P, \qquad A_M = A^N\omega_{NM}, \qquad A^M = A_N\omega^{NM}. \tag{A.3.8}$$

The latter would be the more logical because ω^{MN} are the components of ω^{-1}, while the covariant vector appears on the left in the scalar product. Since most supergravity papers uses the first convention we will follow it. In particular with $\epsilon = 1$ this implies

$$A_M = A^N\omega_{NM}, \qquad A^M = \omega^{MN}A_N. \tag{A.3.9}$$

and the symplectic inner product of two vectors A and B is

$$\langle A, B\rangle = A^M\omega_{MN}B^N = A_M B^M. \tag{A.3.10}$$

In the course of this review we will have several different symplectic spaces: Ω, $\mathbb{C}$, ε. Each will have a different sign ϵ_Ω, $\epsilon_\mathbb{C}$, etc. We choose $\epsilon_\Omega = \epsilon_\mathbb{C} = 1$. The sign is reversed with respect to [7, 8, 79, 96, 104, 118], but the same as in [39, 78, 90, 104].

A.4 Gamma matrices

Gamma matrices form a Clifford algebra

$$[\gamma_\mu, \gamma_\nu] = 2\, g_{\mu\nu}, \qquad [\gamma_a, \gamma_b] = 2\, \eta_{ab}. \tag{A.4.1}$$

The Hermitian conjugate of γ^μ is

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{A.4.2}$$

Antisymmetric products are denoted by

$$\gamma_{a_1\cdots a_n} = \gamma_{[a_1}\cdots\gamma_{a_n]}. \tag{A.4.3}$$

Finally in four dimensions one defines

$$\gamma_5 = i\,\gamma_0\gamma_1\gamma_2\gamma_3, \qquad \varepsilon_{abcd}\gamma^d = i\,\gamma_5\gamma_{abc}. \tag{A.4.4}$$

The left and right projectors are defined by

$$P_L = \frac{1+\gamma_5}{2}, \qquad P_R = \frac{1-\gamma_5}{2}. \tag{A.4.5}$$

A.5 Spinors

Given a Majorana spinor ϵ^α, the chiral left and right Weyl spinors are denoted by [108, sec. 2.1]

$$\varepsilon_\alpha = P_L\,\epsilon^\alpha, \qquad \varepsilon^\alpha = P_R\,\epsilon^\alpha. \tag{A.5.1}$$

The Majorana and Dirac conjugates are

$$\bar\lambda = \lambda^t C, \qquad \bar\lambda = i\,\lambda^\dagger\gamma^0. \tag{A.5.2}$$

The charge conjugation is

$$\lambda^C = B^{-1}\lambda^*, \qquad B = iC\gamma^0. \tag{A.5.3}$$

The matrix C satisfy

$$C^2 = -1, \qquad C^t = -C, \qquad (C\gamma^\mu)^t = C\gamma^\mu. \tag{A.5.4}$$

A.6 Supergravity

Given a Lagrangian $\mathcal{L}$ the dual of the field strength F^Λ is defined by

$$G_\Lambda = \star\left(\frac{\delta\mathcal{L}}{\delta F^\Lambda}\right). \tag{A.6.1}$$

The electric and magnetic charges q_Λ and p^Λ contained in a volume V with boundary Σ are defined by

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \frac{1}{\mathrm{Vol}(\Sigma)} \int_\Sigma \mathcal{F} \tag{A.6.2}$$

where $\mathcal{F} = (F^\Lambda, G_\Lambda)$ are the field strengths. The charges are defined as densities to avoid infinite charges in the case of non-compact surfaces. For compact horizons one takes

$$\mathrm{Vol}(\Sigma) = \mathrm{Vol}(S^2) = 4\pi. \tag{A.6.3}$$

The central charge is defined by

$$\mathcal{Z} = \epsilon_\Omega\, \Gamma(\mathcal{Q}) = \epsilon_\Omega\, \langle \mathcal{V}, \mathcal{Q}\rangle \tag{A.6.4a}$$

$$= L^\Lambda q_\Lambda - M_\Lambda p^\Lambda = \mathrm{e}^{K/2}\, (X^\Lambda q_\Lambda - F_\Lambda p^\Lambda). \tag{A.6.4b}$$

Note that there is a factor 2 in [90, p. 480].[1]

Similarly one defines

$$\mathcal{L}^x = \epsilon_\Omega\, \Gamma(\mathcal{P}^x) = \epsilon_\Omega\, \langle \mathcal{V}, \mathcal{P}^x\rangle \tag{A.6.5a}$$

$$= L^\Lambda P^x_\Lambda - M_\Lambda \widetilde{P}^{x\Lambda}. \tag{A.6.5b}$$

A.7 Topological horizons

Black hole horizons correspond to 2-dimensional (θ, ϕ) sections Σ with spherical, planar or hyperbolic topology [4, 36]. The sign of the curvature is denoted by κ and correspond to

$$\kappa = \begin{cases} +1 & \text{spherical}, \\ 0 & \text{planar}, \\ -1 & \text{hyperbolic}. \end{cases} \tag{A.7.1}$$

In the case $\kappa = 0, -1$ the horizon is non-compact and the full solution describes a black membrane [36].

For a static spacetime the 2-dimensional section is maximally symmetric. The corresponding spaces are the sphere S^2, the euclidean plane $\mathbb{R}^2$ and the hyperboloid H^2 respectively for positive, vanishing and negative curvature (see table A.5). In these cases the uniform metric on Σ reads

$$\mathrm{d}\Sigma^2 = \mathrm{d}\theta^2 + H'(\theta)^2\, \mathrm{d}\phi^2 \tag{A.7.2}$$

where

$$H(\theta) = \begin{cases} -\cos\theta & \kappa = 1, \\ \theta & \kappa = 0, \\ \cosh\theta & \kappa = -1, \end{cases} \qquad H'(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ 1 & \kappa = 0, \\ \sinh\theta & \kappa = -1. \end{cases} \tag{A.7.3}$$

[1] For $\epsilon_\Omega = -1$ one writes $\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V}\rangle$.

The function $H(\theta)$ may be defined by the differential equation

$$H'' + \kappa H = 0, \qquad H(0) = 0, \quad H'(0) = 1. \tag{A.7.4}$$

Another parametrization of the metric is $\chi = H'$

$$\mathrm{d}\Sigma^2 = \frac{\mathrm{d}\chi^2}{1 - \kappa\chi^2} + \chi^2\, \mathrm{d}\phi^2. \tag{A.7.5}$$

The interest of these coordinates is to remove trigonometric/hyperbolic functions for symbolic computations.

topology	Σ	κ	$\mathrm{ISO}(\Sigma)$
spherical	S^2	$+1$	$\mathrm{SO}(3)$
planar	$\mathbb{R}^2$	0	$\mathbb{R}^2$
cylindrical	$\mathbb{R} \times S^1$	0	$\mathbb{R} \times \mathrm{SO}(2)$
toroidal	T^2	0	$\mathrm{SO}(2)^2$
hyperbolic	H^2	-1	$\mathrm{SO}(2,1)$
Riemann surface ($g > 1$)	Σ_g	-1	$\mathrm{SO}(2,1)/\Gamma$

Table A.5: Horizon topology for static spacetime. The last row corresponds to hyperbolic Riemann surfaces; non-hyperbolic surfaces are the sphere S^2 for $g = 0$ and the torus T^2 for $g = 1$.

By definition black holes have a compact (orientable) horizon. These can be obtained by taking the quotient of the isometry group $\mathrm{ISO}(\Sigma)$ by a discrete subgroup Γ. In this case taking the quotient is a global effect and does not affect the fields, and in particular one can work with the above metric. An intermediate case corresponds to a cylindrical black hole with horizon $\mathbb{R} \times S^1$ when the direction ϕ is made compact using the quotient $\mathbb{R}/\mathbb{Z}$. Compact horizons are Riemann surfaces Σ_g where $g \in \mathbb{N}$ denotes the genus. The sphere $g = 0$ is already compact so one does not need to take a quotient. The surface $g = 1$ corresponds to the 2-torus $T^2 \sim S^1 \times S^1$ obtained by the quotient $(\mathbb{R}/\mathbb{Z})^2$, while higher genus surfaces $g > 1$ are obtained by taking the quotient of H^2 by a Fuchsian group Γ, which is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ (see table A.5). The sign of the curvature reads

$$\kappa = \mathrm{sign}(1 - g). \tag{A.7.6}$$

If the black hole is spinning then Σ is deformed as the isometry group is reduced. For example in the case of spherical topology one obtains a spheroid and the isometry is only $\mathrm{SO}(2)$ (corresponding to the Killing vector ∂_ϕ). In particular it is not possible to have a Riemann surface with $g > 0$, but a planar horizon can be reduced to a cylindrical horizon with $\phi \in [0, 2\pi]$ thanks to the ∂_ϕ isometry.

Appendix B

Higher-dimensional supergravities

The number of scalars in the vector multiplets is dimension-dependent (as well as dependent of the number of supercharges).

Some general definitions (charges associated to p-forms, central charges, supersymmetry variations…) can be found in [5, sec. 3.1].

The link between the various theories in different dimensions and their duality groups can be found in [5, sec. 2].

Extended supergravities present central charges that are not Lorentz invariant [5]. Their presence is linked to the existencce of p-branes.

Five-dimensional $N = 2$ sugra was studied in [102], where the Lagrangian and supersymmetric variations are derived, along the possible vector scalar manifolds. Vector multiplets contain only one (real) scalar field, meaning that the corresponding target space is real. Upon dimensional reduction one gets the so-called special real Kähler manifold since, despite the facts that the fields are complex, some properties are linked to the higher-dimensional real manifold [102].

For a list of the scalar manifolds, see [90, tab. 12.3 p. 250] (see also [158, tab. 1 p. 6]).

In higher dimensions not all groups are of E_7-type, which implies that duality invariants are not necessarily quartic [84, sec. 2.1].

Appendix C

Group theory

C.1 Group classification

For some elements see [90, app. B].

C.1.1 Symplectic groups

Given a vector space of dimension $2n$ over a field $\mathbb{K}$ endowed with a skew-symmetric product defined by the 2-form Ω, the set of transformations that preserve this product define the symplectic group $\mathrm{Sp}(2n, \mathbb{K}) \subset \mathrm{SL}(2n, \mathbb{K})$

$$S \in \mathrm{Sp}(2n, \mathbb{K}) \Longrightarrow S^t \Omega S = \Omega. \tag{C.1.1}$$

There are three possible symplectic groups: $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{Sp}(n) \equiv \mathrm{USp}(2n)$. The first two are non-compact while the latter is compact: $\mathrm{USp}(2n)$ is the compact form of $\mathrm{Sp}(2n, \mathbb{R})$, both which being real Lie groups. On the other hand $\mathrm{Sp}(2n, \mathbb{C})$ is complex. They all have n generators and are of dimensions (real or complex) $n(2n+1)$.

The Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$ corresponds to the semi-simple complex algebra C_n, while the others are real forms: $\mathfrak{usp}(n)$ is the compact form and $\mathfrak{sp}(2n, \mathbb{R})$ is the normal (or split) form.

The compact group is isomorphic to

$$\mathrm{U}(n, \mathbb{H}) \equiv \mathrm{USp}(2n) \sim \mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C}). \tag{C.1.2}$$

Note also the isomorphism

$$\mathfrak{sp}(1) \sim \mathfrak{su}(2) \sim \mathfrak{so}(3), \qquad \mathfrak{sp}(2) \sim \mathfrak{so}(5) \tag{C.1.3}$$

Group	Matrices	Group type	compact	π_1
$\mathrm{Sp}(2n, \mathbb{R})$	$\mathbb{R}$	real	no	$\mathbb{Z}$
$\mathrm{Sp}(2n, \mathbb{C})$	$\mathbb{C}$	complex	no	1
$\mathrm{Sp}(n) \equiv \mathrm{USp}(2n)$	$\mathbb{H}$	real	yes	1

Table C.1: Symplectic groups.

C.1.2 Groups on quaternions

Several matrix groups on the quaternions can be defined

$$\mathrm{SO}^*(2n) = \mathrm{O}(n, \mathbb{H}), \tag{C.1.4a}$$
$$\mathrm{SU}^*(2n) = \mathrm{SL}(n, \mathbb{H}), \tag{C.1.4b}$$
$$\mathrm{USp}(2n) = \mathrm{U}(n, \mathbb{H}), \tag{C.1.4c}$$
$$\mathrm{USp}^*(2n_+, 2n_-) = \mathrm{U}(2n_+, 2n_-) \cap \mathrm{Sp}(2n_+, 2n_-, \mathbb{C}). \tag{C.1.4d}$$

C.2 Homogeneous space

A homogeneous space $\mathcal{M}$ of dimension n is a coset manifold

$$\mathcal{M} = \frac{G}{H}, \qquad n = \dim G - \dim H. \tag{C.2.1}$$

It admits $n(n+1)/2$ Killing vectors which is the maximum number in dimension n. In such a space all points are equivalent, i.e. it is always possible to find an isometry transformation that takes a point p to a point p'. Its isometry group is G

$$\mathrm{ISO}(G/H) = G \tag{C.2.2}$$

only if the normalizer of H in G is the trivial group [8, p. 8].

C.2.1 Symmetric space

A symmetric space is a homogeneous space for which the algebra of G can be decomposed as [68]

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{k} \tag{C.2.3}$$

with

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}. \tag{C.2.4}$$

C.3 Solvable algebra

For the following paragraphs, see [181, sec. 1, 184, app. A].

For homogeneous spaces the isometries act transitively on the manifold (i.e. two points can be mapped to each other by a group element). The orbit of a point under the action of the isometry group is locally equivalent to the coset G/H where H is the isotropy group at the corresponding point.

The group G is not semisimple if it is non-symmetric.

A normal space is such that G is non-compact and H corresponds to the maximal compact subgroup, then there exists a solvable subgroup that acts transitively, and whose dimension equals the dimension of the space. In this case there exists a solvable Lie algebra s such that

$$\mathrm{e}^{\mathfrak{s}} = \frac{G}{H}. \tag{C.3.1}$$

This algebra is obtained from the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{s} \tag{C.3.2}$$

and $\dim \mathfrak{s}$ is equal to the rank of the homogeneous space.

When the space is symmetric then the solvable algebra is completed into a simple algebra [184, p. 3].

Appendix D

Formulas

D.1 Quartic invariant identities

The formulas given in this appendix are a consequence of the Jordan algebra's structure of very special geometry and the fact that the duality groups are of E_7-type [33]. While they can be proved using techniques from [33, sec. 4] (see also [9, sec. 3, 30, sec. 2.2, 49, 84]), they have been determined by matching both sides on Mathematica. Some of them appeared already in [79, 105, 118].

The quartic invariant possesses many identities, some of them being given in [30, sec. 2.2].

Given two vectors A and B, any vectors built from them and from $I'_4(\cdot,\cdot,\cdot)$ can be expanded on the following basis

$$\Big\{A, B, I'_4(A), I'_4(A,A,B), I'_4(A,B,B), I'_4(B), I'_4\big(A,A,I'_4(B)\big), I'_4\big(B,B,I'_4(A)\big)\Big\}, \tag{D.1.1}$$

where there are 1, 3 or 5 vectors.

Below is the full list of identities involving respectively $5, 7$ and 9 vectors. They were computed using Mathematica by matching coefficients of both sides by using the explicit expressions of I_4. This has been checked for several cubic models and for the quadratic $n_v = 1$.

We recall two equations involving the section

$$I_4(\operatorname{Re}\mathcal{V}) = I_4(\operatorname{Im}\mathcal{V}) = \frac{1}{16}, \tag{D.1.2a}$$

$$\operatorname{Re}\mathcal{V} = 2\,\epsilon_\Omega\, I'_4(\operatorname{Im}\mathcal{V}) = \epsilon_\Omega\,\frac{I'_4(\operatorname{Im}\mathcal{V})}{2\sqrt{I_4(\operatorname{Im}V)}}, \tag{D.1.2b}$$

$$I'_4(A, \operatorname{Im}\mathcal{V}, \operatorname{Im}\mathcal{V}) = -4\,\langle\operatorname{Im}\mathcal{V}, A\rangle\operatorname{Im}\mathcal{V} - 8\,\langle\operatorname{Re}\mathcal{V}, A\rangle\operatorname{Re}\mathcal{V} - \Omega\mathcal{M}A. \tag{D.1.2c}$$

None of these identities changes when $\mathcal{V}$ is multiplied by a phase.

D.1.1 Symplectic product

$$\langle I'_4(A,A,B), I'_4(A)\rangle = -8\, I_4(A)\,\langle A,B\rangle \tag{D.1.3a}$$

$$\langle I'_4(A,B,B), I'_4(A)\rangle = -\frac{2}{3}\, I_4(A,A,A,B)\,\langle A,B\rangle \tag{D.1.3b}$$

$$\langle I'_4(A,B,B), I'_4(A,A,B)\rangle = 12\,\langle I'_4(A), I'_4(B)\rangle - 4\, I_4(A,A,B,B)\,\langle A,B\rangle \tag{D.1.3c}$$

D.1.2 Order 5

$$
\begin{aligned}
I_4'(I_4'(A),A,A) &= -8\,AI_4(A)\\
I_4'(I_4'(A),A,B) &= 2\,I_4'(A)\,\langle A,B\rangle - \frac{1}{3}\,AI_4(A,A,A,B)\\
I_4'(I_4'(A,A,B),A,A) &= -\frac{4}{3}\,AI_4(A,A,A,B) - 8\,I_4'(A)\,\langle A,B\rangle - 16\,BI_4(A)\\
I_4'(I_4'(A,A,B),A,B) &= -\frac{1}{3}\,2\,BI_4(A,A,A,B) - 2\,AI_4(A,A,B,B) + 2\,I_4'(A,A,B)\,\langle A,B\rangle\\
&\quad - 2\,I_4'(I_4'(A),B,B)\\
I_4'(I_4'(A,B,B),A,A) &= -\frac{4}{3}\,BI_4(A,A,A,B) - 4\,I_4'(A,A,B)\,\langle A,B\rangle + 2\,I_4'(I_4'(A),B,B)
\end{aligned}
$$

D.1.3 Order 7

$$
\begin{aligned}
I_4'(I_4'(A),I_4'(A),A) &= 8\,I_4(A)I_4'(A)\\
I_4'(I_4'(A),I_4'(A),B) &= 4\,I_4(A)I_4'(A,A,B) - \frac{2}{3}\,I_4'(A)I_4(A,A,A,B) - 16\,AI_4(A)\,\langle A,B\rangle\\
I_4'(I_4'(A),I_4'(A,A,B),A) &= 2\,I_4'(A)I_4(A,A,A,B) + 16\,AI_4(A)\,\langle A,B\rangle\\
I_4'(I_4'(A),I_4'(A,B,B),A) &= 2\,I_4'(A)I_4(A,A,B,B) + \frac{4}{3}\,AI_4(A,A,A,B)\,\langle A,B\rangle\\
I_4'(I_4'(A),I_4'(A,A,B),B) &= 8\,I_4(A)I_4'(A,B,B) - 2\,I_4'(A)I_4(A,A,B,B) + \frac{1}{3}\,I_4(A,A,A,B)I_4'(A,A,B)\\
&\quad - 16\,BI_4(A)\,\langle A,B\rangle - \frac{8}{3}\,AI_4(A,A,A,B)\,\langle A,B\rangle\\
I_4'(I_4'(A,A,B),I_4'(A,A,B),A) &= -16\,I_4(A)I_4'(A,B,B) + 8\,I_4'(A)I_4(A,A,B,B) + \frac{4}{3}\,I_4(A,A,A,B)I_4'(A,A,B)\\
&\quad + 64\,BI_4(A)\,\langle A,B\rangle + \frac{16}{3}\,AI_4(A,A,A,B)\,\langle A,B\rangle\\
I_4'(I_4'(A),I_4'(B),A) &= \frac{1}{3}\,I_4'(A)I_4(A,B,B,B) + 2\,A\,\langle I_4'(A),I_4'(B)\rangle
\end{aligned}
$$

$$
\begin{aligned}
I_4'(I_4'(A),I_4'(A,B,B),B) &= -\frac{2}{3}\,I_4'(A)I_4(A,B,B,B) + \frac{1}{3}\,I_4(A,A,A,B)I_4'(A,B,B)\\
&\quad - \frac{4}{3}\,BI_4(A,A,A,B)\,\langle A,B\rangle - 8\,A\,\langle I_4'(A),I_4'(B)\rangle + 16\,I_4(A)I_4'(B)\\
I_4'(I_4'(A,A,B),I_4'(A,A,B),B) &= -\frac{16}{3}\,I_4'(A)I_4(A,B,B,B) + \frac{8}{3}\,I_4(A,A,A,B)I_4'(A,B,B)\\
&\quad - 16\,AI_4(A,A,B,B)\,\langle A,B\rangle - \frac{16}{3}\,BI_4(A,A,A,B)\,\langle A,B\rangle\\
&\quad + 32\,A\,\langle I_4'(A),I_4'(B)\rangle + 32\,I_4(A)I_4'(B)\\
I_4'(I_4'(A,A,B),I_4'(A,B,B),A) &= \frac{16}{3}\,I_4'(A)I_4(A,B,B,B) + 2\,I_4(A,A,B,B)I_4'(A,A,B)\\
&\quad - \frac{2}{3}\,I_4(A,A,A,B)I_4'(A,B,B) + \frac{16}{3}\,BI_4(A,A,A,B)\,\langle A,B\rangle\\
&\quad + 8\,AI_4(A,A,B,B)\,\langle A,B\rangle - 8\,A\,\langle I_4'(A),I_4'(B)\rangle - 32\,I_4(A)I_4'(B)
\end{aligned}
$$

