2*d* gravity with massive matter

Harold Erbin

Lptens, École Normale Supérieure (France)

SCGSC 2017
Ihp, Paris – 17th February 2017

arXiv: 1612.04097, 1511.06150

Outline

Outline: 1. Introduction

Motivations

$2d$ (quantum) gravity is useful for:

- toy model for $4d$ quantum gravity
- spontaneous dimensional reduction [1605.05694, Carlip]
- (non-)critical string theories

Motivations

$2d$ (quantum) gravity is useful for:

- toy model for $4d$ quantum gravity
- spontaneous dimensional reduction [1605.05694, Carlip]
- (non-)critical string theories

Real-world requires massive matter

Goals

- Study classical gravity coupled to massive matter
- Show that (classical) $2d$ gravity is *not* a good toy model
- Derive the spectrum of the Mabuchi action (quantum action for the metric)

Outline: 2. Classical gravity

Total action

Matter ψ + gravity $g_{\mu\nu}$ action

$$S[g,\psi] = S_{\text{grav}}[g] + S_m[g,\psi]$$

Conditions

- renormalizability
- invariance under diffeomorphisms
- no more than first order derivatives
- $S_m[g,\psi]$ obtained from minimal coupling

Note: in $2d$ $g_{\mu\nu}$ has one dynamical component = conformal factor (or Liouville field) ϕ

Gravity action

Gravitational action: two possible terms

$$S_{\text{grav}}[g] = S_{\text{EH}}[g] + S_\mu[g]$$

- Einstein–Hilbert

$$S_{\text{EH}}[g] = \int \mathrm{d}^2\sigma \sqrt{|g|}\, R = 4\pi\chi$$

  topological invariant (Euler number χ) $\to$ not dynamical, ignore it

- Cosmological constant

$$S_\mu[g] = \mu \int \mathrm{d}^2\sigma \sqrt{|g|} = \mu\, A[g]$$

Equations of motion

- Energy–momentum tensor (with traceless and trace components)

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}} = T^{(m)}_{\mu\nu} + 2\pi\mu\, g_{\mu\nu}$$

$$\bar{T}_{\mu\nu} = T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}, \qquad T = g^{\mu\nu} T_{\mu\nu}$$

- Equations of motion

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \qquad \frac{\delta S}{\delta \psi} = 0$$

Equations of motion

- Energy–momentum tensor (with traceless and trace components)

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}} = T^{(m)}_{\mu\nu} + 2\pi\mu\, g_{\mu\nu}$$

$$\bar{T}_{\mu\nu} = T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}, \qquad T = g^{\mu\nu} T_{\mu\nu}$$

- Equations of motion

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \qquad \frac{\delta S}{\delta \psi} = 0$$

- Metric EOM $\rightarrow$ vanishing of $T_{\mu\nu}$

$$\begin{cases} T = T^{(m)} + 4\pi\mu = 0 \\ \bar{T}_{\mu\nu} = \bar{T}^{(m)}_{\mu\nu} = 0 \end{cases}$$

$\rightarrow$ decoupling of traceless component from gravity

Dynamics: conformal matter

- Weyl transformation

$$g_{\mu\nu} = \mathrm{e}^{2\omega(\sigma)} g'_{\mu\nu}$$

conformal invariance $S_m[\eta, \psi] \Longrightarrow$ Weyl invariance $S_m[g, \psi]$

(here $\Longleftarrow$ also holds)

Dynamics: conformal matter

- Weyl transformation

$$g_{\mu\nu} = \mathrm{e}^{2\omega(\sigma)} g'_{\mu\nu}$$

conformal invariance $S_m[\eta, \psi] \Longrightarrow$ Weyl invariance $S_m[g, \psi]$

(here $\Longleftarrow$ also holds)

