Recent developments in superstring field theory ①

Ashoke Sen : 2015-2017

review : 1703.06410 , lecture notes in preparation

with : De Lacroix, Kashyap, Verma

Plan :

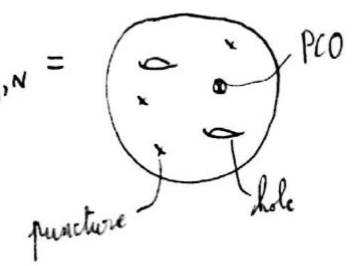
1. Introduction and motivations - divergences in string theory
2. Notations and definitions
3. Off-shell amplitudes and Feynman diagrams
4. Superstring field theory
5. Momentum representation of Green functions
6. Applications : vacuum shift , unitarity, soft theorems ...

1. Introduction

Note: focus on bosonic/heterotic string theory (i.e closed, oriented)

* World-sheet theory

g-loop N-point amplitude (= amputated Green function)

$A_{g,N} =$ 

$$= \int_{\mathcal{M}_{g,N}} \prod_{i=1}^{6g-6+2N} dm_i \; F(m)$$

$N = m + n$, $n \to R$, $m \to NS$

m_i : parameters of the moduli space $\mathcal{M}_{g,N}$ for the Riemann surface $\Sigma_{g,N}$ with genus g and N punctures

F : correlation function

$$F = \Big\langle \prod_{a=1}^{N} V_a \times \underbrace{\text{ghosts} \times \text{PCO}}_{\text{independent of } V_a} \Big\rangle$$

ghosts → superconformal ghosts

$\prod V_a$ → matter : integrated vertex operators

The m_i are equivalent to the Schwinger parameters s_i in QFT

$$\frac{1}{k^2+m^2} = \int_0^\infty ds\, e^{-s(k^2+m^2)} \qquad s \sim \text{proper time}$$

The integration over k becomes Gaussian × polynomial.

* Three types of divergences

- IR : regions $m_i \longrightarrow \infty$ (for $k^2+m^2 \leqslant 0$)
- UV : regions $m_i \longrightarrow 0$ (seen after having integrated k)
- spurious : regions with m_i finite but $A_{g,N}$ diverges due to superconformal ghosts

absent in QFT, bosonic string
present in sugra

* Problems of worldsheet formulation

- divergences $m_i \to \infty$:
  - artificial for $k^2 + m^2 < 0$
  - genuine for $k^2 + m^2 = 0$
    - ↳ IR divergences: quantum effects shift vacuum and masses
- on-shell (BRST / conformal invariance)
  - ↳ prevents from using QFT tools (renormalization, off-shell amplitudes, proof of unitarity)
- one graph at each loop, but real
  - ↳ expects complex result (from unitarity, for unstable resonances) optical theorem
- computations mainly for protected states (BPS, symmetry)..
- not clear $i\varepsilon$ prescription

There are some prescriptions to address these problems, but (often ad hoc)
→ build a string field theory no systematic procedure

* String field theory (SFT):

"Regular" QFT with infinite # of fields s.t.:

- amplitudes agree with the worldsheet ones if the latter can be defined
- genuine (IR) divergences agree but can be handled with the usual QFT tools

Some properties:

- Feynman graph = integral over part of $\mathcal{M}_{g,N}$
- non local interaction $\sim e^{k^2 \#}$

Useful for: proof of background independence, Ward identities and mass degeneracy in shifted vacuum

Notes on construction:

- reverse engineering

  on-shell → off shell → graphs → gauge fixed SFT → BV SFT

- no intrinsic construction of vertices
  - miss minimal area prescription of bosonic SFT
  - use on-shell and world-sheet as much as possible

Recent progresses:

- explicit 1-loop 1-point vertex: Erler-Konopka-Sachs 1704.04210
- generic formulation: Moosavian-Puis 1703.10563

* Achievements (Sen '15-17):

- full quantum SFT action for heterotic/type II
- inclusion of Ramond sector
- define momentum representation of off-shell amplitudes
- explain how to avoid spurious singularities
- computations of vacuum and mass shifts (algo) } address IR
- soft theorems
- proof of unitarity

In progress (with De Lacroix and Puis): proof of crossing symmetry.