D.1.4 Order 9

$$
\begin{aligned}
I_4'(I_4'(A)) &= -16\, I_4(A)^2 A \\
I_4'(I_4'(A), I_4'(A), I_4'(A,A,B)) &= -64\, B I_4(A)^2 - \frac{64}{3}\, I_4(A,A,A,B) A I_4(A) \\
I_4'(I_4'(A), I_4'(A), I_4'(A,B,B)) &= -\frac{16}{3}\, B I_4(A) I_4(A,A,A,B) + \frac{8}{3}\, \langle A,B \rangle\, I_4'(A) I_4(A,A,A,B) \\
&\quad - 16\, I_4(A) I_4(A,A,B,B) A - 16\, \langle A,B \rangle\, I_4(A) I_4'(A,A,B) \\
&\quad + 8\, I_4(A) I_4'(B,B,I_4'(A)) \\
I_4'(I_4'(A), I_4'(A,A,B), I_4'(A,A,B)) &= -\frac{32}{9}\, A I_4(A,A,A,B)^2 - 32\, B I_4(A) I_4(A,A,A,B) \\
&\quad - \frac{16}{3}\, \langle A,B \rangle\, I_4'(A) I_4(A,A,A,B) - 32\, I_4(A) I_4(A,A,B,B) A \\
&\quad + 32\, \langle A,B \rangle\, I_4(A) I_4'(A,A,B) - 16\, I_4(A) I_4'(B,B,I_4'(A)) \\
I_4'(I_4'(A), I_4'(B,B,I_4'(A)), A) &= 32\, A I_4(A)\, \langle A,B \rangle^2 + \frac{4}{3}\, I_4(A,A,A,B) I_4'(A)\, \langle A,B \rangle \\
&\quad - \frac{2}{9}\, I_4(A,A,A,B)^2 A + 8\, A I_4(A) I_4(A,A,B,B) \\
I_4'(I_4'(A), I_4'(A), I_4'(B)) &= -\frac{8}{3}\, I_4(A) A I_4(A,B,B,B) + 4\, \langle I_4'(A), I_4'(B) \rangle\, I_4'(A) \\
&\quad + 4\, I_4(A) I_4'(A,A,I_4'(B)) \\
I_4'(I_4'(A), I_4'(A,A,B), I_4'(A,B,B)) &= -\frac{1}{9}\, 8\, B I_4(A,A,A,B)^2 - \frac{8}{3}\, I_4(A,A,B,B) A I_4(A,A,A,B) \\
&\quad - \frac{4}{3}\, \langle A,B \rangle\, I_4'(A,A,B) I_4(A,A,A,B) \\
&\quad + \frac{4}{3}\, I_4'(B,B,I_4'(A)) I_4(A,A,A,B) - \frac{64}{3}\, I_4(A) I_4(A,B,B,B) A \\
&\quad - 32\, I_4(A) I_4(A,A,B,B) B - 24\, \langle I_4'(A), I_4'(B) \rangle\, I_4'(A) \\
&\quad + 8\, \langle A,B \rangle\, I_4(A,A,B,B) I_4'(A) - 16\, I_4(A) I_4'(A,A,I_4'(B)) \\
&\quad - 16\, \langle A,B \rangle\, I_4(A) I_4'(A,B,B) \\
I_4'(I_4'(A), I_4'(B,B,I_4'(A)), B) &= -32\, I_4(A) B\, \langle A,B \rangle^2 + \frac{2}{9}\, B I_4(A,A,A,B)^2 - 8\, I_4(A) I_4(A,A,B,B) B \\
&\quad + \frac{16}{3}\, A I_4(A) I_4(A,B,B,B) - 12\, \langle I_4'(A), I_4'(B) \rangle\, I_4'(A) \\
&\quad - 8\, I_4(A) I_4'(A,A,I_4'(B)) + \frac{1}{3}\, I_4(A,A,A,B) I_4'(B,B,I_4'(A))
\end{aligned}
$$

$$
\begin{aligned}
I_4'(I_4'(A,A,B),I_4'(B,B,I_4'(A)),A) &= 128\,BI_4(A)\,\langle A,B\rangle^2+\frac{16}{3}\,AI_4(A,A,A,B)\,\langle A,B\rangle^2\\
&\quad+\frac{4}{3}\,I_4(A,A,A,B)I_4'(A,A,B)\,\langle A,B\rangle-\frac{8}{9}\,I_4(A,A,A,B)^2B\\
&\quad+32\,BI_4(A)I_4(A,A,B,B)+\frac{16}{3}\,AI_4(A)I_4(A,B,B,B)\\
&\quad+48\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A)+16\,I_4(A)I_4'(A,A,I_4'(B))\\
&\quad-\frac{2}{3}\,I_4(A,A,A,B)I_4'(B,B,I_4'(A))\\
I_4'(I_4'(A),I_4'(A,A,I_4'(B)),A) &= \frac{8}{3}\,I_4(A)I_4(A,B,B,B)A-12\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A)\\
I_4'(I_4'(A),I_4'(B),I_4'(A,A,B)) &= -64\,I_4(A)AI_4(B)-\frac{4}{9}\,I_4(A,A,A,B)I_4(A,B,B,B)A\\
&\quad-\frac{16}{3}\,I_4(A)I_4(A,B,B,B)B+\frac{4}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A)\\
&\quad-16\,\langle A,B\rangle\,I_4(A)I_4'(B)+2\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B)\\
&\quad+\frac{2}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B))\\
I_4'(I_4'(A),I_4'(A,B,B),I_4'(A,B,B)) &= -128\,I_4(A)AI_4(B)-\frac{16}{9}\,I_4(A,A,A,B)I_4(A,B,B,B)A\\
&\quad-\frac{8}{3}\,I_4(A,A,A,B)I_4(A,A,B,B)B-\frac{64}{3}\,I_4(A)I_4(A,B,B,B)B\\
&\quad+\frac{16}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A)-16\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B)\\
&\quad-\frac{8}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B))\\
&\quad-\frac{8}{3}\,\langle A,B\rangle\,I_4(A,A,A,B)I_4'(A,B,B)\\
&\quad+4\,I_4(A,A,B,B)I_4'(B,B,I_4'(A))\\
I_4'(I_4'(A,A,B),I_4'(A,A,B),I_4'(A,B,B)) &= -16\,AI_4(A,A,B,B)^2-16\,I_4(A,A,A,B)BI_4(A,A,B,B)\\
&\quad-8\,I_4'(B,B,I_4'(A))I_4(A,A,B,B)\\
&\quad-\frac{64}{9}\,I_4(A,A,A,B)I_4(A,B,B,B)A+256\,AI_4(A)I_4(B)\\
&\quad-\frac{64}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A)+128\,\langle A,B\rangle\,I_4(A)I_4'(B)\\
&\quad+16\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B)+\frac{16}{3}\,\langle A,B\rangle\,I_4(A,A,A,B)I_4'(A,B,B)
\end{aligned}
$$

$$
\begin{aligned}
I_4'(I_4'(A,A,B),I_4'(B,B,I_4'(A)),B) = &-\frac{1}{3}\,16\,I_4(A,A,A,B)B\,\langle A,B\rangle^2 - \frac{16}{3}\,I_4(A,B,B,B)I_4'(A)\,\langle A,B\rangle \\
&+ 64\,I_4(A)I_4'(B)\,\langle A,B\rangle - \frac{16}{3}\,I_4(A)I_4(A,B,B,B)B \\
&+ 128\,AI_4(A)I_4(B) + \frac{8}{9}\,AI_4(A,A,A,B)I_4(A,B,B,B) \\
&- 4\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B) - \frac{4}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B)) \\
&+ 2\,I_4(A,A,B,B)I_4'(B,B,I_4'(A)) \\
I_4'(I_4'(A,B,B),I_4'(B,B,I_4'(A)),A) = &\,\frac{32}{3}\,BI_4(A,A,A,B)\,\langle A,B\rangle^2 + 16\,A\,\langle I_4'(A),I_4'(B)\rangle\,\langle A,B\rangle \\
&+ \frac{16}{3}\,I_4(A,B,B,B)I_4'(A)\,\langle A,B\rangle - 64\,I_4(A)I_4'(B)\,\langle A,B\rangle \\
&+ \frac{4}{3}\,I_4(A,A,A,B)I_4'(A,B,B)\,\langle A,B\rangle \\
&- \frac{4}{9}\,I_4(A,A,A,B)I_4(A,B,B,B)A + \frac{64}{3}\,BI_4(A)I_4(A,B,B,B) \\
&+ 16\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B) + \frac{4}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B)) \\
&- 2\,I_4(A,A,B,B)I_4'(B,B,I_4'(A))
\end{aligned}
$$

$$
\begin{aligned}
I_4'(I_4'(A),I_4'(A,A,I_4'(B)),B) = &\,16\,A\,\langle A,B\rangle\,\langle I_4'(A),I_4'(B)\rangle - 4\,I_4'(A,A,B)\,\langle I_4'(A),I_4'(B)\rangle \\
&- \frac{8}{3}\,I_4(A)I_4(A,B,B,B)B + 64\,AI_4(A)I_4(B) \\
&+ \frac{4}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A) - 32\,\langle A,B\rangle\,I_4(A)I_4'(B) \\
&- \frac{1}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B)) \\
I_4'(I_4'(A,A,B),I_4'(A,A,I_4'(B)),A) = &-16\,\langle A,B\rangle\,A\,\langle I_4'(A),I_4'(B)\rangle - 4\,I_4'(A,A,B)\,\langle I_4'(A),I_4'(B)\rangle \\
&+ \frac{32}{3}\,BI_4(A)I_4(A,B,B,B) + \frac{4}{9}\,AI_4(A,A,A,B)I_4(A,B,B,B) \\
&- \frac{16}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A) + 64\,\langle A,B\rangle\,I_4(A)I_4'(B) \\
&+ \frac{2}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B))
\end{aligned}
$$

D.2 Quaternionic gaugings: constraints

For completeness the full set of constraints for the (symplectic) gaugings parameters is listed below [78, sec. 6.1, app. C].

The set of parameters

$$\Theta^\alpha = \{\mathbb{U},\alpha,\widehat{\alpha}^t,\epsilon_+,\epsilon_0,\epsilon_-\} \tag{D.2.1}$$

reads explicitly

$$\alpha = \begin{pmatrix} \alpha^{\Lambda} \\ \alpha_{\Lambda} \end{pmatrix}, = \begin{pmatrix} \begin{pmatrix} \alpha^{A\Lambda} \\ \alpha^{\Lambda}_{A} \end{pmatrix} \\ \begin{pmatrix} \alpha^{A}_{\Lambda} \\ \alpha_{A\Lambda} \end{pmatrix} \end{pmatrix}, \qquad \widehat{\alpha} = \begin{pmatrix} \widehat{\alpha}^{\Lambda} \\ \widehat{\alpha}_{\Lambda} \end{pmatrix}, = \begin{pmatrix} \begin{pmatrix} \widehat{\alpha}^{A\Lambda} \\ \widehat{\alpha}^{\Lambda}_{A} \end{pmatrix} \\ \begin{pmatrix} \widehat{\alpha}^{A}_{\Lambda} \\ \widehat{\alpha}_{A\Lambda} \end{pmatrix} \end{pmatrix},$$
$$\mathbb{U} = \begin{pmatrix} \mathbb{U}^{\Lambda} \\ \mathbb{U}_{\Lambda} \end{pmatrix}, \qquad \epsilon_{\pm} = \begin{pmatrix} \epsilon^{\Lambda}_{\pm} \\ \epsilon_{\pm\Lambda} \end{pmatrix}, \qquad \epsilon_{0} = \begin{pmatrix} \epsilon^{\Lambda}_{0} \\ \epsilon_{0\Lambda} \end{pmatrix}, \tag{D.2.2}$$

where $\mathbb{U}^{\Lambda}$ and $\mathbb{U}_{\Lambda}$ are matrices whose parameters depend on the model.

The number of parameters is (approx.)

$$\#(\text{params}) = n_v(x + 4n_h + 3), \tag{D.2.3}$$

x being the number of independent isometries of the base (this can be of order n_h^2, n_h or 1).

D.2.1 Constraints from abelian algebra

The constraints from the closure of the abelian algebra are

- electric/electric

$$0 = \mathbb{T}(\alpha_{\Lambda}, \hat{\alpha}_{\Sigma}) - \mathbb{T}(\alpha_{\Sigma}, \hat{\alpha}_{\Lambda}), \tag{D.2.4a}$$
$$0 = -(\mathbb{U}_{\Lambda}\alpha_{\Sigma} - \mathbb{U}_{\Sigma}\alpha_{\Lambda}) + (\epsilon_{0\Lambda}\alpha_{\Sigma} - \epsilon_{0\Sigma}\alpha_{\Lambda}) + (\epsilon_{+\Lambda}\widehat{\alpha}_{\Sigma} - \epsilon_{+\Sigma}\widehat{\alpha}_{\Lambda}), \tag{D.2.4b}$$
$$0 = (\mathbb{U}_{\Lambda}\widehat{\alpha}_{\Sigma} - \mathbb{U}_{\Sigma}\widehat{\alpha}_{\Lambda}) + (\epsilon_{-\Lambda}\alpha_{\Sigma} - \epsilon_{-\Sigma}\alpha_{\Lambda}) + (\epsilon_{0\Lambda}\widehat{\alpha}_{\Sigma} - \epsilon_{0\Sigma}\widehat{\alpha}_{\Lambda}), \tag{D.2.4c}$$
$$0 = \alpha^{t}_{\Lambda}\mathbb{C}\alpha_{\Sigma} + 2(\epsilon_{+\Sigma}\epsilon_{0\Lambda} - \epsilon_{+\Lambda}\epsilon_{0\Sigma}), \tag{D.2.4d}$$
$$0 = (\widehat{\alpha}^{t}_{\Lambda}\mathbb{C}\alpha_{\Sigma} - \alpha^{t}_{\Lambda}\mathbb{C}\widehat{\alpha}_{\Sigma}) + 2(\epsilon_{+\Sigma}\epsilon_{-\Lambda} - \epsilon_{+\Lambda}\epsilon_{-\Sigma}), \tag{D.2.4e}$$
$$0 = \widehat{\alpha}^{t}_{\Lambda}\mathbb{C}\widehat{\alpha}_{\Sigma} + 2(\epsilon_{0\Lambda}\epsilon_{-\Sigma} - \epsilon_{0\Sigma}\epsilon_{-\Lambda}). \tag{D.2.4f}$$

- electric/magnetic

$$0 = \mathbb{T}(\alpha_{\Lambda}, \hat{\alpha}^{\Sigma}) - \mathbb{T}(\alpha^{\Sigma}, \hat{\alpha}_{\Lambda}), \tag{D.2.4g}$$
$$0 = -(\mathbb{U}_{\Lambda}\alpha^{\Sigma} - \mathbb{U}^{\Sigma}\alpha_{\Lambda}) + (\epsilon_{0\Lambda}\alpha^{\Sigma} - \epsilon^{\Sigma}_{0}\alpha_{\Lambda}) + (\epsilon_{+\Lambda}\widehat{\alpha}^{\Sigma} - \epsilon^{\Sigma}_{+}\widehat{\alpha}_{\Lambda}), \tag{D.2.4h}$$
$$0 = (\mathbb{U}_{\Lambda}\widehat{\alpha}^{\Sigma} - \mathbb{U}^{\Sigma}\widehat{\alpha}_{\Lambda}) + (\epsilon_{-\Lambda}\alpha^{\Sigma} - \epsilon^{\Sigma}_{-}\alpha_{\Lambda}) + (\epsilon_{0\Lambda}\widehat{\alpha}^{\Sigma} - \epsilon^{\Sigma}_{0}\widehat{\alpha}_{\Lambda}), \tag{D.2.4i}$$
$$0 = \alpha^{t}_{\Lambda}\mathbb{C}\alpha^{\Sigma} + 2(\epsilon^{\Sigma}_{+}\epsilon_{0\Lambda} - \epsilon_{+\Lambda}\epsilon^{\Sigma}_{0}), \tag{D.2.4j}$$
$$0 = (\widehat{\alpha}^{t}_{\Lambda}\mathbb{C}\alpha^{\Sigma} - \alpha^{t}_{\Lambda}\mathbb{C}\widehat{\alpha}^{\Sigma}) + 2(\epsilon^{\Sigma}_{+}\epsilon_{-\Lambda} - \epsilon_{+\Lambda}\epsilon^{\Sigma}_{-}), \tag{D.2.4k}$$
$$0 = \widehat{\alpha}^{t}_{\Lambda}\mathbb{C}\widehat{\alpha}^{\Sigma} + 2(\epsilon_{0\Lambda}\epsilon^{\Sigma}_{-} - \epsilon^{\Sigma}_{0}\epsilon_{-\Lambda}). \tag{D.2.4l}$$

- magnetic/magnetic

$$0 = \mathbb{T}(\alpha^{\Lambda}, \hat{\alpha}^{\Sigma}) - \mathbb{T}(\alpha^{\Sigma}, \hat{\alpha}^{\Lambda}), \tag{D.2.4m}$$
$$0 = -(\mathbb{U}^{\Lambda}\alpha^{\Sigma} - \mathbb{U}^{\Sigma}\alpha^{\Lambda}) + (\epsilon^{\Lambda}_{0}\alpha^{\Sigma} - \epsilon^{\Sigma}_{0}\alpha^{\Lambda}) + (\epsilon^{\Lambda}_{+}\widehat{\alpha}^{\Sigma} - \epsilon^{\Sigma}_{+}\widehat{\alpha}^{\Lambda}), \tag{D.2.4n}$$
$$0 = (\mathbb{U}^{\Lambda}\widehat{\alpha}^{\Sigma} - \mathbb{U}^{\Sigma}\widehat{\alpha}^{\Lambda}) + (\epsilon^{\Lambda}_{-}\alpha^{\Sigma} - \epsilon^{\Sigma}_{-}\alpha^{\Lambda}) + (\epsilon^{\Lambda}_{0}\widehat{\alpha}^{\Sigma} - \epsilon^{\Sigma}_{0}\widehat{\alpha}^{\Lambda}), \tag{D.2.4o}$$
$$0 = \alpha^{t\,\Lambda}\mathbb{C}\alpha^{\Sigma} + 2(\epsilon^{\Sigma}_{+}\epsilon^{\Lambda}_{0} - \epsilon^{\Lambda}_{+}\epsilon^{\Sigma}_{0}), \tag{D.2.4p}$$
$$0 = (\widehat{\alpha}^{t\,\Lambda}\mathbb{C}\alpha^{\Sigma} - \alpha^{t\,\Lambda}\mathbb{C}\widehat{\alpha}^{\Sigma}) + 2(\epsilon^{\Sigma}_{+}\epsilon^{\Lambda}_{-} - \epsilon^{\Lambda}_{+}\epsilon^{\Sigma}_{-}), \tag{D.2.4q}$$
$$0 = \widehat{\alpha}^{t\,\Lambda}\mathbb{C}\widehat{\alpha}^{\Sigma} + 2(\epsilon^{\Lambda}_{0}\epsilon^{\Sigma}_{-} - \epsilon^{\Sigma}_{0}\epsilon^{\Lambda}_{-}). \tag{D.2.4r}$$

We recall the expression of the matrix

$$\mathbb{T}_{\alpha,\hat{\alpha}} = (\alpha^t \partial_\xi)(\hat{\alpha}^t \partial_\xi)\, \mathbb{S}. \tag{D.2.5}$$

The number of (electric) constraints is (approx.)

$$\#(\text{constraints}) = \frac{n_v(n_v-1)}{2}\,(x + 2n_h + 3)\,, \tag{D.2.6}$$

where the front factor comes from the antisymmetric equations on (Λ, Σ), and x is the number of independent entries in the matrix $\mathbb{S}$ (this can be of order n_h^2, n_h or 1).