- Weyl invariance $\rightarrow$ traceless $T^{(m)}_{\mu\nu}$

$$T^{(m)} = 0 \quad \Longrightarrow \quad \mu = 0$$

from gravity (trace) EOM

Dynamics: conformal matter

- Weyl transformation

$$g_{\mu\nu} = \mathrm{e}^{2\omega(\sigma)} g'_{\mu\nu}$$

conformal invariance $S_m[\eta, \psi] \Longrightarrow$ Weyl invariance $S_m[g, \psi]$

(here $\Longleftarrow$ also holds)

- Weyl invariance $\rightarrow$ traceless $T^{(m)}_{\mu\nu}$

$$T^{(m)} = 0 \quad \Longrightarrow \quad \mu = 0$$

from gravity (trace) EOM

Conclusion

Conformal matter coupled to $\mu \neq 0$ gravity is inconsistent.

Dynamics: non-conformal matter (1) – model

- N scalar fields X_i

$$S_m = -\frac{1}{4\pi}\int \mathrm{d}^2\sigma \sqrt{|g|}\left(g^{\mu\nu}\partial_\mu X_i \partial_\nu X_i + V(X_i)\right)$$

- EOM

$$\bar{T}^{(m)}_{\mu\nu} = \partial_\mu X_i \partial_\nu X_i - \frac{1}{2} g_{\mu\nu}(g^{\alpha\beta}\partial_\alpha X_i \partial_\beta X_i) = 0$$
$$V(X) = 4\pi\mu$$
$$-\Delta X_i + \frac{1}{2}\frac{\partial V}{\partial X_i} = 0$$

$\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu$ curved space Laplacian

Dynamics: non-conformal matter (2) – solution

- Conformal gauge (fix diffeomorphisms) $g_{\mu\nu} = \mathrm{e}^{2\phi}\eta_{\mu\nu}$
- Traceless EOM

$$2(\bar{T}_{00} \pm \bar{T}_{01}) = (\partial_0 X_i \pm \partial_1 X_i)^2 = 0$$

$\rightarrow$ sum of squares

$$(\partial_0 \pm \partial_1) X_i = 0 \quad \Longrightarrow \quad \partial_\mu X_i = 0 \quad \Longrightarrow \quad X_i = X_i^0 = \mathrm{cst}$$

Dynamics: non-conformal matter (2) – solution

- Conformal gauge (fix diffeomorphisms) $g_{\mu\nu} = \mathrm{e}^{2\phi}\eta_{\mu\nu}$
- Traceless EOM

$$2(\bar{T}_{00} \pm \bar{T}_{01}) = (\partial_0 X_i \pm \partial_1 X_i)^2 = 0$$

$\to$ sum of squares

$$(\partial_0 \pm \partial_1) X_i = 0 \quad \Longrightarrow \quad \partial_\mu X_i = 0 \quad \Longrightarrow \quad X_i = X_i^0 = \mathrm{cst}$$

- Trace and matter EOM $\to$ constraints on X_i^0

$$\frac{\partial V}{\partial X_i}(X_i^0) = 0, \qquad V(X_i^0) = 4\pi\mu$$

Dynamics: non-conformal matter (2) – solution

- Conformal gauge (fix diffeomorphisms) $g_{\mu\nu} = \mathrm{e}^{2\phi} \eta_{\mu\nu}$
- Traceless EOM

$$2(\bar{T}_{00} \pm \bar{T}_{01}) = (\partial_0 X_i \pm \partial_1 X_i)^2 = 0$$

$\rightarrow$ sum of squares

$$(\partial_0 \pm \partial_1) X_i = 0 \quad \Longrightarrow \quad \partial_\mu X_i = 0 \quad \Longrightarrow \quad X_i = X_i^0 = \mathrm{cst}$$

- Trace and matter EOM $\rightarrow$ constraints on X_i^0

$$\frac{\partial V}{\partial X_i}(X_i^0) = 0, \qquad V(X_i^0) = 4\pi\mu$$

Conclusion

Non-conformal matter coupled to gravity is (at best) trivial.