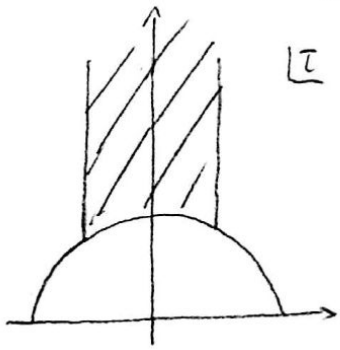
* Other approaches:

- "Munich" (Erler, Sachs, Okawa ...): algebraic construction (L_∞ and A_∞ algebra
- Berkovits: WZW non polynomial action

* More on divergences in string theory

- UV divergences: absent in string theory because $\mathcal{M}_{g,N}$ does not contain the region $m_i \to 0$

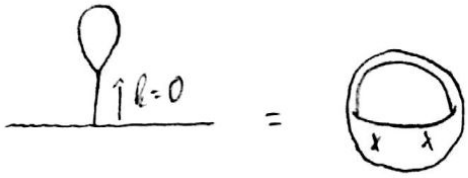
  Caveat: (SFT) amplitudes in Lorentzian signature requires special care.



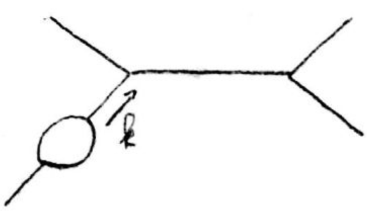
- spurious$^{\text{sing}}$: due to superconformal ghosts

  physically: breakdown of gauge fixing

- IR divergences: degeneration limit (long tubes)
  - tadpole with massless particle = vacuum shift
  - mass renormalization = resummation of 1PR graphs



$$m = 0 \implies \frac{1}{k^2}\bigg|_{k=0} = \infty$$



$$\frac{1}{k^2 + m^2}\bigg|_{k^2=-m^2} = \infty$$

by momentum conservation and on-shell external leg

2. Notations

Heterotic superstring:

- matter $c_m = (26, 15)$
    - D non-compact dimensions (X^μ, ψ^μ)
    - internal CFT (from compactifications with NS fluxes)
- ghosts bc : $c_{bc} = (-26, -26)$
- superghosts $\beta\gamma$: $c_{\beta\gamma} = (0, 11)$

Note: R fluxes could be considered with pure spinor formalism.

Bosonization of $\beta\gamma$ system:

$$\gamma = \eta e^{\varphi} \qquad \beta = \partial\xi\, e^{-\varphi}$$

η, ξ: fermions, φ: Coulomb gas scalar

Quantum numbers: $U(1)$ ghost, $U(1)$ picture, $\mathbb{Z}_2$ GSO

	b	c	β	γ	η	ξ	$e^{p\varphi}$
$(\bar h, h)$	$(0,2)$	$(0,-1)$	$(0,3/2)$	$(0,-1/2)$	$(0,1)$	$(0,0)$	$\left(0, -\frac{p}{2}(p+2)\right)$
N_{gh}	-1	1	-1	1	1	-1	0
N_{pic}	0	0	0	0	-1	1	p
GSO	$+$	$+$	$-$	$-$	$+$	$+$	$(-1)^p$

BRST charge:

$$Q_B = \oint dz\, j_B(z) + \oint d\bar z\, \bar j_B(\bar z)$$

$$j_B(z) = c(T_m + T_{\beta\gamma}) + \gamma T_F + bc\partial c - \frac{1}{4}\gamma^2 b$$

$$T_{bc} = -2b\partial c + c\partial b$$

$$T_{\beta\gamma} = \frac{3}{2}\beta\partial\gamma + \frac{1}{2}\gamma\partial\beta = -\eta\partial\xi - \frac{1}{2}\partial\varphi\partial\varphi - \partial^2\varphi$$

PCO:

$$X(z) = \{Q_B, \xi(z)\} = c\partial\xi + e^{\varphi}T_F - \frac{1}{4}\partial\eta\, e^{2\varphi} b - \frac{1}{4}\partial(\eta e^{2\varphi} b)$$

BRST invariant, $N_{pic} = 1$

note: $\xi \in \mathcal{H}_{large}$
$X \in \mathcal{H}_{small}$
$\Rightarrow$ prevents from acting Q_B on vertex

* Hilbert spaces

- Small Hilbert space
Bosonization involves $\partial\xi$: zero-mode of ξ is absent from the spectrum
$\longrightarrow$ consider states s.t $\eta_0|\psi\rangle = 0$

- (Hilbert space) $\mathcal{H}_T$: GSO even states in small Hilbert space s.t

$$b_0^-|\psi\rangle = 0 \ , \quad L_0^-|\psi\rangle = 0$$

(local coord inv under rot.) (level matching)