D.2.2 Locality constraints

The constraints from locality are

$$0 = \langle \alpha, \alpha^t \rangle = \alpha^\Lambda \alpha^t_\Lambda - \alpha_\Lambda \alpha^{t\Lambda}, \tag{D.2.7a}$$
$$0 = \langle \alpha, \widehat{\alpha}^t \rangle = \alpha^\Lambda \widehat{\alpha}^t_\Lambda - \alpha_\Lambda \widehat{\alpha}^{t\Lambda}, \tag{D.2.7b}$$
$$0 = \langle \widehat{\alpha}, \widehat{\alpha}^t \rangle = \widehat{\alpha}^\Lambda \widehat{\alpha}^t_\Lambda - \widehat{\alpha}_\Lambda \widehat{\alpha}^{t\Lambda}, \tag{D.2.7c}$$
$$0 = \langle \alpha, \epsilon_+ \rangle = \alpha^\Lambda \epsilon_{+\Lambda} - \alpha_\Lambda \epsilon_+^\Lambda, \tag{D.2.7d}$$
$$0 = \langle \alpha, \epsilon_0 \rangle = \alpha^\Lambda \epsilon_{0\Lambda} - \alpha_\Lambda \epsilon_0^\Lambda, \tag{D.2.7e}$$
$$0 = \langle \alpha, \epsilon_- \rangle = \alpha^\Lambda \epsilon_{-\Lambda} - \alpha_\Lambda \epsilon_-^\Lambda, \tag{D.2.7f}$$
$$0 = \langle \widehat{\alpha}, \epsilon_+ \rangle = \widehat{\alpha}^\Lambda \epsilon_{+\Lambda} - \widehat{\alpha}_\Lambda \epsilon_+^\Lambda, \tag{D.2.7g}$$
$$0 = \langle \widehat{\alpha}, \epsilon_0 \rangle = \widehat{\alpha}^\Lambda \epsilon_{0\Lambda} - \widehat{\alpha}_\Lambda \epsilon_0^\Lambda, \tag{D.2.7h}$$

$$0 = \langle \widehat{\alpha}, \epsilon_- \rangle = \widehat{\alpha}^\Lambda \epsilon_{-\Lambda} - \widehat{\alpha}_\Lambda \epsilon_-^\Lambda, \tag{D.2.7i}$$
$$0 = \langle \epsilon_+, \epsilon_- \rangle = \epsilon_+^\Lambda \epsilon_{-\Lambda} - \epsilon_{+\Lambda} \epsilon_-^\Lambda, \tag{D.2.7j}$$
$$0 = \langle \epsilon_+, \epsilon_0 \rangle = \epsilon_+^\Lambda \epsilon_{0\Lambda} - \epsilon_{+\Lambda} \epsilon_0^\Lambda, \tag{D.2.7k}$$
$$0 = \langle \epsilon_0, \epsilon_- \rangle = \epsilon_0^\Lambda \epsilon_{-\Lambda} - \epsilon_{0\Lambda} \epsilon_-^\Lambda, \tag{D.2.7l}$$
$$0 = \langle \mathbb{U}, \epsilon_+ \rangle = \alpha^\Lambda \epsilon_{+\Lambda} - \alpha_\Lambda \epsilon_+^\Lambda, \tag{D.2.7m}$$
$$0 = \langle \mathbb{U}, \epsilon_0 \rangle = \alpha^\Lambda \epsilon_{0\Lambda} - \alpha_\Lambda \epsilon_0^\Lambda, \tag{D.2.7n}$$
$$0 = \langle \mathbb{U}, \epsilon_- \rangle = \alpha^\Lambda \epsilon_{-\Lambda} - \alpha_\Lambda \epsilon_-^\Lambda, \tag{D.2.7o}$$
$$0 = \langle \mathbb{U}, \alpha \rangle = \alpha^\Lambda \epsilon_{0\Lambda} - \alpha_\Lambda \epsilon_0^\Lambda, \tag{D.2.7p}$$
$$0 = \langle \mathbb{U}, \widehat{\alpha} \rangle = \alpha^\Lambda \epsilon_{-\Lambda} - \alpha_\Lambda \epsilon_-^\Lambda \tag{D.2.7q}$$

where

$$\langle \alpha, \alpha^t \rangle = \begin{pmatrix} \langle \alpha^A, \alpha^B \rangle & \langle \alpha^A, \alpha_B \rangle \\ \langle \alpha_A, \alpha^B \rangle & \langle \alpha_A, \alpha_B \rangle \end{pmatrix}, \qquad \langle \alpha, \epsilon_+ \rangle = \begin{pmatrix} \langle \alpha^A, \epsilon_+ \rangle \\ \langle \alpha_A, \epsilon_+ \rangle \end{pmatrix} \tag{D.2.8}$$

and similarly for the others. The notation $\langle \mathbb{U}, X \rangle$ is a shortcut for the product of X with all parameters of $\mathbb{U}$ (by linearity). For example with a cubic prepotential one of the constraint is

$$\langle \beta, X \rangle = 0, \qquad \beta = \begin{pmatrix} \beta^\Lambda \\ \beta_\Lambda \end{pmatrix}. \tag{D.2.9}$$

The numbers of locality constraints is (approx.)

$$\#(\text{locality constraints}) = 3(n_h+1)^2 + x\, n_h(2n_h+3). \tag{D.2.10}$$

Appendix E

Computations

In this section we are collecting long and cumbersome computations.

E.1 Quaternionic isometries: Killing algebra

E.1.1 Computations: duality and extra commutators

The non-vanishing commutators of the algebra are

$$[k_0, k_+] = 2k_+, \qquad [k_0, k_\alpha] = k_\alpha, \qquad \left[k_\alpha, k_\alpha^t\right] = \mathbb{C}\, k_+, \qquad [k_{\mathbb{U}}, k_\alpha] = \mathbb{U}\, k_\alpha. \tag{E.1.1}$$

The evaluation of the last commutator proceeds as

$$\begin{aligned}\left[k_{\mathbb{U}}, k^A\right] &= \frac{1}{2}(\mathbb{U}\xi)^B(\partial_B \xi^A)\partial_a - \Big(\left[\partial^A(\mathbb{U}\xi)^B\right]\partial_B + \left[\partial^A(\mathbb{U}\xi)_B\right]\partial^B\Big) \\ &= \frac{1}{2}\big(v^A{}_B\xi^B + t^{AB}\tilde{\xi}_B\big)\partial_a - t^{BA}\partial_B - u_B{}^A\partial^B \\ &= v^A{}_B\left(\partial^B + \frac{1}{2}\,\xi^B\partial_a\right) - t^{AB}\left(\partial_B - \frac{1}{2}\,\tilde{\xi}_B\partial_a\right).\end{aligned}$$

In components we have

$$\begin{gathered}\left[k_A, h^B\right] = -\delta_A{}^B k_+, \qquad \left[k_0, k^A\right] = k^A, \qquad [k_0, k_A] = k_A, \\ \left[k_{\mathbb{U}}, k^A\right] = (\mathbb{U}\mathbb{C}h)^A, \qquad [k_{\mathbb{U}}, k_A] = (\mathbb{U}\mathbb{C}h)_A.\end{gathered} \tag{E.1.2}$$

E.1.2 Computations: hidden and mixed commutators

We now compute the commutators between hidden and duality symmetries

$$\begin{gathered}[k_0, k_-] = -2\,k_-, \qquad [k_0, k_{\hat\alpha}] = -k_{\hat\alpha}, \qquad [k_-, k_\alpha] = -k_{\hat\alpha}, \\ [k_+, k_-] = -k_0, \qquad [k_+, k_{\hat\alpha}] = k_\alpha, \qquad [k_{\mathbb{U}}, k_{\hat\alpha}] = \mathbb{U}\, k_{\hat\alpha}, \\ \left[k_{\hat\alpha}, k_{\hat\alpha}^t\right] = \mathbb{C}\, k_-, \qquad \left[\widehat{\alpha}^t k_{\hat\alpha}, \alpha^t k_\alpha\right] = \frac{1}{2}\,\widehat{\alpha}\mathbb{C}\alpha\, k_0 + k_{\mathbb{T}_{\alpha,\hat\alpha}}\end{gathered} \tag{E.1.3}$$

where

$$\mathbb{T}_{\alpha,\hat\alpha} = (\alpha^t\mathbb{C}\partial_\xi)(\hat\alpha^t\mathbb{C}\partial_\xi)\,\mathbb{S} = -\frac{1}{2}\,\mathbb{C}(\hat\alpha\alpha^t + \alpha\hat\alpha^t) + \frac{1}{4}\,H''_{\alpha,\hat\alpha}\mathbb{C}, \tag{E.1.4a}$$

$$H''_{\alpha,\hat\alpha} = \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h''_{\alpha,\hat\alpha})^t = (\alpha^t\mathbb{C}\partial_\xi)(\hat\alpha^t\mathbb{C}\partial_\xi)H, \tag{E.1.4b}$$

$$h''_{\alpha,\hat\alpha} = (\alpha^t\mathbb{C}\partial_\xi)(\hat\alpha^t\mathbb{C}\partial_\xi)h. \tag{E.1.4c}$$

We have

$$\begin{aligned}\big[k^A, k_-\big] = a\partial^A - (\mathbb{C}\partial_\xi \partial^A W)^t \partial_\xi - (\partial^A \mathbb{S}\, Z)^t \partial_Z + \text{c.c.} - \frac{1}{2}\xi^A\,\partial_\phi + a\xi^A \partial_a \\ + \frac{1}{2}\,\xi^A \xi^t \partial_\xi - \frac{1}{2}(a\xi^A - \partial^A W)\partial_a .\end{aligned} \tag{E.1.5}$$

Another commutator:

$$\begin{aligned}[k_0, k_-] &= 4\,\mathrm{e}^{-4\phi}\,\partial_a - 2a(-\partial_\phi + 2a\partial_a + \xi^t \partial_\xi) + (\xi^t\partial_\xi - \partial_\phi) W\,\partial_a \\ &\quad - \big(a\xi - \mathbb{C}(\xi^t\partial_\xi - \partial_\phi)\partial_\xi W\big)^t \partial_\xi + \big((\xi^t\partial_\xi)\mathbb{S}Z\big)^t \partial_Z + \text{c.c.} \\ &\quad + 2(a^2 - \mathrm{e}^{-4\phi} - W)\partial_a + (a\xi - \mathbb{C}\partial_\xi W)^t \partial_\xi \\ &= 4(\,\mathrm{e}^{-4\phi} - a^2)\partial_a + 2a\partial_\phi - 2a\,\xi^t\partial_\xi + 4W\,\partial_a - \big(a\xi - 3\mathbb{C}\partial_\xi W\big)^t \partial_\xi \\ &\quad + 2(\mathbb{S}Z)^t\partial_Z + \text{c.c.} + 2(a^2 - \,\mathrm{e}^{-4\phi} - W)\partial_a + (a\xi - \mathbb{C}\partial_\xi W)^t\partial_\xi \\ &= -2\Big[- a\partial_\phi + (a^2 - \,\mathrm{e}^{-4\phi} - W)\partial_a + (a\xi - \mathbb{C}\partial_\xi W)^t\partial_\xi - (\mathbb{S}Z)^t\partial_Z + \text{c.c.}\Big] \\ &= -2\,k_-,\end{aligned}$$

where we used the "homogeneity" of W (12.1.27).

Introducing a set of parameters $\alpha, \widehat{\alpha}$, then we have

$$\begin{aligned}\big[\alpha^t\, k_\alpha, \widehat{\alpha}^t\, k_{\hat{\alpha}}\big] &= \frac{1}{2}\,(-\partial_\phi + a\partial_a)\,(\alpha^t\mathbb{C}\partial_\xi)\hat{\alpha}^t\xi - \frac{1}{2}\,(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\mathbb{C}\partial_\xi W)\,\partial_a \\ &\quad + \frac{1}{2}\,(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\xi\xi^t\partial_\xi) - (\alpha^t\mathbb{C}\partial_\xi)\,\big[\hat{\alpha}^t\mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\partial_\xi\big] \\ &\quad - (\alpha^t\mathbb{C}\partial_\xi)\big[\hat{\alpha}^t(\mathbb{C}\partial_\xi\mathbb{S}\,Z)^t\partial_Z\big] + \cancel{\frac{1}{4}\,(\alpha^t\xi)\,(\hat{\alpha}^t\xi)}\,\partial_a + \frac{1}{2}\,(\alpha^t\xi)\,(\hat{\alpha}^t\mathbb{C}\partial_\xi) \\ &\quad - \frac{a}{2}\,(\hat{\alpha}^t\mathbb{C}\partial_\xi)\,\alpha^t\xi\,\partial_a - \frac{1}{2}\left[\hat{\alpha}^t\left(\cancel{\frac{1}{2}\xi\xi^t} - \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\right)\partial_\xi\right]\,(\alpha^t\xi)\,\partial_a .\end{aligned}$$

The two terms cancel because

$$\hat{\alpha}^t\xi\xi^t\alpha = (\hat{\alpha}^t\xi)(\xi^t\alpha) = (\alpha^t\xi)(\xi^t\hat{\alpha}). \tag{E.1.6}$$

We have

$$(\alpha^t\mathbb{C}\partial_\xi)\hat{\alpha}^t\xi = \alpha^t\mathbb{C}\hat{\alpha}^t \tag{E.1.7}$$

as can be seen by writing the indices explicitly

$$\alpha_i\mathbb{C}_{ij}\partial_j\;\hat{\alpha}_k\xi_k = \alpha_i\mathbb{C}_{ij}\delta_{jk}\hat{\alpha}_k = \alpha_i\mathbb{C}_{ij}\hat{\alpha}_j. \tag{E.1.8}$$

Moreover we can rewrite

$$\hat{\alpha}^t\xi\xi^t\partial_\xi = (\hat{\alpha}^t\xi)(\xi^t\partial_\xi). \tag{E.1.9}$$

and then

$$(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\xi\xi^t\partial_\xi) = (\alpha^t\mathbb{C}\hat{\alpha})(\xi^t\partial_\xi) + (\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\xi). \tag{E.1.10}$$

The expression simplifies to

$$\begin{aligned}[\alpha\, k_\alpha, \widehat{\alpha}\, k_{\hat{\alpha}}] &= -\frac{1}{2}\,\alpha^t\mathbb{C}\hat{\alpha}\,(\partial_\phi - 2a\partial_a - \xi^t\partial_\xi) - \frac{1}{2}\,\cancel{(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\mathbb{C}\partial_\xi W)}\,\partial_a \\ &\quad + \frac{1}{2}\,(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\xi) + \frac{1}{2}\,(\alpha^t\xi)(\hat{\alpha}^t\mathbb{C}\partial_\xi) \\ &\quad - (\alpha^t\mathbb{C}\partial_\xi)\,\big[\hat{\alpha}^t\mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\partial_\xi\big] - (\alpha^t\mathbb{C}\partial_\xi)\big[\hat{\alpha}^t(\mathbb{C}\partial_\xi\mathbb{S}\,Z)^t\partial_Z\big] \\ &\quad + \frac{1}{2}\cancel{\big((\hat{\alpha}^t\mathbb{C}\partial_\xi)(\mathbb{C}\partial_\xi W)^t\alpha\big)}\,\partial_a .\end{aligned}$$

The cancellation occurs since

$$(\hat{\alpha}^t \mathbb{C}\partial_\xi)(\mathbb{C}\partial_\xi W)^t \alpha = (\hat{\alpha}^t \mathbb{C}\partial_\xi)(\alpha^t \mathbb{C}\partial_\xi W)^t = (\hat{\alpha}^t \mathbb{C}\partial_\xi)(\alpha^t \mathbb{C}\partial_\xi W) \tag{E.1.11}$$

the last parenthesis being just a number.

The penultimate in the first expression gives a factor 2 in $2a\partial_a$ since

$$-\frac{a}{2}\,(\hat{\alpha}^t \mathbb{C}\alpha)\,\partial_a = \frac{a}{2}\,(\alpha^t \mathbb{C}\hat{\alpha})\,\partial_a \tag{E.1.12}$$

by antisymmetry of $\mathbb{C}$.

Then we can write

$$\begin{aligned}(\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \xi) + (\alpha^t \xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi) &= (\xi^t \hat{\alpha})(\alpha^t \mathbb{C}\partial_\xi) + (\xi^t \alpha)(\hat{\alpha}^t \mathbb{C}\partial_\xi) \\ &= \xi^t (\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t)\mathbb{C}\,\partial_\xi \\ &= -\big[\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t)\xi\big]^t\,\partial_\xi.\end{aligned}$$

We need to simplify the terms with W and $\mathbb{S}$. Starting with W: this function contains quartic and quadratic terms in ξ, so $(\alpha^t \mathbb{C}\partial_\xi)\,(\hat{\alpha}^t \mathbb{C}\partial_\xi)(\mathbb{C}\partial_\xi W)^t$ is linear in ξ, which implies that it is homogeneous of first order. This linear term is given by the third derivative of h, such that

$$(\alpha^t \mathbb{C}\partial_\xi)\,\big[\hat{\alpha}^t \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W) = \frac{1}{4}\,\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}} \tag{E.1.13}$$

and we have defined

$$h''_{\alpha,\hat{\alpha}} = (\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)h. \tag{E.1.14}$$

As we said its derivative is homogeneous, thus

$$\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}} = \xi^t \partial_\xi(\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}})^t = -\xi^t \mathbb{C} H''_{\alpha,\hat{\alpha}}. \tag{E.1.15}$$

The new symbol we have defined is

$$H''_{\alpha,\hat{\alpha}} = \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}})^t = (\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)H. \tag{E.1.16}$$

Note that the matrix $H''_{\alpha,\hat{\alpha}}$ is constant and symmetric.