Dynamics: non-conformal matter (3) – example

- Free massive scalars

$$V(X_i) = \sum_i m_i^2 X_i^2$$

- Matter EOM

$$m_i^2 X_i^0 = 0 \quad \Longrightarrow \quad X_i^0 = 0 \qquad \forall m_i \neq 0$$

Dynamics: non-conformal matter (3) – example

- Free massive scalars

$$V(X_i) = \sum_i m_i^2 X_i^2$$

- Matter EOM

$$m_i^2 X_i^0 = 0 \quad \Longrightarrow \quad X_i^0 = 0 \qquad \forall m_i \neq 0$$

- Trace EOM

$$\sum_i m_i^2 (X_i^0)^2 = 4\pi\mu \quad \Longrightarrow \quad \mu = 0$$

Dynamics: non-conformal matter (3) – example

- Free massive scalars

$$V(X_i) = \sum_i m_i^2 X_i^2$$

- Matter EOM

$$m_i^2 X_i^0 = 0 \quad \Longrightarrow \quad X_i^0 = 0 \qquad \forall m_i \neq 0$$

- Trace EOM

$$\sum_i m_i^2 (X_i^0)^2 = 4\pi\mu \quad \Longrightarrow \quad \mu = 0$$

Conclusion

Massive free scalar fields coupled to gravity are inconsistent for $\mu \neq 0$, trivial for $\mu = 0$.

Degrees of freedom: conformal matter

- No cosmological constant, $\mu = 0$
- $\exists$ Weyl invariance $\rightarrow$ traceless energy–momentum tensor

$$T^{(m)} = 0$$

Degrees of freedom: conformal matter

- No cosmological constant, $\mu = 0$
- $\exists$ Weyl invariance $\rightarrow$ traceless energy–momentum tensor

$$T^{(m)} = 0$$

- Metric EOM

$$T^{(m)}_{\mu\nu} = 0$$

- Weyl invariant EOM $\rightarrow$ independent of the conformal factor
  $\rightarrow$ 2 constraints on the matter

Degrees of freedom: conformal matter

- No cosmological constant, $\mu = 0$
- $\exists$ Weyl invariance $\to$ traceless energy–momentum tensor

$$T^{(m)} = 0$$

- Metric EOM

$$T^{(m)}_{\mu\nu} = 0$$

- Weyl invariant EOM $\to$ independent of the conformal factor $\to$ 2 constraints on the matter

Conclusion

Gravity reduces the dofs of conformal matter from N to $N - 2$.

Degrees of freedom: non-conformal matter

- Action linear in $g^{\mu\nu}$

$$S_m = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \sqrt{|g|}\, \mathcal{L}, \qquad \mathcal{L} = -\frac{1}{2}\Big(g^{\mu\nu} \mathcal{L}_{\mu\nu}(\psi) + V(\psi) \Big)$$

- Metric EOM

$$\bar{T}_{\mu\nu} = \mathcal{L}_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} \big(g^{\alpha\beta} \mathcal{L}_{\alpha\beta} \big) = 0, \qquad T = -V + 4\pi\mu = 0$$

- Weyl invariant EOM $\rightarrow$ independent of the conformal factor $\rightarrow$ 3 constraints on the matter

Degrees of freedom: non-conformal matter

- Action linear in $g^{\mu\nu}$

$$S_m = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \sqrt{|g|}\, \mathcal{L}, \qquad \mathcal{L} = -\frac{1}{2}\Big(g^{\mu\nu}\mathcal{L}_{\mu\nu}(\psi) + V(\psi)\Big)$$

- Metric EOM

$$\bar{T}_{\mu\nu} = \mathcal{L}_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\big(g^{\alpha\beta}\mathcal{L}_{\alpha\beta}\big) = 0, \qquad T = -V + 4\pi\mu = 0$$

- Weyl invariant EOM $\rightarrow$ independent of the conformal factor $\rightarrow$ 3 constraints on the matter
- Abolishing gauge invariance (Weyl) *removes* dofs

Conclusion

Gravity reduces the dofs of generic non-conformal matter from N to $N-3$, instead of $N-1$.