$b_0^\pm = b_0 \pm \bar{b}_0$
$c_0^\pm = \frac{1}{2}(c_0 \pm \bar{c}_0)$
$L_0^\pm = L_0 \pm \bar{L}_0$

- $\mathcal{H}_n \subset \mathcal{H}_T$: $N_{pic}(\psi) = n$

- Natural states for off-shell and SFT :

$$\hat{\mathcal{H}}_T = \mathcal{H}_{-1} \oplus \mathcal{H}_{-1/2}$$

$$\tilde{\mathcal{H}}_T = \underset{\substack{\downarrow \\ \text{NS}}}{\mathcal{H}_{-1}} \oplus \underset{\substack{\downarrow \\ \text{R}}}{\mathcal{H}_{-3/2}}$$

special role : $|p\rangle = e^{p\varphi(0)}|0\rangle$
$\beta_n|p\rangle = 0 \quad n \geqslant -p-\frac{1}{2}$
$\gamma_n|p\rangle = 0 \quad n \geqslant p+\frac{3}{2}$

- Physical states :

$$|\psi\rangle \in \hat{\mathcal{H}}_T : \quad Q_B|\psi\rangle = 0 \ , \quad N_{gh}(\psi) = 2 .$$

ex : $\psi = e^{-\varphi}\bar{c}cV$ with $h_0(V) = (1,1)$ primary in CFT_m
ex : $V = e^{ikX}$, $h = \frac{k^2}{4}$ (bosonic)

* **Basis states** : $|\varphi_r\rangle \in \hat{\mathcal{H}}_T$, $|\varphi_r^c\rangle \in \tilde{\mathcal{H}}_T$

$$\langle \varphi_r^c | c_0^- | \varphi_s \rangle = \delta_{rs}$$

* **Anomalies** : correlation functions vanish excep if :
  - $N_{gh} = 6-6g$
  - $N_{pic} = 2g-2$

$N_{pic}(A_{g,m,n}) = -m - \frac{n}{2} \implies$ inserts PCO to match

3. Off-shell amplitudes and Feynman diagrams

z implicit

* Tree-level 3-point function

$$A_{0,3} = \left\langle \prod_{i=1}^{3} \bar{c}c V_i(z_i) \right\rangle_{S^2} \propto (z_1 - z_2)^{h_3 - h_1 - h_2 + 1} \times \text{perms} \times \text{c.c} =$$

No integration since $\dim \mathcal{M}_{0,3} = 0$.

Independent of z_i if on-shell state, i.e $h_i = 1$

For $h_i \neq 1$ $A_{0,3}$ is not invariant under $SL(2,\mathbb{C})$

$$z \longmapsto \frac{az+b}{cz+d}$$

This is a consequence of the presence of punctures: the metric is changed around each puncture.

Solution: introduce local coordinates w_i with flat metric $|dw_i|^2$ around each puncture. $\to$ recovers inv.

There are maps: $z = f_i(w_i)$ s.t. $z_i = f_i(0)$

For a primary operator:

$$f \circ \varphi(w) = f'(w)^h \, \overline{f'(w)}^{\bar{h}} \, \varphi(f(w)) + \text{secondaries}$$

Then:

$$A_{0,3} = \left\langle \prod_{i=1}^{3} f_i \circ V_i(0) \right\rangle_{S^2} = \left(\prod_{i=1}^{3} f_i'(0)^{h_i} \, \overline{f_i'(0)}^{\bar{h}_i} \right) \left\langle \prod_{i=1}^{3} V_i(f_i(0)) \right\rangle_{S^2}$$

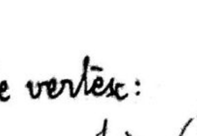
is invariant under $SL(2,\mathbb{C})$. But it depends on the local coordinate choice. through $f_i'(0)$

Proof: One has $\left\langle \prod_i V_i(f_i(0)) \right\rangle \propto (f_1 - f_2)^{h_3 - h_1 - h_2} \times \text{perms} \times \text{cc}$

and $f_i \longmapsto \dfrac{a f_i + b}{c f_i + d} \implies f_i' \longrightarrow \dfrac{f_i'}{(c f_i + d)^2}$

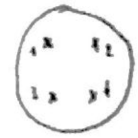
$$f_i - f_j \longrightarrow \frac{f_i - f_j}{(c f_i + d)(c f_j + d)}$$

Define the vertex:

$$\{A_1, A_2, A_3\}_0 =$$ 

$= A_{0,3}$

* Tree-level 4-point function

$$A_{0,4} = \int d^2 z_4 \left\langle \prod_{i=1}^{3} \bar{c}cV_i(z_i)\, V_4(z_4) \right\rangle_{S^2} =$$ 

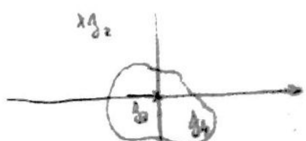
On-shell: independent of z_1, z_2, z_3, but there are divergences for $z_4 \longrightarrow z_1, z_2, z_3$ (collision of punctures).