Using all this we can simplify the W term as

$$(\alpha^t \mathbb{C}\partial_\xi)\,\big[\hat{\alpha}^t \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t \partial_\xi\big] = \frac{1}{4}\,(H''_{\alpha,\hat{\alpha}}\mathbb{C}\xi)^t \partial_\xi. \tag{E.1.17}$$

After all this the computation for $\mathbb{S}$ is straightforward:

$$\begin{aligned}(\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)\mathbb{S} &= \frac{1}{2}(\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)\left(\xi\xi^t + \frac{1}{2}\,H\right)\mathbb{C} \\ &= -\frac{1}{2}\,\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t) + \frac{1}{4}\,H''_{\alpha,\hat{\alpha}}\mathbb{C}.\end{aligned}$$

The new expression is

$$\begin{aligned}[\alpha\, k_\alpha, \widehat{\alpha}\, k_{\hat{\alpha}}] = &-\frac{1}{2}\,k_0 + \frac{1}{2}\big[\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t)\xi\big]^t\,\partial_\xi \\ &- \frac{1}{4}\,(H''_{\alpha,\hat{\alpha}}\mathbb{C}\xi)^t\partial_\xi + \frac{1}{2}\big[\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t)Z\big]^t\,\partial_Z - \frac{1}{4}\,(H''_{\alpha,\hat{\alpha}}\mathbb{C}Z)^t\partial_Z.\end{aligned}$$

We recognize the vector $k_{-\mathbb{U}_{\alpha,\hat{\alpha}}}$ with parameters

$$\mathbb{U}_{\alpha,\hat{\alpha}} = -\frac{1}{2}\,\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t) + \frac{1}{4}\,H''_{\alpha,\hat{\alpha}}\mathbb{C}. \tag{E.1.18}$$

E.2 Gauged supergravity

E.2.1 Computations : constraints from algebra closure

We compute first the various pieces:

$$
\begin{aligned}
[k_{\mathbb{U}_\Lambda}, k_\Sigma] &= \Big[k_{\mathbb{U}_\Lambda}, \alpha^t_\Sigma \mathbb{C} k_\alpha + \widehat{\alpha}^t_\Sigma \mathbb{C} \widehat{k}_\alpha\Big] = \alpha^t_\Sigma \mathbb{C}\mathbb{U}_\Lambda\, k_\alpha + \widehat{\alpha}^t_\Sigma \mathbb{C}\mathbb{U}_\Lambda\, \widehat{k}_\alpha, \\
\big[\alpha^t_\Lambda \mathbb{C} k_\alpha, k_\Sigma\big] &= \Big[\alpha^t_\Lambda \mathbb{C} k_\alpha, k_{\mathbb{U}_\Sigma} + \alpha^t_\Sigma \mathbb{C} k_\alpha + \widehat{\alpha}^t_\Sigma \mathbb{C} \widehat{k}_\alpha + \epsilon_{0\Sigma} k_0 + \epsilon_{-\Sigma} k_-\Big] \\
&= -\alpha^t_\Lambda \mathbb{C}\mathbb{U}_\Sigma\, k_\alpha + \alpha^t_\Lambda \mathbb{C}\alpha_\Sigma\, k_+ - \frac{1}{2}\,\alpha^t_\Lambda \mathbb{C}\widehat{\alpha}_\Sigma\, k_0 - k_{\mathbb{U}(\alpha_\Lambda, \hat{\alpha}_\Sigma)} \\
&\quad - \epsilon_{0\Sigma}\alpha^t_\Lambda \mathbb{C}\, k_\alpha + \epsilon_{-\Sigma}\alpha^t_\Lambda \mathbb{C}\, \widehat{k}_\alpha, \\
\Big[\widehat{\alpha}^t_\Lambda \mathbb{C}\widehat{k}_\alpha, k_\Sigma\Big] &= \Big[\widehat{\alpha}^t_\Lambda \mathbb{C}\widehat{k}_\alpha, k_{\mathbb{U}_\Sigma} + \alpha^t_\Sigma \mathbb{C} k_\alpha + \widehat{\alpha}^t_\Sigma \mathbb{C} \widehat{k}_\alpha + \epsilon_{+\Sigma} k_+ + \epsilon_{0\Sigma} k_0\Big] \\
&= -\widehat{\alpha}^t_\Lambda \mathbb{C}\mathbb{U}_\Sigma\, \widehat{k}_\alpha + \frac{1}{2}\,\widehat{\alpha}^t_\Lambda \mathbb{C}\alpha_\Sigma\, k_0 + k_{\mathbb{T}(\alpha_\Lambda, \hat{\alpha}_\Sigma)} + \widehat{\alpha}^t_\Lambda \mathbb{C}\widehat{\alpha}_\Sigma\, k_- \\
&\quad - \epsilon_{+\Sigma}\widehat{\alpha}^t_\Lambda \mathbb{C}\, k_\alpha + \epsilon_{0\Sigma}\widehat{\alpha}^t_\Lambda \mathbb{C}\, \widehat{k}_\alpha, \\
[\epsilon_{+\Lambda} k_+, k_\Sigma] &= \Big[\epsilon_{+\Sigma} k_+, \widehat{\alpha}^t_\Sigma \mathbb{C}\widehat{k}_\alpha + \epsilon_{0\Sigma} k_0 + \epsilon_{-\Sigma} k_-\Big] \\
&= \epsilon_{+\Lambda}\widehat{\alpha}^t_\Sigma \mathbb{C}\, k_\alpha - 2\epsilon_{+\Lambda}\epsilon_{0\Sigma}\, k_+ - \epsilon_{+\Lambda}\epsilon_{-\Sigma}\, k_0, \\
[\epsilon_{0\Lambda} k_0, k_\Sigma] &= \epsilon_{0\Lambda}\alpha^t_\Sigma \mathbb{C}\, k_\alpha - \epsilon_{0\Lambda}\widehat{\alpha}^t_\Sigma \mathbb{C}\, \widehat{k}_\alpha + 2\epsilon_{+\Sigma}\epsilon_{0\Lambda}\, k_+ - 2\epsilon_{0\Lambda}\epsilon_{-\Sigma}\, k_-, \\
[\epsilon_{-\Lambda} k_-, k_\Sigma] &= \big[\epsilon_{-\Sigma} k_-, \mathbb{C}\alpha^t_\Sigma k_\alpha + \epsilon_{0\Sigma} k_0 + \epsilon_{+\Sigma} k_+\big] \\
&= -\epsilon_{-\Lambda}\alpha^t_\Sigma \mathbb{C}\, \widehat{k}_\alpha + 2\epsilon_{0\Sigma}\epsilon_{-\Lambda}\, k_- + \epsilon_{+\Sigma}\epsilon_{-\Lambda}\, k_0.
\end{aligned}
$$

Adding everything we get

$$
\begin{aligned}
[k_\Lambda, k_\Sigma] = k_{\mathbb{T}(\alpha_\Lambda, \hat{\alpha}_\Sigma)} &+ \big(\alpha^t_\Sigma \mathbb{C}\mathbb{U}_\Lambda + \epsilon_{+\Lambda}\widehat{\alpha}^t_\Sigma \mathbb{C} + \epsilon_{0\Lambda}\alpha^t_\Sigma \mathbb{C}\big)\, k_\alpha \\
&+ \big(\widehat{\alpha}^t_\Sigma \mathbb{C}\mathbb{U}_\Lambda + \epsilon_{-\Sigma}\alpha^t_\Lambda \mathbb{C} + \epsilon_{0\Sigma}\widehat{\alpha}^t_\Lambda \mathbb{C}\big)\, \widehat{k}_\alpha \\
&+ \big(\alpha^t_\Lambda \mathbb{C}\alpha_\Sigma + 2\epsilon_{+\Sigma}\epsilon_{0\Lambda}\big)\, k_+ + \left(\frac{1}{2}\,\widehat{\alpha}^t_\Lambda \mathbb{C}\alpha_\Sigma + \epsilon_{+\Sigma}\epsilon_{-\Lambda}\right) k_0 \\
&+ \big(\widehat{\alpha}^t_\Lambda \mathbb{C}\widehat{\alpha}_\Sigma + 2\epsilon_{0\Lambda}\epsilon_{-\Sigma}\big)\, k_- - (\Lambda \leftrightarrow \Sigma).
\end{aligned}
\tag{E.2.1}
$$

We will take the transpose and use that

$$
\mathbb{U}^t\mathbb{C} + \mathbb{C}\mathbb{U} = 0. \tag{E.2.2}
$$

E.3 Static BPS solutions

E.3.1 Ansatz

We take the following ansatz for the metric and the gauge fields

$$
\mathrm{d}s^2 = \mathrm{e}^{2U}\mathrm{d}t^2 - \mathrm{e}^{-2U}\mathrm{d}r^2 - \mathrm{e}^{2(V-U)}\,\mathrm{d}\Sigma_g^2, \tag{E.3.1a}
$$

$$
A^\Lambda = \tilde{q}^\Lambda\,\mathrm{d}t - \kappa p^\Lambda F'(\theta)\mathrm{d}\phi. \tag{E.3.1b}
$$

The functions $U, V, \tilde{q}$ and p depend only on r. The space Σ_g is a Riemann surface.[1]

[1] The convention are slightly different from the one in the appendix A.7. One needs to make the replacement $(H, H') \to (-\kappa H', H)$.

Ansatz: Vierbein and spin connections

Recall the metric

$$\mathrm{d}s^2 = \mathrm{e}^{2U}\mathrm{d}t^2 - \mathrm{e}^{-2U}\mathrm{d}r^2 - \mathrm{e}^{2(V-U)}\big(\mathrm{d}\theta^2 + F^2\mathrm{d}\phi^2\big). \tag{E.3.2}$$

We introduce the following vierbein

$$e^0 = \mathrm{e}^U\mathrm{d}t, \qquad e^1 = \mathrm{e}^{-U}\mathrm{d}r, \qquad e^2 = \mathrm{e}^{V-U}\mathrm{d}\theta, \qquad e^3 = F\,\mathrm{e}^{V-U}\mathrm{d}\phi. \tag{E.3.3}$$

We compute the differential

$$\begin{aligned}
\mathrm{d}e^0 &= U'\mathrm{d}r \wedge e^0,\\
\mathrm{d}e^1 &= 0,\\
\mathrm{d}e^2 &= (V' - U')\,\mathrm{e}^{V-U}\mathrm{d}r \wedge \mathrm{d}\theta,\\
\mathrm{d}e^3 &= F\,(V' - U')\,\mathrm{e}^{V-U}\mathrm{d}r \wedge \mathrm{d}\phi + F'\,\mathrm{e}^{V-U}\mathrm{d}\theta \wedge \mathrm{d}\phi.
\end{aligned}$$

Using (A.7.4) and the vierbein expressions (E.3.3), we can replace all the differentials by the vierbein

$$\mathrm{d}e^0 = U'\,\mathrm{e}^U e^1 \wedge e^0, \tag{E.3.4a}$$

$$\mathrm{d}e^1 = 0, \tag{E.3.4b}$$

$$\mathrm{d}e^2 = (V' - U')\mathrm{e}^U e^1 \wedge e^2, \tag{E.3.4c}$$

$$\mathrm{d}e^3 = (V' - U')\mathrm{e}^U e^1 \wedge e^3 + \frac{F'}{F}\,\mathrm{e}^{U-V} e^2 \wedge e^3. \tag{E.3.4d}$$

Using Cartan formula

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0 \tag{E.3.5}$$

we obtain the following spin connections

$$\begin{aligned}
\omega^0{}_1 &= U'\,\mathrm{e}^U e^0, \qquad \omega^2{}_1 = (V' - U')\,\mathrm{e}^U e^2,\\
\omega^3{}_1 &= (V' - U')\,\mathrm{e}^U e^3, \qquad \omega^3{}_2 = \frac{F'}{F}\,\mathrm{e}^{U-V} e^3.
\end{aligned} \tag{E.3.6}$$

The explicit components

$$\omega^a{}_b = \omega_\mu{}^a{}_b\,\mathrm{d}x^\mu \tag{E.3.7}$$

are

$$\omega_{001} = U'\,\mathrm{e}^U, \qquad \omega_{212} = \omega_{313} = (V' - U')\,\mathrm{e}^U, \qquad \omega_{323} = \frac{F'}{F}\,\mathrm{e}^{U-V}. \tag{E.3.8}$$

Field strength

Recall the gauge fields

$$A^\Lambda = \tilde{q}^\Lambda\,\mathrm{d}t - \kappa p^\Lambda F'\,\mathrm{d}\phi. \tag{E.3.9}$$

In terms of the vierbein (E.3.3) we have

$$A^\Lambda = \tilde{q}^\Lambda\;\mathrm{e}^U e^0 - \kappa\,\frac{F'}{F}\,\mathrm{e}^{U-V} p^\Lambda\, e^3. \tag{E.3.10}$$

Now we compute the field strength as

$$F^\Lambda = \mathrm{d}A^\Lambda = \tilde{q}'^\Lambda\,\mathrm{d}r \wedge \mathrm{d}t + (p^\Lambda - 2b\tilde{q}^\Lambda)F\,\mathrm{d}\theta \wedge \mathrm{d}\phi - \kappa p'^\Lambda F'\,\mathrm{d}r \wedge \mathrm{d}\phi \tag{E.3.11a}$$

$$= -\tilde{q}'^\Lambda\, e^0 \wedge e^1 - \kappa\,\frac{F'}{F}\,p'^\Lambda\,\mathrm{e}^{2U-V}\, e^1 \wedge e^3 + p^\Lambda\,\mathrm{e}^{2(U-V)} e^2 \wedge e^3. \tag{E.3.11b}$$

The Hodge dual field strength is

$$\star F^\Lambda = \tilde{q}'^\Lambda\, e^2 \wedge e^3 + \kappa\, \frac{F'}{F}\, p'^\Lambda\, \mathrm{e}^{2U-V}\, e^0 \wedge e^2 + p^\Lambda\, \mathrm{e}^{2(U-V)} e^0 \wedge e^1 \tag{E.3.12a}$$

$$= \tilde{q}'^\Lambda\, \mathrm{e}^{2(V-U)} F\, \mathrm{d}\theta \wedge \mathrm{d}\phi + \kappa\, \frac{F'}{F}\, p'^\Lambda\, \mathrm{e}^{2U}\, \mathrm{d}t \wedge \mathrm{d}\theta - p^\Lambda\, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \mathrm{d}t. \tag{E.3.12b}$$

Finally the anti-self dual form is

$$\mathcal{F}^{-\Lambda} = \frac{1}{2}\big(F^\Lambda - i \star F^\Lambda\big) = \tilde{F}^\Lambda (e^0 \wedge e^1 + i\, e^2 \wedge e^3) + \frac{F'}{F}\, \tilde{G}^\Lambda (e^1 \wedge e^3 + i\, e^0 \wedge e^2) \tag{E.3.13}$$

where

$$\tilde{F}^\Lambda = -\frac{1}{2}\, \tilde{q}^\Lambda - \frac{i}{2}\, p^\Lambda\, \mathrm{e}^{2(U-V)}, \qquad \tilde{G}^\Lambda = -\kappa\, \mathrm{e}^{2U-V}\, p'^\Lambda. \tag{E.3.14}$$

The symplectic dual G_Λ of F^Λ is defined by

$$G_\Lambda = \frac{\delta \mathcal{L}}{\delta F^\Lambda} = \mathcal{R}_{\Lambda\Sigma}\, F^\Sigma - \mathcal{I}_{\Lambda\Sigma} \star F^\Sigma. \tag{E.3.15}$$

It reads explictly (with a matrix/vector notation)

$$\begin{aligned} G = \mathcal{R}\ &(\tilde{q}'\, \mathrm{d}r \wedge \mathrm{d}t + p\, F\, \mathrm{d}\theta \wedge \mathrm{d}\phi - \kappa p' F'\, \mathrm{d}r \wedge \mathrm{d}\phi) \\ &- \mathcal{I} \left(\tilde{q}'\, \mathrm{e}^{2(V-U)} F\, \mathrm{d}\theta \wedge \mathrm{d}\phi + \kappa\, \frac{F'}{F}\, p'\, \mathrm{e}^{2U}\, \mathrm{d}t \wedge \mathrm{d}\theta - p\, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \mathrm{d}t\right), \end{aligned} \tag{E.3.16}$$

or after simplification

$$\begin{aligned} G = \big(\mathcal{R}\tilde{q}' + \mathcal{I} p\, \mathrm{e}^{2(U-V)}\big) \mathrm{d}t + \big(\mathcal{R}p - \mathcal{I}\tilde{q}'\, \mathrm{e}^{2(V-U)}\big) F\, \mathrm{d}\theta \wedge \mathrm{d}\phi \\ - \kappa\, F' \big(\mathcal{R}\, \mathrm{d}r \wedge \mathrm{d}\phi + \mathcal{I}\, \mathrm{e}^{2U}\, \mathrm{d}t \wedge \mathrm{d}\theta\big)\, p'. \end{aligned} \tag{E.3.17}$$

The "conserved" electric and magnetic charges are defined by [104]

$$p^\Lambda = \frac{1}{4\pi} \int_{S^2} F^\Lambda, \qquad q_\Lambda = \frac{1}{4\pi} \int_{S^2} G_\Lambda. \tag{E.3.18}$$

The pair

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \tag{E.3.19}$$

forms the correct symplectic vector of charges.[2]

We obtain the explicit expressions

$$q_\Lambda = \mathcal{R}_{\Lambda\Sigma}\, p^\Sigma - \mathrm{e}^{2(V-U)} \mathcal{I}_{\Lambda\Sigma} \tilde{q}'^\Sigma. \tag{E.3.20}$$

We can solve for $\tilde{q}'^\Lambda$ in terms of p^Λ and q_Λ

$$\tilde{q}'^\Lambda = \mathrm{e}^{2(U-V)} (\mathcal{I}^{-1})^{\Lambda\Sigma} (\mathcal{R}_{\Sigma\Delta}\, p^\Delta - q_\Sigma). \tag{E.3.21}$$

If $p'^\Lambda = 0$ we can obtain the field strength and its Hodge dual in terms of the symplectic charges (we use a matrix/vector notation)

$$\begin{aligned} F &= \mathrm{e}^{2(U-V)} (\mathcal{I}^{-1} \mathcal{R}\, p - \mathcal{I}^{-1} q)\, \mathrm{d}r \wedge \mathrm{d}t + p\, F\, \mathrm{d}\theta \wedge \mathrm{d}\phi, \\ \star F &= -p\, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \mathrm{d}t + \mathcal{I}^{-1} (\mathcal{R}p - q)\, F \mathrm{d}\theta \wedge \mathrm{d}\phi. \end{aligned}$$

[2] Note that [104] forgets to add κ in the formula: the presence of κ here can be traced to the fact that it is absent in (E.3.1b), and ultimately the reason is that the gauge field should be defined with the integral of F, and not its derivative; see [4] for comparison.

From here we compute the symplectic dual of F^Λ

$$\begin{aligned} G = \mathcal{R} & \left(\mathrm{e}^{2(U-V)} \mathcal{I}^{-1} (\mathcal{R}p - q) \, \mathrm{d}r \wedge \mathrm{d}t + p \, F \, \mathrm{d}\theta \wedge \mathrm{d}\phi \right) \\ & - \mathcal{I} \left(-p \, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \mathrm{d}t + \mathcal{I}^{-1} (\mathcal{R}p - q) \, F \mathrm{d}\theta \wedge \mathrm{d}\phi \right) \end{aligned} \tag{E.3.23}$$

and after replacing the charges

$$G = \mathrm{e}^{2(U-V)} \big((\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R}) p - \mathcal{R}\mathcal{I}^{-1} q \big) \mathrm{d}t + q F \, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{E.3.24}$$

We can gather both vectors into a symplectic vector using the expression of $\mathcal{M}$ [90, p. 515]

$$\mathcal{F} = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} = \mathrm{e}^{2(V-U)} \mathcal{M} \mathcal{Q} \, \mathrm{d}r \wedge \mathrm{d}t + \mathcal{Q} \, F \, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{E.3.25}$$

Note that it does not seem possible to write such an expression if $p' \neq 0$.

Dirac quantization condition implies that [104, sec. 2]

$$p^\Lambda P^3_\Lambda \in \mathbb{Z}, \qquad p^\Lambda k^u_\Lambda \in \mathbb{Z}. \tag{E.3.26}$$

Supersymmetry restricts the integers to be

$$p^\Lambda P^3_\Lambda = \kappa, \qquad p^\Lambda k^u_\Lambda = 0. \tag{E.3.27}$$

It seems that for $P^1, P^2 \neq 0$ one has [78, app. D]

$$(p^\Lambda P^x_\Lambda)^2 = \kappa^2. \tag{E.3.28}$$

E.3.2 Symplectic extension

Almost all the BPS equations we obtained in the previous sections are already symplectic invariant since they are given in terms of symplectic invariant quantities.

We replace the charges by $\mathcal{Q}$. To replace $\tilde{q}'^\Lambda$ we note that

$$\mathrm{e}^{-2(U-V)} \tilde{q}'^\Lambda = (\mathcal{I}^{-1})^{\Lambda\Sigma} (\mathcal{R}_{\Sigma\Delta} \, p^\Delta - q_\Sigma) \tag{E.3.29}$$

corresponds to the first component of $-\mathcal{M}\mathcal{Q}$.