Outline: 3. Quantum gravity

Functional integration

- Partition functions

$$Z = \int \mathrm{d}_g g_{\mu\nu} \, \mathrm{e}^{-S_\mu[g]} Z_m|g]$$

$$Z_m[g] = \int \mathrm{d}_g \psi \, \mathrm{e}^{-S_m[g,\psi]}$$

- Quantum effects $\rightarrow$ dynamics for the conformal factor

- For computations: fix diffeomorphisms

Conformal gauge

- Conformal gauge

$$g = \mathrm{e}^{2\phi} g_0$$

ϕ Liouville mode, g_0 (fixed) background metric

- Partition function

$$Z[\phi] = \mathrm{e}^{-S_{\text{grav}}[g_0,\phi]}\, Z_m[g_0], \quad S_{\text{grav}} = -\ln \frac{Z_m[\mathrm{e}^{2\phi} g_0]}{Z_m[g_0]}$$

(ignore ghosts from gauge fixing)

Conformal gauge

- Conformal gauge

$$g = \mathrm{e}^{2\phi} g_0$$

ϕ Liouville mode, g_0 (fixed) background metric

- Partition function

$$Z[\phi] = \mathrm{e}^{-S_{\text{grav}}[g_0,\phi]}\, Z_m[g_0], \quad S_{\text{grav}} = -\ln \frac{Z_m[\mathrm{e}^{2\phi} g_0]}{Z_m[g_0]}$$

(ignore ghosts from gauge fixing)

- Typically [1112.1352, Ferrari-Klevtsov-Zelditch]

$$S_{\text{grav}} = S_\mu + \frac{c}{6}\, S_L + \beta^2\, S_M + \cdots$$

S_μ cosmological constant, S_L Liouville action, S_M Mabuchi action

Outline: 4. Mabuchi spectrum

Mabuchi action

- Kähler potential (work at fixed area)

$$e^{2\phi} = \frac{A}{A_0}\left(1 + \frac{A_0}{2\pi\chi}\,\Delta_0 K\right)$$

- Mabuchi action (Euclidean) [Mabuchi '86]

$$S_M = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\sqrt{g_0}\left[-g_0^{\mu\nu}\partial_\mu K\partial_\nu K + \left(\frac{4\pi\chi}{A_0} - R_0\right)K + \frac{4\pi\chi}{A}\,\phi e^{2\phi}\right]$$

Mabuchi action

- Kähler potential (work at fixed area)

$$e^{2\phi} = \frac{A}{A_0}\left(1 + \frac{A_0}{2\pi\chi}\,\Delta_0 K\right)$$

- Mabuchi action (Euclidean) [Mabuchi '86]

$$S_M = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\sqrt{g_0}\left[-g_0^{\mu\nu}\partial_\mu K\partial_\nu K + \left(\frac{4\pi\chi}{A_0} - R_0\right)K + \frac{4\pi\chi}{A}\,\phi e^{2\phi}\right]$$

- EOM (same as Liouville)

$$R = \frac{4\pi\chi}{A}$$

- Note: ill-defined on the torus/cylinder ($\chi = 0$)

Minisuperspace model

Minisuperspace (background = cylinder)

$$\phi = \phi(t), \qquad K = K(t), \qquad g_0 = \eta$$

Conjectured action (variable area, Lorentzian signature)

$$S_M = -\frac{1}{2}\int \mathrm{d}t \left[\dot{K}^2 - \ddot{K}\ln\left(\frac{\ddot{K}}{4\pi\mu}\right) + \ddot{K}\right], \qquad \mathrm{e}^{2\phi} = \frac{\ddot{K}}{4\pi\mu}$$