Ex: tachyons $V_i = e^{ik_i X(z_i)}$

$$A_{0,4} \propto \left(\int d^2 z_4 \prod_{i=1}^{3} |z_4 - z_i|^{k_4 \cdot k_i} \right) \prod_{i,j=1}^{3} |z_i - z_j|^{2 + k_i \cdot k_j}$$

The integral diverges for $k_i \cdot k_4 \leqslant 0$ if $z_4 \to z_i$: this can happen for physical values of the k_i.

Idea: cut regions around z_1, z_2, z_3 in z_4 plane, and change interpretation of these contributions.



Consider $z_4 \longrightarrow z_3 = 0$ case, write $z_4 = q y_4$ with y_4 fixed

$$\int \frac{d^2 q}{|q|^2} \left\langle \bar{c}cV_1(z_1)\, \bar{c}cV_2(z_2)\, \bar{c}cV_3(0)\, |q y_4|^2\, V_4(\lambda y_4) \right\rangle \quad \text{radial ordering}$$

$$= \int \frac{d^2 q}{|q|^2} \left\langle \bar{c}cV_1(z_1)\, \bar{c}cV_2(z_2) \oint_{|w| = q^{1/2}} dw\, w\, b(w) \oint_{|w| = q^{1/2}} d\bar{w}\, \bar{w}\, \bar{b}(\bar{w})\, \bar{c}cV_4(\lambda y_4)\, \bar{c}cV_3(0) \right\rangle$$

$$= \int \frac{d^2 q}{|q|^2} \left\langle \bar{c}cV_1(z_1)\, \bar{c}cV_2(z_2) \oint dw\, w\, b(w) \oint d\bar{w}\, \bar{w}\, \bar{b}(\bar{w})\, q^{L_0} \bar{q}^{\bar{L}_0}\, \bar{c}cV_4(y_4)\, \bar{c}cV_3(0) \right\rangle$$

Insert complete set of states

$$= \sum_{r,s} \left\langle \bar{c}cV_1(z_1)\, \bar{c}cV_2(z_2) \varphi_r(0) \right\rangle \left\langle \bar{c}cV_3(z_3)\, \bar{c}cV_4(z_4) \varphi_s(0) \right\rangle \int \frac{d^2 q}{|q|^2} \left\langle \varphi_r^c\, \bar{c}_0 c_0\, q^{L_0} \bar{q}^{\bar{L}_0}\, b_0 \bar{b}_0\, \bar{c}_0 c_0\, \varphi_s^c \right\rangle$$

Looks like two 3-point functions linked by a propagator.

Make this more precise:

$$q = e^{-s-i\theta}, \quad s \in [s_0, \infty[\,, \quad \theta \in [0, 2\pi[$$

$\hookrightarrow$ cut-off

$$\int \frac{d^2q}{|q|^2}\, q^{L_0} \bar{q}^{\bar{L}_0} = \int_{s_0}^{\infty} ds \int_0^{2\pi} d\theta\, e^{-s(L_0+\bar{L}_0)}\, e^{-i\theta(L_0-\bar{L}_0)}$$

$$= \frac{1}{L_0+\bar{L}_0}\, e^{-s_0(L_0+\bar{L}_0)}\, \delta_{L_0,\bar{L}_0} \quad ; \quad b_0 \bar{b}_0 = -\frac{1}{2} b_0^+ b_0^- \,, \quad c_0^- b_0 \bar{b}_0 c_0^- \sim c_0^- b_0^+$$

Choose $|\varphi_r\rangle$ to be eigenstates of L_0: $\quad L_0 = \bar{L}_0 = \frac{k^2}{4} + c$