The symplectic invariant equations are

$$\langle \mathcal{Q}, \mathcal{G} \rangle = -\kappa, \tag{E.3.30a}$$

$$\mathrm{Re}(\mathrm{e}^{-i\psi} \mathcal{L}) = \mathrm{e}^{2(U-V)} \, \mathrm{Im}(\mathrm{e}^{-i\psi} \mathcal{Z}) \tag{E.3.30b}$$

$$\psi' = -A_r + 2 \, \mathrm{e}^{-U} \, \mathrm{Re}(\mathrm{e}^{-i\psi} \mathcal{L}), \tag{E.3.30c}$$

$$\begin{aligned} 2 \, \mathrm{e}^{2V} \partial_r \big(\mathrm{e}^{-U} \, \mathrm{Im}(\mathrm{e}^{-i\psi} \mathcal{V}) \big) = & -8 \, \mathrm{e}^{2(V-U)} \, \mathrm{Re}(\mathrm{e}^{-i\psi} \mathcal{L}) \, \mathrm{Re}(\mathrm{e}^{-i\psi} \mathcal{V}) \\ & - \mathcal{Q} - \mathrm{e}^{2(V-U)} \mathcal{M} \mathcal{G}, \end{aligned} \tag{E.3.30d}$$

$$(\mathrm{e}^V)' = -2 \, \mathrm{e}^{V-U} \, \mathrm{Im}(\mathrm{e}^{-i\psi} \mathcal{L}). \tag{E.3.30e}$$

We also have the equation

$$2 \, \partial_r \big(\mathrm{e}^U \, \mathrm{Re}(\mathrm{e}^{-i\psi} \mathcal{V}) \big) = \mathrm{e}^{2(U-V)} \mathcal{M} \mathcal{Q} + \mathcal{G}. \tag{E.3.31}$$

The second term cannot be seen from the original equation since g^Λ was set to zero, but we could get it by computing explicitly the derivative of M_Λ.

The equation (E.3.30d) can be modified using (E.3.30e) to include one factor e^V inside the derivative. The LHS is

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) &= 2\,\mathrm{e}^{V}\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) - 2\,\mathrm{e}^{V-U}\partial_r(\mathrm{e}^{V})\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &= 2\,\mathrm{e}^{V}\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) + 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \end{aligned}$$

and it combines with the RHS to

$$\begin{aligned} 2\,\mathrm{e}^{V}\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = &- 8\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &- 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &- \mathcal{Q} - \mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G}. \end{aligned} \tag{E.3.32}$$

Finally we recall the equations for U' and z'^i

$$(\mathrm{e}^{U})' = -g_\Lambda\tilde{p}^\Lambda\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}), \tag{E.3.33a}$$

$$(z^i)' = \mathrm{e}^{-U}\,\mathrm{e}^{i\psi}g^{i\bar{\jmath}}\big(\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\mathcal{Z} + i\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\big). \tag{E.3.33b}$$

E.3.3 Fayet–Iliopoulos gauging

We write

$$\mathcal{G} = \mathcal{P}^3 = \begin{pmatrix} g^\Lambda \\ g_\Lambda \end{pmatrix} \tag{E.3.34}$$

to really distinguish between non-constant and constant prepotentials.

Equations from special geometry

We can use several identities involving the quartic invariant in order to express all equations in terms of $\mathrm{Im}\,\mathcal{V}$ and V uniquely.

We define

$$\widetilde{\mathcal{V}} = \mathrm{e}^{V-U}\,\mathrm{e}^{-i\psi}\,\mathcal{V}. \tag{E.3.35}$$

The first step is to use the identity (7.3.13) in (E.3.32)

$$2\,\mathrm{e}^{V}\partial_r\,\mathrm{Im}\,\widetilde{\mathcal{V}} = -\mathcal{Q} + I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{G}), \tag{E.3.36}$$

Then using (7.3.12) and (7.3.14) as

$$I_4(\mathrm{Im}\,\widetilde{\mathcal{V}}) = \frac{1}{16}\,\mathrm{e}^{4(V-U)}, \qquad \mathrm{Re}\,\widetilde{\mathcal{V}} = -2\,\mathrm{e}^{2(U-V)}\,I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}). \tag{E.3.37}$$

we can replace $\mathrm{Re}(\widetilde{\mathcal{V}})$ and e^U

$$\mathrm{e}^{2U-V}\,\mathrm{Re}\,\widetilde{\mathcal{V}} = -2\,\mathrm{e}^{4U-3V}\,I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}) = -\frac{1}{8}\,\mathrm{e}^{V}\,\frac{I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}})}{I_4(\mathrm{Im}\,\widetilde{\mathcal{V}})}. \tag{E.3.38}$$

In terms of this new variable the equations (E.3.30d) and (E.3.30e) become

$$2\,\mathrm{e}^{V}\partial_r\big(\mathrm{Im}\,\widetilde{\mathcal{V}})\big) = -\mathcal{Q} + I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{G}), \tag{E.3.39a}$$

$$(\mathrm{e}^{V})' = -2\left\langle \mathcal{G}, \mathrm{Im}\,\widetilde{\mathcal{V}} \right\rangle. \tag{E.3.39b}$$

E.4 NUT black hole

E.4.1 Ansatz

We consider $N = 2$ gauged supergravity with n_v vector multiplets. Fayet–Iliopoulos gaugings are denoted by g_Λ.

We take the following ansatz for the metric and the gauge fields[3]

$$\mathrm{d}s^2 = \mathrm{e}^{2U}\big(\mathrm{d}t + 2\kappa n F'(\theta)\,\mathrm{d}\phi\big)^2 - \mathrm{e}^{-2U}\mathrm{d}r^2 - \mathrm{e}^{2(V-U)}\big(\mathrm{d}\theta^2 + F(\theta)^2\mathrm{d}\phi^2\big), \tag{E.4.1a}$$

$$A^\Lambda = \tilde{q}^\Lambda\big(\mathrm{d}t + 2\kappa n F'(\theta)\,\mathrm{d}\phi\big) - \kappa\tilde{p}^\Lambda F'(\theta)\mathrm{d}\phi. \tag{E.4.1b}$$

$U, V, \tilde{q}$ and p are only function of r, while

$$F(\theta) = \begin{cases} \sin\theta & \kappa = 1 \\ \theta & \kappa = 0 \\ \sinh\theta & \kappa = -1 \end{cases}, \qquad \kappa = \mathrm{sign}(1-g) \tag{E.4.2}$$

where g is the genus of the surface. We note that the second derivative of F satisfies

$$F'' = -\kappa F. \tag{E.4.3}$$

E.4.2 Vierbein and spin connections

Recall the metric

$$\mathrm{d}s^2 = \mathrm{e}^{2U}\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big)^2 - \mathrm{e}^{-2U}\mathrm{d}r^2 - \mathrm{e}^{2(V-U)}\big(\mathrm{d}\theta^2 + F^2\mathrm{d}\phi^2\big). \tag{E.4.4}$$

We introduce the following vierbein

$$e^0 = \mathrm{e}^U\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big), \qquad e^1 = \mathrm{e}^{-U}\mathrm{d}r, \qquad e^2 = \mathrm{e}^{V-U}\mathrm{d}\theta, \qquad e^3 = F\,\mathrm{e}^{V-U}\mathrm{d}\phi. \tag{E.4.5}$$

We compute the differential

$$\begin{aligned}
\mathrm{d}e^0 &= U'\mathrm{d}r \wedge e^0 + 2\kappa n\,F''\,\mathrm{e}^U\mathrm{d}\theta \wedge \mathrm{d}\phi, \\
\mathrm{d}e^1 &= 0, \\
\mathrm{d}e^2 &= (V' - U')\,\mathrm{e}^{V-U}\mathrm{d}r \wedge \mathrm{d}\theta, \\
\mathrm{d}e^3 &= F\,(V' - U')\,\mathrm{e}^{V-U}\mathrm{d}r \wedge \mathrm{d}\phi + F'\,\mathrm{e}^{V-U}\mathrm{d}\theta \wedge \mathrm{d}\phi.
\end{aligned}$$

Using (E.4.3) and the vierbein expressions (E.4.5), we can replace all the differential by the vierbein

$$\mathrm{d}e^0 = U'\,\mathrm{e}^U e^1 \wedge e^0 - 2n\,\mathrm{e}^{3U-2V}e^2 \wedge e^3, \tag{E.4.6a}$$

$$\mathrm{d}e^1 = 0, \tag{E.4.6b}$$

$$\mathrm{d}e^2 = (V' - U')e^U e^1 \wedge e^2, \tag{E.4.6c}$$

$$\mathrm{d}e^3 = (V' - U')e^U e^1 \wedge e^3 + \frac{F'}{F}\,\mathrm{e}^{U-V}e^2 \wedge e^3. \tag{E.4.6d}$$

Using the Cartan formula

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0 \tag{E.4.7}$$

[3] Nick is defining $N = \kappa n$.

we obtain the following spin connections

$$\begin{gathered}\omega^{0}{}_{1} = U'\,\mathrm{e}^{U} e^{0}, \qquad \omega^{0}{}_{2} = -n\,\mathrm{e}^{3U-2V} e^{3}, \qquad \omega^{0}{}_{3} = n\,\mathrm{e}^{3U-2V} e^{2}, \\ \omega^{2}{}_{1} = (V' - U')\,\mathrm{e}^{U} e^{2}, \qquad \omega^{3}{}_{1} = (V' - U')\,\mathrm{e}^{U} e^{3}, \\ \omega^{3}{}_{2} = \frac{F'}{F}\,\mathrm{e}^{U-V} e^{3} + n\,\mathrm{e}^{3U-2V} e^{0}.\end{gathered} \tag{E.4.8}$$

The last term in $\omega^{3}{}_{2}$ comes from the fact that

$$0 = \mathrm{d}e^{3} + \omega^{3}{}_{2} e^{2} + \omega^{3}{}_{0} e^{0} = \mathrm{d}e^{3} + \omega^{3}{}_{2} e^{2} + n\,\mathrm{e}^{3U-2V} e^{2} \wedge e^{0}. \tag{E.4.9}$$

since $\omega^{3}{}_{0} = \omega^{0}{}_{3}$.

The explicit components

$$\omega^{a}{}_{b} = \omega_{\mu}{}^{a}{}_{b}\mathrm{d}x^{\mu} \tag{E.4.10}$$

are

$$\begin{gathered}\omega_{001} = U'\,\mathrm{e}^{U}, \qquad \omega_{203} = -\omega_{302} = n\,\mathrm{e}^{3U-2V}, \\ \omega_{212} = \omega_{313} = (V' - U')\,\mathrm{e}^{U}, \qquad \omega_{323} = \frac{F'}{F}\,\mathrm{e}^{U-V}, \\ \omega_{023} = n\,\mathrm{e}^{3U-2V}.\end{gathered} \tag{E.4.11}$$

E.4.3 Gauge fields

Recall the gauge fields

$$A^{\Lambda} = \tilde{q}^{\Lambda}\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) - \kappa\tilde{p}^{\Lambda} F'\,\mathrm{d}\phi \tag{E.4.12a}$$

$$= \tilde{q}^{\Lambda}\,\mathrm{d}t - \kappa p^{\Lambda} F'\,\mathrm{d}\phi. \tag{E.4.12b}$$

where we have defined

$$p^{\Lambda} = \tilde{p}^{\Lambda} - 2n\tilde{q}^{\Lambda}. \tag{E.4.13}$$

For $n = 0$ we obviously recover the formula from [104], and for this reason formulas written in terms of $^{\Lambda}$ in terms of $\tilde{p}^{\Lambda}$ should be equivalent to this case.

In terms of the vierbein (E.4.5) we have

$$A^{\Lambda} = \tilde{q}^{\Lambda}\,\mathrm{e}^{U} e^{0} - \kappa\,\frac{F'}{F}\,\mathrm{e}^{U-V}\tilde{p}^{\Lambda} e^{3}. \tag{E.4.14}$$

Field strengths

Electric field strength Now we compute the field strength

$$F^{\Lambda} = \mathrm{d}A^{\Lambda} \tag{E.4.15}$$

and we get

$$F^{\Lambda} = \tilde{q}'^{\Lambda}\,\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + p^{\Lambda} F\,\mathrm{d}\theta \wedge \mathrm{d}\phi - \kappa\tilde{p}'^{\Lambda} F'\,\mathrm{d}r \wedge \mathrm{d}\phi \tag{E.4.16a}$$

$$= -\tilde{q}'^{\Lambda}\,\mathrm{d}t \wedge \mathrm{d}r + p^{\Lambda}\,F\,\mathrm{d}\theta \wedge \mathrm{d}\phi - \kappa p'^{\Lambda} F'\,\mathrm{d}r \wedge \mathrm{d}\phi, \tag{E.4.16b}$$

or in terms of the tetrads

$$F^{\Lambda} = -\tilde{q}'^{\Lambda}\,e^{0} \wedge e^{1} + p^{\Lambda}\,\mathrm{e}^{2(U-V)} e^{2} \wedge e^{3} - \kappa\,\tilde{p}'^{\Lambda}\frac{F'}{F}\,\mathrm{e}^{2U-V}\,e^{1} \wedge e^{3}. \tag{E.4.16c}$$

In particular it is trivial to see that the Bianchi identity is satisfied

$$\mathrm{d}F = p'^{\Lambda}\,F\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi + p'^{\Lambda} F\,\mathrm{d}\theta \wedge \mathrm{d}r \wedge \mathrm{d}\phi = 0. \tag{E.4.17}$$

Hodge field strength Using the facts that

$$\star(e^\mu \wedge e^\nu) = \frac{1}{2}\,\varepsilon^{\mu\nu}{}_{\rho\sigma}\, e^\rho \wedge e^\sigma, \tag{E.4.18}$$

and

$$\varepsilon^{01}{}_{23} = \varepsilon^{13}{}_{02} = -1, \qquad \varepsilon^{23}{}_{01} = 1, \tag{E.4.19}$$

the Hodge dual field strength is found to be

$$\star F^\Lambda = p^\Lambda\, \mathrm{e}^{2(U-V)} e^0 \wedge e^1 + \tilde q'^\Lambda\, e^2 \wedge e^3 + \kappa\, \tilde p'^\Lambda \frac{F'}{F}\; \mathrm{e}^{2U-V}\, e^0 \wedge e^2 \tag{E.4.20a}$$

or by replacing the tetrads

$$\begin{aligned}\star F^\Lambda = &- p^\Lambda\, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\, \mathrm{d}\phi\big) + \tilde q'^\Lambda\, \mathrm{e}^{2(V-U)} F\, \mathrm{d}\theta \wedge \mathrm{d}\phi \\ &- \kappa\, \tilde p'^\Lambda \frac{F'}{F}\; \mathrm{e}^{2U}\, \big(\mathrm{d}t + 2\kappa n F'\, \mathrm{d}\phi\big) \wedge \mathrm{d}\theta.\end{aligned} \tag{E.4.20b}$$

We can also expand in order to get all components

$$\begin{aligned}\star F^\Lambda = p^\Lambda\, \mathrm{e}^{2(U-V)} \mathrm{d}t \wedge \mathrm{d}r &+ \left(\tilde q'^\Lambda\, \mathrm{e}^{2(V-U)} + 2n\, \tilde p'^\Lambda \frac{F'^2}{F^2}\; \mathrm{e}^{2U}\right) F\, \mathrm{d}\theta \wedge \mathrm{d}\phi \\ &- 2\kappa n\, p^\Lambda\, \mathrm{e}^{2(U-V)} F'\, \mathrm{d}r \wedge \mathrm{d}\phi - \kappa\, \tilde p'^\Lambda \frac{F'}{F}\; \mathrm{e}^{2U}\, \mathrm{d}t \wedge \mathrm{d}\theta.\end{aligned} \tag{E.4.20c}$$

(Anti-)self dual field strength The anti-self dual form is

$$\mathcal{F}^{-\Lambda} = \frac{1}{2}\big(F^\Lambda - i\star F^\Lambda\big) = \tilde F^\Lambda (e^0 \wedge e^1 + i\, e^2 \wedge e^3) + \frac{F'}{F}\, \tilde G^\Lambda (e^1 \wedge e^3 + i\, e^0 \wedge e^2) \tag{E.4.21}$$

where

$$\tilde F^\Lambda = -\frac{1}{2}\, \tilde q^\Lambda - \frac{i}{2}\, p^\Lambda\, \mathrm{e}^{2(U-V)}, \qquad \tilde G^\Lambda = -\kappa\, \mathrm{e}^{2U-V}\, \tilde p'^\Lambda. \tag{E.4.22}$$

Magnetic field strength The symplectic dual G_Λ of F^Λ is defined by

$$G_\Lambda = \star\left(\frac{\delta \mathcal{L}}{\delta F^\Lambda}\right) = \mathcal{R}_{\Lambda\Sigma}\, F^\Sigma - \mathcal{I}_{\Lambda\Sigma}\, \star F^\Sigma. \tag{E.4.23}$$

It reads explictly (with a matrix/vector notation)

$$\begin{aligned}G = \mathcal{R}\, &\big(\tilde q'\, \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\, \mathrm{d}\phi\big) + p\, F\, \mathrm{d}\theta \wedge \mathrm{d}\phi - \kappa \tilde p' F'\, \mathrm{d}r \wedge \mathrm{d}\phi\big) \\ &- \mathcal{I}\bigg(\tilde q'\, \mathrm{e}^{2(V-U)} F\, \mathrm{d}\theta \wedge \mathrm{d}\phi + \kappa\, \frac{F'}{F}\, \tilde p'\, \mathrm{e}^{2U}\, \big(\mathrm{d}t + 2\kappa n F'\, \mathrm{d}\phi\big) \wedge \mathrm{d}\theta \\ &\qquad + p\, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\, \mathrm{d}\phi\big)\bigg),\end{aligned} \tag{E.4.24}$$

or after simplication (in the last term we moved $\tilde p'$ in front of the expression since all matrices are symmetric)

$$\begin{aligned}G = \big(\mathcal{R}\tilde q' + \mathcal{I} p\, \mathrm{e}^{2(U-V)}\big)\, \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\, \mathrm{d}\phi\big) + \big(\mathcal{R} p - \mathcal{I} \tilde q'\, \mathrm{e}^{2(V-U)}\big) F\, \mathrm{d}\theta \wedge \mathrm{d}\phi \\ - \kappa\, \tilde p'\, F'\, \big(\mathcal{R}\, \mathrm{d}r \wedge \mathrm{d}\phi + \mathcal{I}\, \mathrm{e}^{2U}\, \big(\mathrm{d}t + 2\kappa n F'\, \mathrm{d}\phi\big) \wedge \mathrm{d}\theta\big)\,.\end{aligned} \tag{E.4.25}$$

Electromagnetic charges

The electric and magnetic charges are defined by [104]

$$p^\Lambda = \frac{1}{4\pi}\int_{S^2} F^\Lambda, \qquad q_\Lambda = \frac{1}{4\pi}\int_{S^2} G_\Lambda. \tag{E.4.26}$$

The pair

$$\mathcal{Q} = (p^\Lambda, q_\Lambda) \tag{E.4.27}$$

forms the correct symplectic vector of charges.[4]

We obtain the explicit expressions

$$p^\Lambda = \tilde{p}^\Lambda - 2n\,\tilde{q}^\Lambda, \tag{E.4.28a}$$

$$q_\Lambda = \mathcal{R}_{\Lambda\Sigma}\,p^\Sigma - \mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{q}'^\Sigma + 2n\,\mathcal{I}_{\Lambda\Sigma}\tilde{p}'^\Sigma\,\mathrm{e}^{2U}\int \frac{F'^2}{F}\,\mathrm{d}\theta, \tag{E.4.28b}$$

which justify a posteriori that we identified p^Λ above.

The last integral can be done as

$$\int_0^{\theta_{\max}} \frac{F'^2}{F}\,\mathrm{d}\theta = \int_0^{F_{\max}} \frac{F'}{F}\,\mathrm{d}F = \ln F(\theta_{\max}) - \ln F(0). \tag{E.4.29}$$

Since $F(0) = 0$ the last piece is divergent so we should require that

$$n = 0 \qquad \text{or} \qquad \tilde{p}'^\Lambda = 0. \tag{E.4.30}$$

Since we want that our black holes carry a NUT charge we require

$$\boxed{\tilde{p}'^\Lambda = 0.} \tag{E.4.31}$$

Another evidence for imposing this equation is that the field strength (E.4.16) and its dual (E.4.20) do not respect the isometries of the spacetime if $\tilde{p}^\Lambda \neq 0$. Moreover if this equation does not hold it is not possible to construct the symplectic vector of field strengths. Finally we will see that supersymmetry imposes naturally this constraint. For the rest of the section we will consider that this term is absent.