Minisuperspace model

Minisuperspace (background = cylinder)

$$\phi = \phi(t), \qquad K = K(t), \qquad g_0 = \eta$$

Conjectured action (variable area, Lorentzian signature)

$$S_M = -\frac{1}{2} \int \mathrm{d}t \left[\dot{K}^2 - \ddot{K} \ln \left(\frac{\ddot{K}}{4\pi\mu} \right) + \ddot{K} \right], \qquad \mathrm{e}^{2\phi} = \frac{\ddot{K}}{4\pi\mu}$$

Motivations:

- reproduce the features of the full action
- reproduce minisuperspace EOM
- different ways to infer this action

Hamiltonian

1. Conjugate momentum to $\dot{K}$

$$P = \frac{\delta S_M}{\delta \ddot{K}} = \frac{1}{2} \ln\left(\frac{\ddot{K}}{4\pi\mu}\right) = \phi$$

2. Canonical transformation

$$P = \phi, \qquad \dot{K} = -\Pi$$

3. Hamiltonian

$$H_M = \frac{\Pi^2}{2} + 2\pi\mu e^{2\phi} = H_L$$

Spectrum

- Canonical quantization [Braaten et al. '84]

$$\hat{H}_M \psi_p = 2p^2 \, \psi_p$$

- Wave functions

$$\psi_p(\phi) = \frac{2(\pi\mu)^{-ip}}{\Gamma(-2ip)} \, K_{2ip}(2\sqrt{\pi\mu}\, \mathrm{e}^{\phi}) \sim_{-\infty} \mathrm{e}^{2ip\phi} + R_0(p)\mathrm{e}^{-2ip\phi}$$

$p \in \mathbb{R}$: orthonormal set

Spectrum

- Canonical quantization [Braaten et al. '84]

$$\hat{H}_M \psi_p = 2p^2 \, \psi_p$$

- Wave functions

$$\psi_p(\phi) = \frac{2(\pi\mu)^{-ip}}{\Gamma(-2ip)} \, K_{2ip}(2\sqrt{\pi\mu}\,\mathrm{e}^{\phi}) \sim_{-\infty} \mathrm{e}^{2ip\phi} + R_0(p)\mathrm{e}^{-2ip\phi}$$

$p \in \mathbb{R}$: orthonormal set

- 3-point function (limit of DOZZ $b \to 0$)

$$C_0(p_1, p_2, p_3) = \int_{-\infty}^{\infty} \mathrm{d}\phi \; \psi_{p_1}(\phi)\mathrm{e}^{-2ip_2\phi}\psi_{p_3}(\phi)$$

$$= (\pi\mu)^{-2\tilde{p}} \, \Gamma(2\tilde{p}) \prod_i \frac{\Gamma((-1)^i 2\tilde{p}_i)}{\Gamma(2p_i)}$$

$$2\tilde{p} = \sum_i p_i, \qquad \tilde{p}_i = \tilde{p} - p_i, \qquad i = 1, 2, 3$$

Outline: 5. Conclusion

Conclusion

Main results

- No-go theorems for classical gravity
- Dof counting for classical gravity
- Minisuperspace dynamics of Mabuchi action = Liouville
- Mabuchi spectrum: $\mathrm{e}^{2ip\phi}$, $p \in \mathbb{R}$

Conclusion

Main results

- No-go theorems for classical gravity
- Dof counting for classical gravity
- Minisuperspace dynamics of Mabuchi action = Liouville
- Mabuchi spectrum: $\mathrm{e}^{2ip\phi}$, $p \in \mathbb{R}$

Open problems:

- Few physical properties of Mabuchi action known (1-loop string susceptibility)
- Mabuchi should not be a CFT, but zero-mode dynamics is a CFT: how the full dynamics of Mabuchi differs from Liouville?
- Find rigorous formulation at variable area and on the torus/cylinder (Kähler formalism not appropriate)
- Comparison with matrix models, CDT...