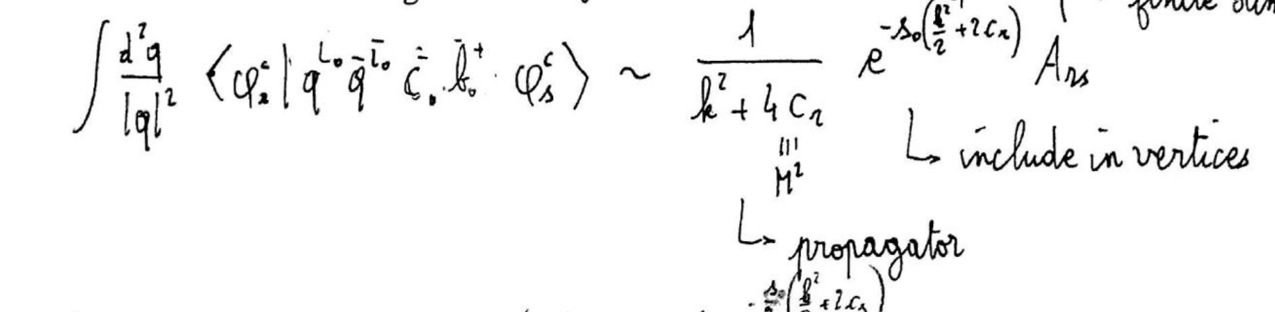
$$\int \frac{d^2q}{|q|^2} \langle \varphi_r^c | q^{L_0} \bar{q}^{\bar{L}_0} c_0^- b_0^+ \varphi_s^c \rangle \sim \frac{1}{k^2 + 4c_r}\, e^{-s_0\left(\frac{k^2}{2}+2c_r\right)} A_{rs}$$

A_{rs} $\to$ finite dim matrix

$e^{-s_0\left(\frac{k^2}{2}+2c_r\right)}$ $\hookrightarrow$ include in vertices

$k^2 + 4c_r = M^2$ $\hookrightarrow$ propagator

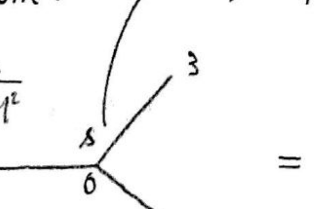
Feynman diagram:



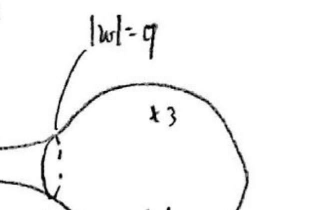
Convergence: # of states $\propto e^{K\sqrt{c_r}}$ (Cardy) but factor $e^{-s_0 c_r}$ dumps the growth

Do the same thing for $z_4 \sim z_1, z_2$:

t-channel:

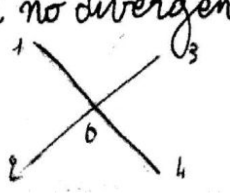


u-channel:



Remaining region with no divergences $\to$ fundamental vertex

$$\}A_1, \ldots, A_4\{_0 =$$



Notes:

- the sum of the 4 contribution is independent of s.
- the complete basis contains off-shell states

Riemann surface interpretation of propagator: plumbing fixture

$\longrightarrow$ glue two Riemann surfaces according to one respective punctures Σ_1 and Σ_2

at $w^{(1)} = 0$, $w^{(2)} = 0$

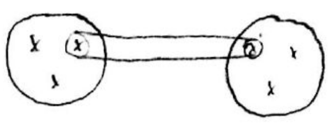
s.t $w^{(1)} w^{(2)} = q$ induce local coord + PCO locations

Superstring: if gluing R punctures then one picture is missing

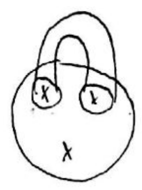
$\longrightarrow$ insert PCO $\mathcal{X}_0 = \oint \frac{\mathcal{X}(z)}{z} dz$

Propagator $\frac{\mathcal{G}}{L_0 + \bar{L}_0}$ where $\mathcal{G} = \begin{cases} 1 & NS \\ \mathcal{X}_0 & R \end{cases}$

Separating

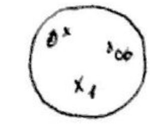


Non-separating



* Esplicit vertex construction

⑦ 1704.01210
Erler-Konopka-Sachs



$\{A_1, A_2, A_3\}_0 = \langle f_0 \circ A_1(0)\, f_1 \circ A_2(0)\, f_\infty \circ A_3(0) \rangle$

Global coordinate z : punctures at $z = 0, 1, \infty$.

Local coordinates : w_0, w_1, w_∞, maps : $f_0(0) = 0$, $f_1(0) = 1$, $f_\infty(0) = \infty$.