Imposing (E.4.31) we obtain the electromagnetic charges

$$p^\Lambda = \tilde{p}^\Lambda - 2n\,\tilde{q}^\Lambda, \tag{E.4.32a}$$

$$q_\Lambda = \mathcal{R}_{\Lambda\Sigma}\,p^\Sigma - \mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{q}'^\Sigma. \tag{E.4.32b}$$

We can solve for $\tilde{q}'^\Lambda$ in terms of p^Λ and q_Λ

$$\boxed{\tilde{q}'^\Lambda = \mathrm{e}^{2(U-V)}(\mathcal{I}^{-1})^{\Lambda\Sigma}(\mathcal{R}_{\Sigma\Delta}\,p^\Delta - q_\Sigma).} \tag{E.4.33}$$

We note that the above relation corresponds to

$$\tilde{q}'^\Lambda = -\,\mathrm{e}^{2(U-V)}(\mathcal{M}\mathcal{Q})^\Lambda, \tag{E.4.34}$$

and we may use this relation for obtaning symplectic covariant formulas.

[4] Note that [104] forgets to add κ in the formula: the presence of κ here can be traced to the fact that it is absent in (E.4.1b), and ultimately the reason is that the gauge field should be defined with the integral of F, and not its derivative; see [4] for comparison.

Symplectic field strengths

Imposing the condition (E.4.31), the expression (E.4.16) for the field strength becomes

$$F^\Lambda = \tilde{q}'^\Lambda \, \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \, \mathrm{d}\phi\big) + p^\Lambda F \, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{E.4.35}$$

The Bianchi identity reads

$$\mathrm{d}F^\Lambda = (p' + 2n \, \tilde{q}'^\Lambda) F \, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi = \tilde{p}'^\Lambda F \, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi = 0 \tag{E.4.36}$$

which is solved by (E.4.31) and this is consistent.

The Hodge dual (E.4.20) reads

$$\star F^\Lambda = -p^\Lambda \, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \, \mathrm{d}\phi\big) + \tilde{q}'^\Lambda \, \mathrm{e}^{2(V-U)} F \, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{E.4.37}$$

Finally the magnetic field strength (E.4.25) is

$$G = \big(\mathcal{R}\tilde{q}' + \mathcal{I} p \, \mathrm{e}^{2(U-V)}\big) \, \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \, \mathrm{d}\phi\big) + \big(\mathcal{R}p - \mathcal{I}\tilde{q}' \, \mathrm{e}^{2(V-U)}\big) F \, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{E.4.38}$$

Then we can use the expression (E.4.33) for removing $\tilde{q}'$ in F^Λ and G_Λ (we use a matrix/vector notation)

$$F = \mathrm{e}^{2(U-V)} (\mathcal{I}^{-1}\mathcal{R} \, p - \mathcal{I}^{-1} q) \, \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \, \mathrm{d}\phi\big) + p \, F \, \mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{E.4.39a}$$

$$G = \mathrm{e}^{2(U-V)} \big((\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})p - \mathcal{R}\mathcal{I}^{-1} q\big) \, \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \, \mathrm{d}\phi\big) + q \, F \, \mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{E.4.39b}$$

where G is obtained from the simplification of

$$\begin{aligned} G = \mathcal{R} \, \Big(&\mathrm{e}^{2(U-V)} \mathcal{I}^{-1} (\mathcal{R}p - q) \, \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \, \mathrm{d}\phi\big) + p \, F \, \mathrm{d}\theta \wedge \mathrm{d}\phi\Big) \\ &- \mathcal{I} \Big(-p \, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \, \mathrm{d}\phi\big) + \mathcal{I}^{-1} (\mathcal{R}p - q) \, F \mathrm{d}\theta \wedge \mathrm{d}\phi\Big). \end{aligned} \tag{E.4.40}$$

Note that we also have

$$\star F = -p \, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \, \mathrm{d}\phi\big) + \mathcal{I}^{-1} (\mathcal{R}p - q) \, F \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{E.4.41}$$

Looking at (E.4.39) we can gather F and G into a symplectic vector using (E.4.34)

$$\mathcal{F} = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} = \mathrm{e}^{2(V-U)} \mathcal{M}\mathcal{Q} \, \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \, \mathrm{d}\phi\big) + \mathcal{Q} \, F \, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{E.4.42}$$

As explained above we cannot obtain this symplectic vector if $\tilde{p}' \neq 0$.

Maxwell equation

Maxwell equation reads

$$\mathrm{d}G_\Lambda = 0. \tag{E.4.43}$$

From the expression (E.4.39) we obtain

$$\mathrm{d}G = \Big[2n \, \mathrm{e}^{2(U-V)} \big((\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})p - \mathcal{R}\mathcal{I}^{-1} q\big) + q'\Big] \, F \, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{E.4.44}$$

or in components

$$q' = -2n \, \mathrm{e}^{2(U-V)} \big((\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})p - \mathcal{R}\mathcal{I}^{-1} q\big). \tag{E.4.45}$$

This computation is much more complicated if one keeps $\tilde{p}' \neq 0$ (the hope would be to get $\tilde{p}' = 0$ as a second equation).

The constraint (E.4.31) and the Bianchi identity

$$\mathrm{d}F^\Lambda = 0 \tag{E.4.46}$$

both read

$$\tilde{p}' = p' + 2n\,\tilde{q}' = 0. \tag{E.4.47}$$

Using the expression (E.4.33) one obtains

$$p' = -2n\ \mathrm{e}^{2(U-V)}\mathcal{I}^{-1}(\mathcal{R}\,p - q). \tag{E.4.48}$$

The equations for p' and q' can be gathered into a symplectic equation as

$$\mathcal{Q}' = -2n\,\mathrm{e}^{2(U-V)}\,\mathcal{M}\mathcal{Q} \tag{E.4.49}$$

using the expression for $\mathcal{M}$. This result can also be straightforwardly derived from the symplectic field strength (E.4.42).

Central charge

The central charge is defined by

$$\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V}\rangle = p^\Lambda M_\Lambda - q_\Lambda L^\Lambda. \tag{E.4.50}$$

where $\mathcal{Q} = (p^\Lambda, q_\Lambda)$. Using (E.4.32), the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and $M_\Lambda = \mathcal{N}_{\Lambda\Sigma}L^\Sigma$ we can find another expression

$$\mathcal{Z} = p^\Lambda(\mathcal{R}_{\Lambda\Sigma} + i\mathcal{I}_{\Lambda\Sigma})L^\Sigma - \big(\mathcal{R}_{\Lambda\Sigma}\,p^\Lambda -\ \mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{q}'^\Lambda\big)L^\Sigma,$$

and after simplifcation we get

$$\mathcal{Z} = \mathcal{I}_{\Lambda\Sigma}\big(\,\mathrm{e}^{2(V-U)}\tilde{q}'^\Lambda + ip^\Lambda\big)L^\Sigma. \tag{E.4.51}$$

Now we can deduce its relation with $\tilde{F}^\Lambda$ from (E.4.22)

$$\mathcal{Z} = -2\,\mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{F}^\Lambda L^\Sigma. \tag{E.4.52}$$

Let's now compute the derivative of the central charge

$$\mathcal{Z}_i \equiv \mathrm{D}_i\mathcal{Z} = \langle \mathcal{Q}, U_i\rangle\,. \tag{E.4.53}$$

We have

$$\mathcal{Z}_i = p^\Lambda(\mathcal{R}_{\Lambda\Sigma} - i\mathcal{I}_{\Lambda\Sigma})f_i^\Sigma - \big(\mathcal{R}_{\Lambda\Sigma}\,p^\Lambda -\ \mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{q}'^\Lambda\big)f_i^\Sigma,$$

since now $h_{i\Lambda} = \bar{\mathcal{N}}_{\Lambda\Sigma}f_i^\Sigma$, simplification gives

$$\mathcal{Z}_i = \mathcal{I}_{\Lambda\Sigma}\big(\,\mathrm{e}^{2(V-U)}\tilde{q}'^\Lambda - ip^\Lambda\big)f_i^\Sigma. \tag{E.4.54}$$

On the other hand we will have

$$\mathcal{Z}_{\bar{\imath}} = \mathcal{I}_{\Lambda\Sigma}\big(\,\mathrm{e}^{2(V-U)}\tilde{q}'^\Lambda + ip^\Lambda\big)\bar{f}_{\bar{\imath}}^\Sigma. \tag{E.4.55}$$

Finally we introduce a last quantity

$$\mathcal{L} = \langle \mathcal{G}, \mathcal{V}\rangle = g^\Lambda M_\Lambda - g_\Lambda L^\Lambda. \tag{E.4.56}$$

where $\mathcal{G} = (g^\Lambda, g_\Lambda)$ (recall that $g^\Lambda = 0$ for the moment).
Inverting (E.4.50) we get

$$\mathcal{I}_{\Lambda\Sigma}\tilde{F}^\Lambda L^\Sigma = -\frac{1}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}. \tag{E.4.57}$$

We also define

$$\mathcal{I}_{\Lambda\Sigma}\tilde{G}^\Lambda L^\Sigma = -\frac{1}{2}\,\mathcal{Y}. \tag{E.4.58}$$

E.5 BPS equations for NUT black hole

We obtain the equation from [104] by taking $P^3_\Lambda = g_\Lambda$. We take the scalars and spinors to depend only on r. The ansatz for the spinors is

$$\varepsilon_A(r) = \mathrm{e}^{\frac{1}{2}(H+i\alpha)}\varepsilon_{0A} \tag{E.5.1}$$

with H and α both functions of r, and ε_{0A} is a constant spinor.

E.5.1 Gravitino equation

The gravitino variation is

$$\delta\psi_{\mu A} = \mathcal{D}_\mu\varepsilon_A + iS_{AB}\,\gamma_\mu\varepsilon^B + T^-_{\mu\nu}\gamma^\nu\varepsilon_{AB}\varepsilon^B = 0 \tag{E.5.2}$$

where

$$\mathcal{D}_\mu\varepsilon_A = D_\mu\varepsilon_A + \frac{i}{2}\,g_\Lambda A^\Lambda_\mu \sigma^3{}_A{}^B\varepsilon_B, \tag{E.5.3a}$$

$$D_\mu = \partial_\mu + \frac{1}{4}\,\omega_{\mu ab}\gamma^{ab} + \frac{i}{2}\,A_\mu, \tag{E.5.3b}$$

$$A_\mu = \frac{1}{2i}(K_i\partial_\mu z^i - K_{\bar{\imath}}\partial_\mu z^{\bar{\imath}}), \tag{E.5.3c}$$

$$S_{AB} = -\frac{i}{2}\,\mathcal{L}\,\sigma^3{}_A{}^C\varepsilon_{BC}, \tag{E.5.3d}$$

$$T^-_{\mu\nu} = 2i\,\mathcal{I}_{\Lambda\Sigma}L^\Sigma\mathcal{F}^{-\Lambda}_{\mu\nu}. \tag{E.5.3e}$$

More precisely we will look at the components of $\gamma^a\delta\psi_{aA}$ (no sum over a).

We can obtain another expression for T^- from (E.4.21)

$$\begin{aligned}
T^- &= 2i\,\mathcal{I}_{\Lambda\Sigma}L^\Sigma\mathcal{F}^{-\Lambda} \\
&= 2i\,\mathcal{I}_{\Lambda\Sigma}\tilde{F}^\Lambda L^\Sigma(e^0\wedge e^1 + i\,e^2\wedge e^3) + 2i\,\frac{F'}{F}\,\mathcal{I}_{\Lambda\Sigma}\tilde{G}^\Lambda L^\Sigma(e^1\wedge e^3 + i\,e^0\wedge e^2) \\
&= -i\;\mathrm{e}^{2(U-V)}\mathcal{Z}\,(e^0\wedge e^1 + i\,e^2\wedge e^3) - i\,\frac{F'}{F}\,\mathcal{Y}\,(e^1\wedge e^3 + i\,e^0\wedge e^2)
\end{aligned}$$

using the expressions (E.4.52) and (E.4.58). By contracting this expression with γ^b and multiplying by γ^a (thus with no sum over a) with

$$\gamma^a\gamma^b = \frac{1}{2}\,\gamma^{ab}, \tag{E.5.4}$$

we can see that only one term will remain for each value of a, and the factor will be ± 1 or $\pm i$.

The components of the variation read

$$\gamma^0\delta\psi_{0A} = \frac{1}{2}\big(U'\,\mathrm{e}^{U}\gamma^1 + n\,\mathrm{e}^{3U-2V}\gamma^{023}\big)\varepsilon_A + \frac{i}{2}\,g_\Lambda\tilde{q}^\Lambda\;\mathrm{e}^{-U}\gamma^0\sigma^3{}_A{}^B\varepsilon_B + iS_{AB}\varepsilon^B \qquad \text{(E.5.5a)}$$

$$- \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B - \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,$$

$$\gamma^1\delta\psi_{1A} = e^U\left(\partial_r + \frac{i}{2}\,A_r\right)\gamma^1\varepsilon_A + iS_{AB}\varepsilon^B - \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B \qquad \text{(E.5.5b)}$$

$$+ i\,\frac{F'}{F}\,\mathcal{Y}\,\gamma^{13}\varepsilon_{AB}\varepsilon^B,$$

$$\gamma^2\delta\psi_{2A} = \frac{1}{2}\big((V'-U')e^U\gamma^1 - n\,\mathrm{e}^{3U-2V}\gamma^{023}\big)\varepsilon_A + iS_{AB}\varepsilon^B \qquad \text{(E.5.5c)}$$

$$+ \frac{1}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{23}\varepsilon_{AB}\varepsilon^B - \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,$$

$$\gamma^3\delta\psi_{3A} = \frac{1}{2}\left((V'-U')e^U\gamma^1 - n\,\mathrm{e}^{3U-2V}\gamma^{023} + \frac{F'}{F}\;\mathrm{e}^{U-V}\gamma^2\right)\varepsilon_A + iS_{AB}\varepsilon^B \qquad \text{(E.5.5d)}$$

$$- \frac{i}{2}\,\frac{F'}{F}\,\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{U-V}\gamma^3\sigma^3{}_A{}^B\varepsilon_B + \frac{1}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{23}\varepsilon_{AB}\varepsilon^B + i\,\frac{F'}{F}\,\mathcal{Y}\,\gamma^{13}\varepsilon_{AB}\varepsilon^B.$$

We use the fact that $\gamma^a\gamma^b = \gamma^{ab}/2$ in all the last terms. Also we introduce curved index r for derivatives by using the inverse tetrad for the 1-component. We can rewrite γ^{023} and γ^{13}, and we simplify the equations

$$\gamma^0\delta\psi_{0A} = \frac{\mathrm{e}^U}{2}\big(U' - in\,\mathrm{e}^{2(U-V)}\big)\gamma^1\varepsilon_A + \frac{i}{2}\,g_\Lambda\tilde{q}^\Lambda\;\mathrm{e}^{-U}\gamma^0\sigma^3{}_A{}^B\varepsilon_B + iS_{AB}\varepsilon^B \qquad \text{(E.5.6a)}$$

$$- \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B - \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,$$

$$\gamma^1\delta\psi_{1A} = e^U\left(\partial_r + \frac{i}{2}\,A_r\right)\gamma^1\varepsilon_A + iS_{AB}\varepsilon^B - \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B \qquad \text{(E.5.6b)}$$

$$- \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,$$

$$\gamma^2\delta\psi_{2A} = \frac{\mathrm{e}^U}{2}\big((V'-U') + in\,\mathrm{e}^{2(U-V)}\big)\gamma^1\varepsilon_A + iS_{AB}\varepsilon^B + \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B \qquad \text{(E.5.6c)}$$

$$- \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,$$

$$\gamma^3\delta\psi_{3A} = \gamma^2\delta\psi_{2A} + \frac{1}{2}\,\frac{F'}{F}\;\mathrm{e}^{U-V}\big(\gamma^2\varepsilon_A - i\kappa\,g_\Lambda\tilde{p}^\Lambda\,\gamma^3\sigma^3{}_A{}^B\varepsilon_B\big). \qquad \text{(E.5.6d)}$$

First we see that each equation contains a θ-dependent term which should vanish since we have only r-dependent functions, thus

$$\mathcal{Y} = \mathcal{I}_{\Lambda\Sigma}\tilde{G}^\Lambda L^\Sigma = 0 \Longrightarrow \mathcal{I}_{\Lambda\Sigma}\tilde{p}'^\Lambda L^\Sigma = 0. \qquad \text{(E.5.7)}$$

We note that (E.5.6d) and (E.5.6c) differ only by a θ-dependent term, which gives a first projector equation

$$\gamma^2\varepsilon_A - i\kappa\,g_\Lambda\tilde{p}^\Lambda\,\gamma^3\sigma^3{}_A{}^B\varepsilon_B = 0. \qquad \text{(E.5.8)}$$

Taking the difference of (E.5.6a) and (E.5.6b) gives

$$\mathrm{e}^U\left(\partial_r + \frac{i}{2}\,A_r\right)\varepsilon_A = \frac{\mathrm{e}^U}{2}\big(U' - in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A + \frac{i}{2}\,g_\Lambda\tilde{q}^\Lambda\;\mathrm{e}^{-U}\gamma^{01}\sigma^3{}_A{}^B\varepsilon_B. \qquad \text{(E.5.9)}$$

Finally we need to take (E.5.6a) minus (E.5.6c)

$$\big(2U' - V' - 2in\,\mathrm{e}^{2(U-V)}\big)\gamma^1\varepsilon_A + i\,g_\Lambda \tilde{q}^\Lambda\ \mathrm{e}^{-2U}\,\gamma^0\sigma^3{}_A{}^B\varepsilon_B - 2i\,\mathrm{e}^{U-2V}\,\mathcal{Z}\gamma^{01}\varepsilon_{AB}\varepsilon^B = 0. \quad \text{(E.5.10)}$$

We multiply (E.5.6c) by gamma matrices and we replace S_{AB} to get

$$\frac{i}{2}\,\mathcal{L}\,\gamma^{01}\sigma^3{}_A{}^C\varepsilon_{BC}\varepsilon^B = \frac{1}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\varepsilon_{AB}\varepsilon^B + \frac{i}{2}\,\mathrm{e}^U\big(V' - U' + in\,\mathrm{e}^{2(U-V)}\big)\gamma^0\varepsilon_A. \quad \text{(E.5.11)}$$

Let's summarize the equations we need to solve[5]

$$0 = \mathcal{I}_{\Lambda\Sigma}\tilde{p}'^\Lambda L^\Sigma, \quad \text{(E.5.12a)}$$

$$\left(\partial_r + \frac{i}{2}\,A_r\right)\varepsilon_A = \frac{1}{2}\big(U' - in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A + \frac{i}{2}\,g_\Lambda\tilde{q}^\Lambda\ \mathrm{e}^{-2U}\gamma^{01}\sigma^3{}_A{}^B\varepsilon_B, \quad \text{(E.5.12b)}$$

$$\big(2U' - V' - 2in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A = -2i\,\mathrm{e}^{U-2V}\,\mathcal{Z}\gamma^0\varepsilon_{AB}\varepsilon^B - i\,g_\Lambda\tilde{q}^\Lambda\ \mathrm{e}^{-2U}\,\gamma^{01}\sigma^3{}_A{}^B\varepsilon_B, \quad \text{(E.5.12c)}$$

$$\varepsilon_A = -\kappa\,g_\Lambda\tilde{p}^\Lambda\,\gamma^{01}\sigma^3{}_A{}^B\varepsilon_B, \quad \text{(E.5.12d)}$$

$$i\,\mathcal{L}\,\gamma^{01}\sigma^3{}_A{}^C\varepsilon_{BC}\varepsilon^B = \mathrm{e}^{2(U-V)}\mathcal{Z}\,\varepsilon_{AB}\varepsilon^B - i\,\mathrm{e}^U\big(V' - U' + in\,\mathrm{e}^{2(U-V)}\big)\gamma^0\varepsilon_A. \quad \text{(E.5.12e)}$$

These equations are equivalent to the ones in [104] if we replace

$$U' \longrightarrow U' - in\,\mathrm{e}^{2(U-V)}. \quad \text{(E.5.13)}$$

There are four equations with projectors, and we need to reduce two of them to bosonic equations in order to get 1/4-BPS solutions.