The vertex should be symmetric under eschange of the punctures:

subgroup $S_3 \subset SL(2,\mathbb{C})$ s.t:

$$0 \leftrightarrow 1 : \quad a(z) = 1 - z$$

$$0 \leftrightarrow \infty : \quad b(z) = \frac{1}{z}$$

$$1 \leftrightarrow \infty : \quad a \circ b \circ a(z) = \frac{z}{z-1}$$

Take $f_i \in SL(2,\mathbb{C})$. Symmetric vertexesrequire

$$\sigma \circ f_i(w) = f_{\sigma(i)}(e^{i\theta_{i,\sigma}} w) \qquad \sigma \in S_3$$

(identification of the local coordinates after σ up to a phase)

One has : $a \circ b \circ a \circ f_0(w) = f_0(e^{i\theta} w)$

Then $f_0 \in SL(2,\mathbb{C})$ and $f_0(0) = 0$ implies

$$f_0(w) = \frac{2w}{w + 2\beta} = z_0 \qquad (e^{i\theta} = -1)$$

This gives

$$f_1(w) = a \circ f_0(w) = \frac{2\beta - w}{2\beta + w} = z_1$$

$$f_\infty(w) = b \circ f_0(w) = \frac{w + 2\beta}{2w} = z_\infty$$

The disks covered by local coordinates should not overlap

$$\Longrightarrow \quad 3 \leq 2\beta \quad , \text{ parametrizes } \beta = \frac{3}{2} e^{\omega}$$

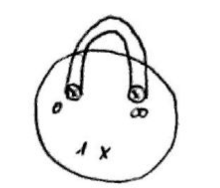
$s_0 = \frac{\omega}{2}$

Note: $e^{-\omega L_0^+} |A\rangle = e^{-\omega} \circ A(0)|0\rangle$

↳ worldsheet tube of length ω

* Propagator diagram

Plumbing fixture : $w_0 w_\infty = e^{-s-i\theta}$



This gives an identification between the regions with points z_0 and z_∞ :

$$z_\infty = \frac{1 + A(s,\theta) z_0}{2 - z_0} \qquad A(s,\theta) = \frac{9\, e^{2\omega + s + i\theta} - 1}{2}$$

Goal : introduce uniformization coordinate ξ, relate ω, s and θ to the torus modulus τ :

$$\xi \sim \xi + 1 \qquad \xi \sim \xi + \tau .$$

Holomorphic one-form : $\omega(z)\, dz$

- normalization : $\int_A \omega(z)\, dz = 1$
- modulus : $\int_B dz\, \omega(z) = \tau$
- Abel map : $\xi = W(z) = \int_1^z dz' \omega(z')$
  ↳ locate puncture at $\xi = 0$

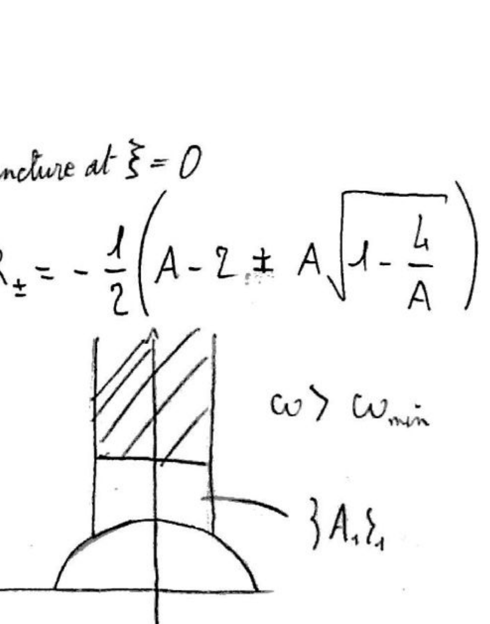
One finds :

$$\omega(z)\, dz = \frac{1}{2\pi i}\, d\left(\ln \frac{z - R_-}{z - R_+}\right) \qquad R_\pm = -\frac{1}{2}\left(A - 2 \pm A\sqrt{1 - \frac{4}{A}}\right)$$

$$\Longrightarrow \quad \tau = \frac{1}{2\pi i} \ln \frac{2 - R_-}{2 - R_+}$$

Single covering if $\omega > \omega_{min} \simeq 2{,}74$.

$$\tau \simeq \frac{i}{\pi}\left(\omega + \ln \frac{3}{2}\right) + \frac{i}{2\pi}(s + i\theta) + O(e^{-2\omega})$$



$$W(z) = \frac{1}{2\pi i} \ln\left(\frac{1 - R_+}{1 - R_-} \, \frac{z - R_-}{z - R_+}\right)$$

Local map for the puncture (induced from gluing)

$$h_1(w) = W \circ f_1(w) \simeq -\frac{e^{-\omega}}{3\pi i}\, w + O(e^{-3\omega})$$

Not covered : build elementary vertex $\{A_1 \Sigma_1\}$ for the other τ.