We can plug (E.5.12d) into itself and find the following consistency condition[6]

$$(\kappa g_\Lambda\tilde{p}^\Lambda)^2 = 1 \Longrightarrow \boxed{g_\Lambda\tilde{p}^\Lambda = \pm\kappa.} \quad \text{(E.5.14)}$$

For simplicity we will keep the expression

$$\boxed{\varepsilon_A = -\kappa\,g_\Lambda\tilde{p}^\Lambda\gamma^{01}\sigma^3{}_A{}^B\varepsilon_B} \quad \text{(E.5.15)}$$

for the projector and simplify the sign only at the end. If g_Λ is fixed, then we can pick a sign and obtain the other just by inverting the other charges. An equivalent formulation gives

$$\kappa\,g_\Lambda\tilde{p}^\Lambda\,\varepsilon_A = -\gamma^{01}\sigma^3{}_A{}^B\varepsilon_B \quad \text{(E.5.16)}$$

by multiplying (E.5.15) on both side by $\kappa g_\Lambda p^\Lambda$ and using (E.5.14).

We can use it to simplify (E.5.12c)

$$\big(2U' - V' - 2in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A = -2i\,\mathrm{e}^{U-2V}\,\mathcal{Z}\,\gamma^0\varepsilon_{AB}\varepsilon^B + i\,c\,\varepsilon_A \quad \text{(E.5.17)}$$

where we have introduced the shortcut notation

$$c = \kappa\,g_\Lambda\tilde{p}^\Lambda\,g_\Sigma\tilde{q}^\Sigma\,\mathrm{e}^{-2U} = \pm g_\Lambda\tilde{q}^\Lambda\,\mathrm{e}^{-2U}. \quad \text{(E.5.18)}$$

We rewrite the equation as

$$\big(2U' - V' - i\tilde{c}\big)\varepsilon_A = -2i\,\mathrm{e}^{U-2V}\,\mathcal{Z}\,\gamma^0\varepsilon_{AB}\varepsilon^B \quad \text{(E.5.19)}$$

[5] We obtain five equations from four because we got one additional constraint by requiring that the θ-dependent term in each equation vanishes.

[6] We could have not included κ into this equation but this choice allows to remove all κ from the equations, and it appears that it is necessary for finding a solution.

where

$$\tilde{c} = c + 2n\,\mathrm{e}^{2(U-V)} = \kappa\, g_\Lambda \tilde{p}^\Lambda\, g_\Sigma \tilde{q}^\Sigma\, \mathrm{e}^{-2U} + 2n\,\mathrm{e}^{2(U-V)}. \tag{E.5.20}$$

Hence we can interpret the effect of n as shifting c instead of U'.

We can now look for consistency of this last equation by plugging it into itself. First take the complex conjugate

$$(2U' - V' + i\tilde{c})\varepsilon^A = 2i\,\mathrm{e}^{U-2V}\,\bar{\mathcal{Z}}\,\gamma^0 \varepsilon^{AB}\varepsilon_B. \tag{E.5.21}$$

Now use this result into the first equation

$$|2U' - V' + i\tilde{c}|^2 = 4|\mathcal{Z}|^2\,\mathrm{e}^{2U-4V}, \tag{E.5.22}$$

or written differently

$$|\mathcal{Z}|^2 = \frac{\mathrm{e}^{4V-2U}}{4}\big((2U' - V')^2 + \tilde{c}^2\big). \tag{E.5.23}$$

We define the phase[7] $\psi(r)$ by the equation

$$2\,\mathrm{e}^{U-2V}\,\mathrm{e}^{-i\psi}\mathcal{Z} = 2U' - V' - i\tilde{c}, \tag{E.5.24}$$

or by replacing $\tilde{c}$

$$\boxed{2\,\mathrm{e}^{U-2V}\,\mathrm{e}^{-i\psi}\mathcal{Z} = 2U' - V' - i\big(\kappa\, g_\Lambda \tilde{p}^\Lambda\, g_\Sigma \tilde{q}^\Sigma\, \mathrm{e}^{-2U} + 2n\,\mathrm{e}^{2(U-V)}\big).} \tag{E.5.25}$$

The real and imaginary parts of this equation are respectively

$$2\,\mathrm{e}^{U-2V}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}) = 2U' - V', \tag{E.5.26a}$$

$$2\,\mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) = -\kappa\, g_\Lambda \tilde{p}^\Lambda\, g_\Sigma \tilde{q}^\Sigma\, \mathrm{e}^{-2U} - 2n\,\mathrm{e}^{2(U-V)}. \tag{E.5.26b}$$

The second equation will help us to replace $\tilde{q}^\Lambda$ everywhere.

The projector then becomes

$$\boxed{\varepsilon_A = i\,\mathrm{e}^{i\psi}\,\gamma^0 \varepsilon_{AB}\varepsilon^B.} \tag{E.5.27}$$

The version with indices up is

$$\varepsilon^A = i\,\mathrm{e}^{-i\psi}\,\gamma^0 \varepsilon^{AB}\varepsilon_B. \tag{E.5.28}$$

The phase ψ which appears here is the same as the one of the spinor in (E.5.1), as can be seen by comparing the phases of (E.5.27), thus

$$\alpha = \psi. \tag{E.5.29}$$

Inserting the projector (E.5.15) into (E.5.12b) turns it into a bosonic equation

$$\partial_r \varepsilon_A = \frac{1}{2}\Big(U' - i\big(A_r + c + n\,\mathrm{e}^{2(U-V)}\big)\Big)\varepsilon_A \tag{E.5.30a}$$

$$= \frac{1}{2}\Big(U' - i\big(A_r + \tilde{c} - n\,\mathrm{e}^{2(U-V)}\big)\Big)\varepsilon_A. \tag{E.5.30b}$$

Plugging the ansatz (E.5.1) for the spinor, we get a differential equation for the phase

$$\psi' = -\big(A_r + c + n\,\mathrm{e}^{2(U-V)}\big) \tag{E.5.31}$$

[7] We know that both sides of the equation differ by this phase because of the above value for $|\mathcal{Z}|$.

from the imaginary part, while the real part tells us that $H' = U'$, and setting to zero the integration constant we have

$$H = U. \tag{E.5.32}$$

Replacing c we have

$$\psi' = -\big(A_r + \kappa\, g_\Lambda \tilde{p}^\Lambda\, g_\Sigma \tilde{q}^\Sigma\, \mathrm{e}^{-2U} + n\, \mathrm{e}^{2(U-V)}\big). \tag{E.5.33}$$

and it simplifies with (E.5.26b)

$$\boxed{\psi' = -A_r + 2\, \mathrm{e}^{U-2V}\, \mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) + n\, \mathrm{e}^{2(U-V)}.} \tag{E.5.34}$$

The last step is to simplify (E.5.12e)

$$\begin{aligned}
i\,\mathcal{L}\,\gamma^{01}\sigma^3{}_A{}^C\varepsilon_{BC}\varepsilon^B &= \mathrm{e}^{2(U-V)}\mathcal{Z}\,\varepsilon_{AB}\varepsilon^B - i\,\mathrm{e}^U\big(V' - U' + in\,\mathrm{e}^{2(U-V)}\big)\gamma^0\varepsilon_A,\\
-i\,\mathcal{L}\,\gamma^{01}\sigma^3{}_A{}^C\gamma^0\varepsilon_{CB}\varepsilon^B &= \mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^0\varepsilon_{AB}\varepsilon^B - i\,\mathrm{e}^U\big(V' - U' + in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A,\\
-\,\mathrm{e}^{-i\psi}\mathcal{L}\,\gamma^{01}\sigma^3{}_A{}^C\varepsilon_C &= -i\,\mathrm{e}^{2(U-V)}\,\mathrm{e}^{-i\psi}\mathcal{Z}\,\varepsilon_A - i\,\mathrm{e}^U\big(V' - U' + in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A,\\
\kappa\, g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{-i\psi}\mathcal{L}\,\varepsilon_A &= -i\,\mathrm{e}^{2(U-V)}\,\mathrm{e}^{-i\psi}\mathcal{Z}\,\varepsilon_A - i\,\mathrm{e}^U\big(V' - U' + in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A.
\end{aligned}$$

In the first step we multiplied by γ^0 and reversed ε_{BC}, then we used the projector (E.5.27), and finally we used the other projector (E.5.16). After simplifcation we obtain a bosonic equation

$$i\kappa\, g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{-i\psi}\mathcal{L} = \mathrm{e}^{2(U-V)}\,\mathrm{e}^{-i\psi}\mathcal{Z} + \mathrm{e}^U\Big(V' - U' + in\,\mathrm{e}^{2(U-V)}\Big). \tag{E.5.35}$$

The real part and imaginary parts read

$$\kappa\, g_\Lambda\tilde{p}^\Lambda\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}) = -\,\mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) - \mathrm{e}^U(V' - U'), \tag{E.5.36a}$$

$$\boxed{\kappa\, g_\Lambda\tilde{p}^\Lambda\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) = \mathrm{e}^{2(U-V)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{3U-2V}.} \tag{E.5.36b}$$

From the equation (E.5.26a)

$$\mathrm{e}^U V' = 2\big((\,\mathrm{e}^U)' - \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z})\big), \tag{E.5.37}$$

we can simplify the first equation

$$\kappa\, g_\Lambda\tilde{p}^\Lambda\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}) = -\,\mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) - \big(2(\,\mathrm{e}^U)' - 2\,\mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) - (\,\mathrm{e}^U)'\big) \tag{E.5.38}$$

and get a differential equation for U'

$$\boxed{(\,\mathrm{e}^U)' = -\kappa\, g_\Lambda\tilde{p}^\Lambda\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z}).} \tag{E.5.39}$$

Plugging this equation back we obtain a differential equation for V'

$$\boxed{(\,\mathrm{e}^V)' = -2\kappa\, g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}).} \tag{E.5.40}$$

We can solve these two equations instead of (E.5.26a) and (E.5.36a).

Adding (E.5.35) to (E.5.25) gives

$$\mathrm{e}^{2(U-V)}\,\mathrm{e}^{-i\psi}\mathcal{Z} + i\kappa\, g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{-i\psi}\mathcal{L} = \mathrm{e}^U\Big(U' - i\big(\kappa\, g_\Lambda\tilde{p}^\Lambda\, g_\Sigma\tilde{q}^\Sigma\,\mathrm{e}^{-2U} + n\,\mathrm{e}^{2(U-V)}\big)\Big). \tag{E.5.41}$$

This equation is just a rewriting of previous equations.

E.5.2 Gaugino variation

The gaugino variation is given by

$$\delta\lambda^{iA} = i\partial_\mu z^i\,\gamma^\mu\varepsilon^A - g^{i\bar{\jmath}}\bar{f}_{\bar{\jmath}}^{\Sigma}\,\mathcal{I}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\,\gamma^{\mu\nu}\varepsilon^{AB}\varepsilon_B + ig_\Lambda g^{i\bar{\jmath}}\bar{f}_{\bar{\jmath}}^{\Lambda}\;\sigma^3{}_C{}^B\varepsilon^{CA}\varepsilon_B = 0. \tag{E.5.42}$$

The variation becomes[8]

$$\begin{aligned}\delta\lambda^{iA} = i\,\mathrm{e}^U\partial_r z^i\;\gamma^1\varepsilon^A &+ \frac{1}{2}\,\mathrm{e}^{2(U-V)}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}(\gamma^{01}+i\gamma^{23})\varepsilon^{AB}\varepsilon_B + ig^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\;\sigma^3{}_C{}^B\varepsilon^{CA}\varepsilon_B\\ &+ 2\,\frac{F'}{F}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Y}(\gamma^{13}+i\,\gamma^{02})\varepsilon^{AB}\varepsilon_B.\end{aligned} \tag{E.5.43}$$

The last term is the only θ dependence and it should cancel

$$g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Y} = g^{i\bar{\jmath}}\,\mathcal{I}_{\Lambda\Sigma}\tilde{G}^\Lambda\bar{f}_{\bar{\jmath}}^{\Sigma} = 0 \Longrightarrow \mathcal{I}_{\Lambda\Sigma}\tilde{G}^\Lambda\bar{f}_{\bar{\jmath}}^{\Sigma} = 0. \tag{E.5.44}$$

Adding this to the previous equation (E.5.12a), we see that G^Λ is orthogonal to the n_v+1 base vectors $(L^\Lambda, f_{\bar{\jmath}}^{\Lambda})$ which implies that it vanishes. We deduce that

$$\boxed{\tilde{p}'^\Lambda = 0 \Longrightarrow \tilde{p} = \mathrm{cst.}} \tag{E.5.45}$$

We can simplify the rest of (E.5.43)

$$\begin{aligned}
i\,\mathrm{e}^U\partial_r z^i\;\gamma^1\varepsilon^A &= -\frac{1}{2}\,\mathrm{e}^{2(U-V)}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}(\gamma^{01}+i\gamma^{23})\varepsilon^{AB}\varepsilon_B - ig^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\;\sigma^3{}_C{}^B\varepsilon^{CA}\varepsilon_B\\
i\,\mathrm{e}^U\partial_r z^i\;\gamma^1\varepsilon^A &= -\,\mathrm{e}^{2(U-V)}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}\;\gamma^{01}\varepsilon^{AB}\varepsilon_B - ig^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\;\sigma^3{}_C{}^B\varepsilon^{CA}\varepsilon_B\\
i\,\mathrm{e}^U\partial_r z^i\;\varepsilon^A &= \;\mathrm{e}^{2(U-V)}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}\;\gamma^0\varepsilon^{AB}\varepsilon_B + ig^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\;\gamma^0\varepsilon^{CA}\gamma^{01}\sigma^3{}_C{}^B\varepsilon_B\\
i\,\mathrm{e}^U\partial_r z^i\;\varepsilon^A &= i\,\mathrm{e}^{2(U-V)}\,\mathrm{e}^{i\psi}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}\,\varepsilon^A - i\kappa\,g_\Lambda\tilde{p}^\Lambda\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\;\gamma^0\varepsilon^{CA}\varepsilon_C\\
i\,\mathrm{e}^U\partial_r z^i\;\varepsilon^A &= i\,\mathrm{e}^{2(U-V)}\,\mathrm{e}^{i\psi}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}\,\varepsilon^A - \kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{i\psi}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\,\varepsilon^A.
\end{aligned}$$

First we replaced γ^{23} by γ^{01}, then we multiplied by γ^1 and we introduced $(\gamma^0)^2=1$, after what we used projectors (E.5.27) and (E.5.16) respectively for the first and second terms of the RHS, and finally we used again (E.5.27) for the last term after changing $\varepsilon^{CA} = -\varepsilon^{AC}$.

Cleaning up this equation gives finally

$$\boxed{\mathrm{e}^{-i\psi}\,\mathrm{e}^U\partial_r z^i = g^{i\bar{\jmath}}\Big(\,\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\mathcal{Z} + i\,\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\Big).} \tag{E.5.46}$$

We want to rewrite it in terms of the sections. It is easier to proceed if we replace

$$\mathrm{D}_i\mathcal{Z} = \mathcal{I}_{\Lambda\Sigma}\big(\,\mathrm{e}^{2(V-U)}\tilde{q}'^\Lambda + ip^\Lambda\big)f_i^\Sigma, \qquad \mathrm{D}_i\mathcal{L} = -g_\Sigma f_i^\Sigma, \tag{E.5.47}$$

using (E.4.54), to get

$$\mathrm{e}^{-i\psi}\,\mathrm{e}^U\partial_r z^i = g^{i\bar{\jmath}}\bar{f}_{\bar{\jmath}}^{\Sigma}\Big(\,\mathrm{e}^{2(U-V)}\,\mathcal{I}_{\Lambda\Sigma}\big(\,\mathrm{e}^{2(V-U)}\tilde{q}'^\Lambda + ip^\Lambda\big) - i\,\kappa\,g_\Lambda\tilde{p}^\Lambda\,g_\Sigma\Big) \tag{E.5.48}$$

We contract both sides with f_i^Δ. Using the relation

$$-\,g^{i\bar{\jmath}}\bar{f}_{\bar{\jmath}}^{\Sigma}f_i^\Delta = \frac{1}{2}\,(\mathcal{I}^{-1})^{\Sigma\Delta} + L^\Sigma\bar{L}^\Delta \tag{E.5.49}$$

[8] The contraction is antisymmetric and should give a factor 2; but we wrote $\tilde{F}e^0e^1$, and we did not write the component e^1e^0, thus we do not take it into account (or we could by multiplying by a factor 1/2).

we find

$$
\begin{aligned}
\mathrm{e}^{-i\psi}\,\mathrm{e}^{U} f_i^{\Delta}\partial_r z^i &= -\left(\frac{1}{2}\,(\mathcal{I}^{-1})^{\Sigma\Delta} + L^{\Sigma}\bar{L}^{\Delta}\right)\left(\mathrm{e}^{2(U-V)}\,\mathcal{I}_{\Lambda\Sigma}\big(\mathrm{e}^{2(V-U)}\tilde{q}'^{\Lambda} + ip^{\Lambda}\big) - i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,g_{\Sigma}\right)\\
&= -\frac{1}{2}\big(\tilde{q}'^{\Delta} + i\,\mathrm{e}^{2(U-V)}p^{\Delta}\big) + \frac{i}{2}\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_{\Sigma} + i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,\bar{L}^{\Delta}\,L^{\Sigma}g_{\Sigma}\\
&\quad - \mathrm{e}^{2(U-V)}\,\mathcal{I}_{\Lambda\Sigma}L^{\Sigma}\bar{L}^{\Delta}\big(\mathrm{e}^{2(V-U)}\tilde{q}'^{\Lambda} + ip^{\Lambda}\big)\\
&= -\frac{1}{2}\big(\tilde{q}'^{\Delta} + i\,\mathrm{e}^{2(U-V)}p^{\Delta}\big) + \frac{i}{2}\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_{\Sigma} - i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,\bar{L}^{\Delta}\,\mathcal{L}\\
&\quad - \mathrm{e}^{2(U-V)}\,\bar{L}^{\Delta}\mathcal{Z}
\end{aligned}
$$

where we used the expression of $\mathcal{Z}$ and $\mathcal{L}$

$$
\begin{aligned}
&= -\frac{1}{2}\big(\tilde{q}'^{\Delta} + i\,\mathrm{e}^{2(U-V)}p^{\Delta}\big) + \frac{i}{2}\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_{\Sigma}\\
&\quad - \bar{L}^{\Delta}\big(\mathrm{e}^{2(U-V)}\,\mathcal{Z} + i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,\mathcal{L}\big)\\
&= -\frac{1}{2}\big(\tilde{q}'^{\Delta} + i\,\mathrm{e}^{2(U-V)}p^{\Delta}\big) + \frac{i}{2}\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_{\Sigma}\\
&\quad - \bar{L}^{\Delta}\,\mathrm{e}^{i\psi}\,\mathrm{e}^{U}\Big(U' - i\big(c + n\,\mathrm{e}^{2(U-V)}\big)\Big)
\end{aligned}
$$

by using (E.5.41). We now consider the LHS

$$
\begin{aligned}
f_i^{\Lambda}\partial_r z^i = \partial_r z^i\left(\partial_i L^{\Lambda} + \frac{1}{2}(\partial_i K)L^{\Lambda}\right) &= \partial_r L^{\Lambda} + \frac{1}{2}(z'^i\partial_i K - z'^{\bar{\imath}}\partial_{\bar{\imath}}K)L^{\Lambda}\\
&= \partial_r L^{\Lambda} + iA_r L^{\Lambda} = \partial_r L^{\Lambda} - i\big(\psi' + c + n\,\mathrm{e}^{2(U-V)}\big)L^{\Lambda}
\end{aligned}
$$

from (E.5.34) and from

$$
\partial_r L^{\Lambda} = z'^i\partial_i L^{\Lambda} + z'^{\bar{\imath}}\partial_{\bar{\imath}}L^{\Lambda} = z'^i\partial_i L^{\Lambda} + z'^{\bar{\imath}}\partial_{\bar{\imath}}(\mathrm{e}^{\frac{K}{2}}X^{\Lambda}) = z'^i\partial_i L^{\Lambda} + \frac{1}{2}\,z'^{\bar{\imath}}L^{\Lambda}\partial_{\bar{\imath}}K
$$

(explained with words, L^{Λ} depends on $\bar{z}$ by the Kähler potential).