* General description

Amplitude contains 3 types of data : moduli parameters, local coord. choices, PCO locations

Build fiber bundle : extra data are fibers, $\mathcal{M}_{g,m,n}$ is the base

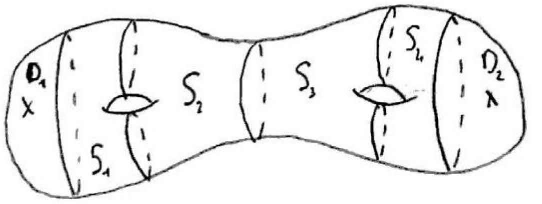
$$\mathcal{M}_{g,m,n} \xrightarrow{\text{local coord}} \hat{\mathcal{P}}_{g,m,n} \xrightarrow{\text{PCO}} \tilde{\mathcal{P}}_{g,m,n}$$

m: NS , n: R
$N = m + n$

Parametrization of $\hat{\mathcal{P}}_{g,m,n}$: $\Sigma_{g,m,n}$ can be seen as the union of N disks and $2g-2+N$ 3-punctured spheres, joined along $3g-3+2N$ circles C_s.

Transition functions $\underbrace{\sigma_s}_{\text{left of } C_s} = F_s(\underbrace{\tau_s}_{\text{right of } C_s})$ parametrizes $\hat{\mathcal{P}}_{g,m,n}$.

note : - dim $\hat{\mathcal{P}}_{g,m,n} = \infty$
- $\exists$ redundancy



$\Sigma_{2,2}$

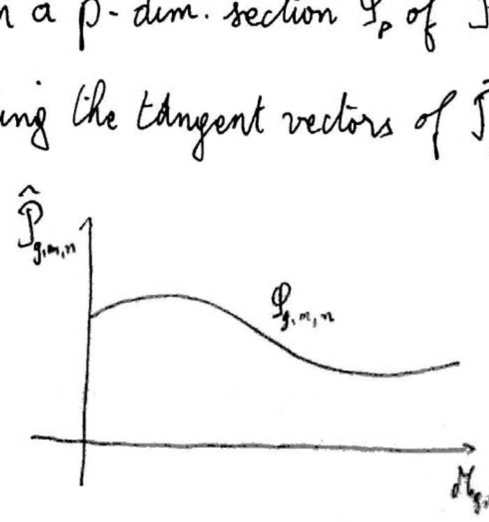
$\tilde{\mathcal{P}}_{g,m,n} = \hat{\mathcal{P}}_{g,m,n} +$ PCO locations.

Define p-forms s.t. it can be integrated on a p-dim. section $\mathcal{S}_p$ of $\tilde{\mathcal{P}}_{g,m,n}$.

$$\int_{\mathcal{S}_p} \omega_p = \#.$$ Constructed by finding the tangent vectors of $\tilde{\mathcal{P}}_{g,m,n}$.

Amplitude : $p = 6g - 6 + 2N$.

$$A_{g,m,n} = \int_{\mathcal{S}_{g,m,n}} \Omega^{(g,m,n)}_{6g-6+2N}(\underbrace{K_i}_{\text{NS states}}, \underbrace{L_j}_{\text{R states}})$$



Remember : $K_i \in \mathcal{H}_{-1}$, $L_j \in \mathcal{H}_{-1/2}$

$$\Omega^{(g,m,n)}_{6g-6+2N} \sim \Big\langle \bigwedge_{i=1}^{6g-6+2N} \mathcal{B}_i\, dt_i \prod_{\alpha=1}^{2g-2+m+\frac{n}{2}} \mathcal{X}(y_\alpha) \prod_{i=1}^{m} K_i \prod_{j=1}^{n} L_j \Big\rangle_{\Sigma_{g,m,n}}$$