Gluing the two sides we find

$$
\begin{aligned}
&\mathrm{e}^{-i\psi}\,\mathrm{e}^{U}\Big(\partial_r L^{\Delta} - i\big(\psi' + c + n\,\mathrm{e}^{2(U-V)}\big)L^{\Delta}\Big) + \bar{L}^{\Delta}\,\mathrm{e}^{i\psi}\,\mathrm{e}^{U}\Big(U' - i\big(c + n\,\mathrm{e}^{2(U-V)}\big)\Big)\\
&\qquad\qquad = -\frac{1}{2}\big(\tilde{q}'^{\Delta} + i\,\mathrm{e}^{2(U-V)}p^{\Delta}\big) + \frac{i}{2}\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_{\Sigma}. \qquad \text{(E.5.50)}
\end{aligned}
$$

We focus on the LHS

$$
\begin{aligned}
&\mathrm{e}^{-i\psi}\;\mathrm{e}^{U}\Big(\partial_r L^{\Delta} - i\big(\psi' + c + n\,\mathrm{e}^{2(U-V)}\big)L^{\Delta}\Big) + \bar{L}^{\Delta}\,\mathrm{e}^{i\psi}\,\mathrm{e}^{U}\Big(U' - i\big(c + n\,\mathrm{e}^{2(U-V)}\big)\Big)\\
&\quad = \mathrm{e}^{-i\psi}\,\mathrm{e}^{U}\big(\partial_r L^{\Delta} - i\psi' L^{\Delta}\big) + U'\,\mathrm{e}^{U}\,\mathrm{e}^{i\psi}\bar{L}^{\Delta} - i\,\mathrm{e}^{U}\big(c + n\,\mathrm{e}^{2(U-V)}\big)\big(\mathrm{e}^{-i\psi}L^{\Delta} + \mathrm{e}^{i\psi}\bar{L}^{\Delta}\big)\\
&\quad = \mathrm{e}^{U}\partial_r\big(\mathrm{e}^{-i\psi}L^{\Delta}\big) + U'\,\mathrm{e}^{U}\,\mathrm{e}^{i\psi}\bar{L}^{\Delta} + 2i\,\mathrm{e}^{U}\big(2\,\mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{2(U-V)}\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^{\Delta})\\
&\quad = \mathrm{e}^{U}\partial_r\big(\mathrm{e}^{-i\psi}L^{\Delta}\big) + U'\,\mathrm{e}^{U}\big(\mathrm{Re}(\mathrm{e}^{-i\psi}L^{\Delta}) - i\,\mathrm{Im}(\mathrm{e}^{-i\psi}L^{\Delta})\big)\\
&\qquad + 2i\,\mathrm{e}^{U}\big(2\,\mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{2(U-V)}\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^{\Delta})
\end{aligned}
$$

using (E.5.26b) and that $\mathrm{Im}(x^*) = -\,\mathrm{Im}\,x$ to replace c. We multiply each side by 2 and using the fact that $(\mathrm{e}^{\pm U})' = \pm U'\,\mathrm{e}^{U}$ we decompose this equation into real and imaginary parts[9]

$$
\begin{aligned}
2\,\partial_r\big(\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda)\big) &= -\tilde{q}'^\Lambda, && \text{(E.5.51a)}\\
2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}L^\Lambda)\big) &= -p^\Lambda + \kappa\,\mathrm{e}^{2(V-U)}g_\Delta\tilde{p}^\Delta\,(\mathcal{I}^{-1})^{\Sigma\Lambda}g_\Sigma && \text{(E.5.51b)}\\
&\quad - 4\big(2\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{U}\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Delta).
\end{aligned}
$$

The first equation is directly integrated to give

$$
\boxed{\tilde{q}^\Lambda = -2\,\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda).} \tag{E.5.52}
$$

Finally we can use (E.5.36b) to get

$$
\boxed{
\begin{aligned}
2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}L^\Lambda)\big) =&- 4\big(2\kappa g_\Delta\tilde{p}^\Delta\;\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) - n\,\mathrm{e}^{U}\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Delta)\\
&- p^\Lambda + \kappa\,\mathrm{e}^{2(V-U)}g_\Delta\tilde{p}^\Delta\,(\mathcal{I}^{-1})^{\Sigma\Lambda}g_\Sigma.
\end{aligned}
} \tag{E.5.53}
$$

E.5.3 Summary

We found two projectors

$$
\begin{aligned}
\varepsilon^A &= i\,\mathrm{e}^{-i\psi}\,\gamma^0\varepsilon^{AB}\varepsilon_B, && \text{(E.5.54a)}\\
\varepsilon_A &= -\kappa g_\Lambda\tilde{p}^\Lambda\,\gamma^{01}\sigma^3{}_A{}^B\varepsilon_B. && \text{(E.5.54b)}
\end{aligned}
$$

We have algebraic

$$
\begin{aligned}
g_\Lambda\tilde{p}^\Lambda &= \varepsilon_D\kappa, && \text{(E.5.55a)}\\
\kappa\,g_\Lambda\tilde{p}^\Lambda\;\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) &= \mathrm{e}^{2(U-V)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{3U-2V} && \text{(E.5.55b)}
\end{aligned}
$$

and differential equations

$$
\begin{aligned}
\psi' &= -A_r + 2\,\mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{2(U-V)}, && \text{(E.5.55c)}\\
(\mathrm{e}^{U})' &= -\kappa\,g_\Lambda\tilde{p}^\Lambda\;\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}), && \text{(E.5.55d)}\\
(\mathrm{e}^{V})' &= -2\kappa\,g_\Lambda\tilde{p}^\Lambda\;\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}), && \text{(E.5.55e)}\\
(z^i)' &= \mathrm{e}^{-U}\;\mathrm{e}^{i\psi}g^{i\bar{\jmath}}\big(\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\mathcal{Z} + i\,\kappa\,g_\Lambda\tilde{p}^\Lambda\;\mathrm{D}_{\bar{\jmath}}\mathcal{L}\big). && \text{(E.5.55f)}
\end{aligned}
$$

We have

$$
\varepsilon_D = \pm 1 \tag{E.5.56}
$$

and both signs correspond to different branches of BPS solutions. In general one can study the solution with $\varepsilon_D = -1$ [76, 96, 105] and the other branch can be found by flipping the sign of the charges – and apparently e^{U} – once $\mathcal{G}$ is fixed (see [97, app. B, 104, p. 6]). In particular this choice agrees with [35, p. 8]. Note that setting κ to the RHS is necessary (if one wants a solution) even if we do not see this from the equations.

The equations (E.5.55d) and (E.5.55f) can be gathered into

$$
\begin{aligned}
2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}L^\Lambda)\big) =&- 8\kappa\,g_\Delta\tilde{p}^\Delta\;\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda)\\
&+ 4n\,\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda) - p^\Lambda + \kappa\,g_\Delta\tilde{p}^\Delta\;\mathrm{e}^{2(V-U)}(\mathcal{I}^{-1})^{\Sigma\Lambda}g_\Sigma.
\end{aligned} \tag{E.5.57}
$$

[9] For the imaginary part we need to multiply by $\mathrm{e}^{2(V-U)}$.

One needs also to impose Maxwell equations (E.4.49)

$$\mathcal{Q}' = -2n\, e^{2(U-V)}\, \mathcal{M}\mathcal{Q}. \tag{E.5.58}$$

It includes the equation

$$\tilde{p}'^{\Lambda} = 0 \tag{E.5.59}$$

and the charges $\tilde{q}^{\Lambda}$ are given by the equation (E.5.52)

$$\tilde{q}^{\Lambda} = -2\, e^{U}\, \mathrm{Re}(\, e^{-i\psi} L^{\Lambda}). \tag{E.5.60}$$

Note that (E.5.55a) reduces to Dirac quantization condition from [104] when $n = 0$. Using the definition (E.4.32)

$$\tilde{p}^{\Lambda} = p^{\Lambda} + 2n\, \tilde{q}^{\Lambda} \tag{E.5.61}$$

and the equation (E.5.60)

$$\tilde{q}^{\Lambda} = -2\, e^{U}\, \mathrm{Re}(\, e^{-i\psi} L^{\Lambda}), \tag{E.5.62}$$

we obtain[10] a new expression for (E.5.55a) which depends only on the electromagnetic charges

$$g_{\Lambda} p^{\Lambda} - 4n\ e^{U} g_{\Lambda}\, \mathrm{Re}(\, e^{-i\psi} L^{\Lambda}) = \kappa. \tag{E.5.63}$$

We can use (E.5.55b) in order to get an expression for $e^{i\psi}$. This last expression will not help to solve the equation since it is complicated, but it means that we can always integrate the differential equation for the phase (E.5.55c), and we can obtain the expression if we know all other quantities. From (E.5.55b) we have[11]

$$\big(\, e^{-i\psi}\mathcal{L} + \, e^{i\psi}\bar{\mathcal{L}}\big) = -i\, e^{2(U-V)}\big(\, e^{-i\psi}\mathcal{Z} - \, e^{i\psi}\bar{\mathcal{Z}}\big) + 2n\, e^{3U-2V}. \tag{E.5.64}$$

We multiply by $e^{i\psi}$ in order to get a second order equation

$$e^{2i\psi}\big(\bar{\mathcal{L}} - i\, e^{2(U-V)}\bar{\mathcal{Z}}\big) - 2n\, e^{3U-2V}\, e^{i\psi} + \big(\mathcal{L} + i\, e^{2(U-V)}\mathcal{Z}\big) = 0 \tag{E.5.65}$$

whose solutions are

$$e^{i\psi} = -\frac{n\, e^{3U-2V}}{\bar{\mathcal{L}} - i\, e^{2(U-V)}\bar{\mathcal{Z}}} \pm 2\sqrt{\left(\frac{n\, e^{3U-2V}}{\bar{\mathcal{L}} - i\, e^{2(U-V)}\bar{\mathcal{Z}}}\right)^2 - \frac{\mathcal{L} + i\, e^{2(U-V)}\mathcal{Z}}{\bar{\mathcal{L}} - i\, e^{2(U-V)}\bar{\mathcal{Z}}}}. \tag{E.5.66}$$

For $n = 0$ it reduces to [76, eq. (2.39)]

$$e^{2i\psi} = \frac{e^{2(U-V)}\mathcal{Z} - i\mathcal{L}}{e^{2(U-V)}\bar{\mathcal{Z}} + i\bar{\mathcal{L}}}\,. \tag{E.5.67}$$

E.5.4 Symplectic extension

Almost all the BPS equations we obtained in the previous sections are already symplectic invariant since they are given in terms of symplectic invariant quantities. The symplectic covariant expression of Dirac quantization condition can be read from (E.5.63).

The symplectic invariant equations are

$$\langle \mathcal{Q}, \mathcal{G}\rangle + 4n\ e^{U}\, \mathrm{Re}(\, e^{-i\psi}\mathcal{L}) = \varepsilon_D\, \kappa, \tag{E.5.68a}$$

$$\varepsilon_D\ \mathrm{Re}(\, e^{-i\psi}\mathcal{L}) = \, e^{2(U-V)}\, \mathrm{Im}(\, e^{-i\psi}\mathcal{Z}) + n\, e^{3U-2V} \tag{E.5.68b}$$

$$\begin{aligned} 2\, e^{2V}\partial_r\big(\, e^{-U}\, \mathrm{Im}(\, e^{-i\psi}\mathcal{V})\big) &= \Big(4n\ e^{U} - 8\varepsilon_D\ e^{2(V-U)}\, \mathrm{Re}(\, e^{-i\psi}\mathcal{L})\Big)\, \mathrm{Re}(\, e^{-i\psi}\mathcal{V}) \\ &\quad - \mathcal{Q} - \varepsilon_D\ e^{2(V-U)}\mathcal{M}\mathcal{G}, \end{aligned} \tag{E.5.68c}$$

$$(\, e^{V})' = -2\varepsilon_D\, e^{V-U}\, \mathrm{Im}(\, e^{-i\psi}\mathcal{L}), \tag{E.5.68d}$$

$$\mathcal{Q}' = -2n\, e^{2(U-V)}\, \mathcal{M}\mathcal{Q}. \tag{E.5.68e}$$

[10] Since the formula contained $\tilde{q}$ and not $\tilde{q}'$ we could not use (E.4.34) to replace it.

[11] To lighten notations we take $g_{\Lambda}\tilde{p}^{\Lambda} = \kappa$

We also have the derivative of equation (E.5.60)

$$2\,\partial_r\big(\,\mathrm{e}^{U}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = -\mathcal{G} - \,\mathrm{e}^{2(U-V)}\mathcal{M}\mathcal{Q}. \tag{E.5.69}$$

The first term cannot be seen from (E.5.60) since g^{Λ} was set to zero, but we could get it by computing explicitly the derivative of M_{Λ}.

Finally we recall the equations for ψ', U' and z'^{i}

$$\psi' = -A_r - 2\,\mathrm{e}^{-U}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) - n\,\mathrm{e}^{2(U-V)}, \tag{E.5.70a}$$

$$(\,\mathrm{e}^{U})' = -\varepsilon_D\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}) + \,\mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z}), \tag{E.5.70b}$$

$$(z^{i})' = \,\mathrm{e}^{-U}\,\mathrm{e}^{i\psi}g^{i\bar{\jmath}}\big(\,\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\mathcal{Z} + i\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\big). \tag{E.5.70c}$$

Other equations

The equation (E.5.68c) can be modified using (E.5.68d) to include one factor e^{V} inside the derivative. The LHS is

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\,\mathrm{e}^{-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) &= 2\,\mathrm{e}^{V}\partial_r\big(\,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) - 2\,\mathrm{e}^{V-U}\partial_r(\,\mathrm{e}^{V})\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ &= 2\,\mathrm{e}^{V}\partial_r\big(\,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) + 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \end{aligned}$$

and it combines with the RHS to

$$\begin{aligned} 2\,\mathrm{e}^{V}\partial_r\big(\,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = 4\,\big(n\,\,\mathrm{e}^{U} - 2\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ - 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ - \mathcal{Q} - \,\mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G}. \end{aligned} \tag{E.5.71}$$

One can also use Maxwell equation (E.5.68e) to rewrite (E.5.69) as

$$2\,\partial_r\big(\,\mathrm{e}^{U}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = \frac{1}{2n}\mathcal{Q}' - \mathcal{G}. \tag{E.5.72}$$

It is then straightforward to integrate this equation

$$4n\,\mathrm{e}^{U}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) = \mathcal{Q} - 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} \tag{E.5.73}$$

where $\widehat{\mathcal{Q}}$ is the integration constant. In turn one can use this to get the expression for $\mathcal{Q}$ if one knows the other quantities. Moreover plugging this result into Dirac quantization equation (E.5.68a) gives

$$\langle \mathcal{Q}, \mathcal{G}\rangle + 4n\,\,\mathrm{e}^{U}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) = \Big\langle \widehat{\mathcal{Q}}, \mathcal{G}\Big\rangle = \varepsilon_D\,\kappa \tag{E.5.74}$$

which shows that the LHS of Dirac equation is constant.

Finally one can use this expression for $\mathcal{Q}$ in order to rewrite the equation (E.5.68c) for the imaginary part of $\mathcal{V}$

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\,\mathrm{e}^{-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = \Big(8n\,\,\mathrm{e}^{U} - 8\varepsilon_D\,\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\Big)\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ - 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} - \varepsilon_D\,\,\mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G}. \end{aligned} \tag{E.5.75}$$

The main advantage is that $\mathcal{Q}$ has been replaced by the constant $\widehat{\mathcal{Q}}$, while the extra term $\mathcal{G}\,r$ is not a big problem.

Another formulation

We can use the second equation to replace n everywhere: we then get a set of equations which is the same as for $n=0$, and any solution of this set should satisfy the additional constraint (E.5.68b). The new equations are

$$\psi' = -A_r + \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + \mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}), \tag{E.5.76a}$$

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) &= -4\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &\quad - \mathcal{Q} - \mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G}, \end{aligned} \tag{E.5.76b}$$

$$\mathcal{Q}' = 2\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) - \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathcal{M}\mathcal{Q}. \tag{E.5.76c}$$

If we multiply (E.5.76b) by $\mathcal{M}$ (which is real) we get

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{M}\mathcal{V})\big) &= -2\big(\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{M}\mathcal{V}) \\ &\quad - \mathcal{M}\mathcal{Q} + \mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{M}\mathcal{G} \\ &\quad + 2\,\mathrm{e}^{-U}\,\mathrm{Im}\left(\mathrm{e}^{-i\psi}\partial_r(\mathrm{e}^{2V}\mathcal{M})\mathcal{V}\right) \\ -2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(i\,\mathrm{e}^{-i\psi}\mathcal{V})\big) &= -2\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathrm{Re}(i\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ &\quad - \mathcal{M}\mathcal{Q} - \mathrm{e}^{2(V-U)}\mathcal{G} + 2\,\mathrm{e}^{-U}\,\mathrm{Im}\left(\mathrm{e}^{-i\psi}\partial_r(\mathrm{e}^{2V}\mathcal{M})\mathcal{V}\right) \end{aligned}$$

since $\mathcal{M}^2=-1$. We obtain

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V})\big) &= -\,2\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &\quad + \mathcal{M}\mathcal{Q} + \mathrm{e}^{2(V-U)}\mathcal{G} + 2\,\mathrm{e}^{-U}\,\mathrm{Im}\left(\mathrm{e}^{-i\psi}\partial_r(\mathrm{e}^{2V}\mathcal{M})\mathcal{V}\right). \end{aligned} \tag{E.5.77}$$

Equations from special geometry

We can use several identities involving the quartic invariant in order to express all equations in terms of $\mathrm{Im}\,\mathcal{V}$ and V uniquely.

We define

$$\widetilde{\mathcal{V}} = \mathrm{e}^{V-U}\,\mathrm{e}^{-i\psi}\,\mathcal{V}. \tag{E.5.78}$$

The first step is to use the identity (D.1.2c) in (E.5.71)

$$2\,\mathrm{e}^{V}\partial_r\,\mathrm{Im}\,\widetilde{\mathcal{V}} = -\mathcal{Q} + I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{G}) + 4n\;\mathrm{e}^{2U-V}\,\mathrm{Re}\,\widetilde{\mathcal{V}}, \tag{E.5.79}$$

Then using (D.1.2a) and (D.1.2b) as

$$I_4(\mathrm{Im}\,\widetilde{\mathcal{V}}) = \frac{1}{16}\,\mathrm{e}^{4(V-U)}, \qquad \mathrm{Re}\,\widetilde{\mathcal{V}} = -2\,\mathrm{e}^{2(U-V)}\,I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}). \tag{E.5.80}$$

we can replace $\mathrm{Re}(\widetilde{\mathcal{V}})$ and e^U

$$\mathrm{e}^{2U-V}\,\mathrm{Re}\,\widetilde{\mathcal{V}} = -2\,\mathrm{e}^{4U-3V}\,I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}) = -\frac{1}{8}\,\mathrm{e}^{V}\,\frac{I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}})}{I_4(\mathrm{Im}\,\widetilde{\mathcal{V}})}. \tag{E.5.81}$$

In terms of this new variable the equations (E.5.68c) and (E.5.68d) become

$$2\,\mathrm{e}^{V}\partial_r\big(\mathrm{Im}\,\widetilde{\mathcal{V}})\big) = -\mathcal{Q} + I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{G}) - \frac{n}{2}\,\mathrm{e}^{V}\,\frac{I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}})}{I_4(\mathrm{Im}\,\widetilde{\mathcal{V}})}, \tag{E.5.82a}$$

$$(\mathrm{e}^{V})' = -2\left\langle \mathcal{G}, \mathrm{Im}\,\widetilde{\mathcal{V}}\right\rangle. \tag{E.5.82b}$$

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