where $\quad t_i$: coord. of the section

$$\mathcal{B}_i = \underbrace{\sum_s \oint \frac{\partial F_s}{\partial t_i}\, d\sigma_s\, b(\sigma_s) + \sum_s \oint \frac{\partial \bar{F}_s}{\partial t_i}\, d\bar{\sigma}_s\, \bar{b}(\bar{\sigma}_s)}_{\text{deformation of transition functions}} - \underbrace{\sum_\alpha \frac{1}{\mathcal{X}(y_\alpha)} \frac{\partial y_\alpha}{\partial t_i} \partial\xi(y_\alpha)}_{\text{if } y_\alpha = y_\alpha(m_i)}$$

m_i: moduli

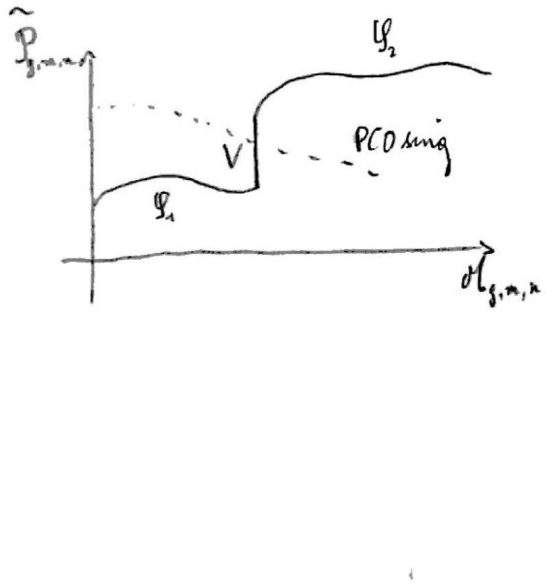
Some properties for $Q_B K_i = Q_B L_i = 0$:

- $A(Q_B\Lambda, K_1, \dots, K_m, L_1, \dots, L_n) = 0$
- $\int_{\mathcal{S}_p} \Omega_p = \int_{\mathcal{S}'_p} \Omega_p$

$\Longleftrightarrow$ A independent of local coord + PCO location

Caveat: existence of spurious poles. Then Ω_p has a singularity when integrating over $\mathcal{S}_p$ (typically $\nexists\, \mathcal{S}_p$ without singularity)

$\longrightarrow$ cannot consider continuous sections, use vertical integration



$$A = \int_{\mathcal{S}_1} \Omega_p + \int_{\mathcal{S}_2} \Omega_p + \text{boundary corrections}$$

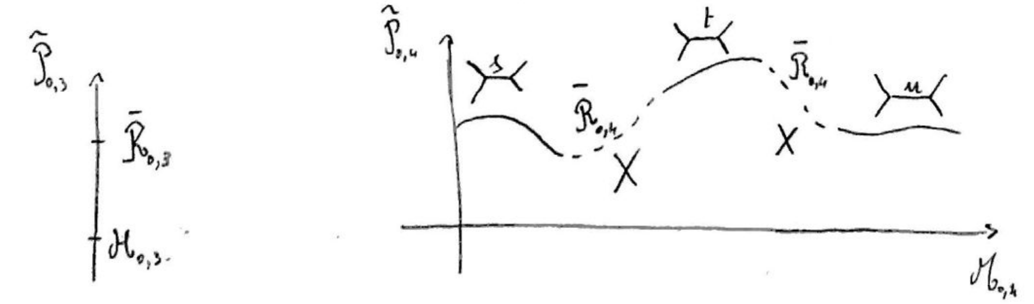
4. Superstring field theory

* Gauge fixed SFT

Divide the section $\mathcal{S}_{g,m,n}$ in cells:

- each cell = Feynman diagram, the sum of all diagrams reproduces $\mathcal{S}_{g,m,n}$
- plumbing fixture between two punctures = propagator between two legs
  - same diagrams: ↗ loop
  - two diagrams: ↗ legs

⟶ cells with no propagators = fundamental vertices $\bar{\mathcal{R}}_{g,m,n}$



Multilinear string products:

$$\{K_1 \dots K_m L_1 \dots L_n\}_g = \int_{\bar{\mathcal{R}}_{g,m,n}} \Omega^{(g,m,n)}_{6g-6+2N}(K_i, L_j)$$

$$\{K_1 \dots L_n\} = \sum_{g=0}^{\infty} g_s^{2g} \{K_1 \dots L_n\}_g$$

⟶ interaction terms in the action

Note: open-closed bosonic SFT: vertices can be defined by metric of minimal area with constraints:
- length of any non-trivial open curve with end points at boundary $\geq \pi$
- " " " closed curve $\geq 2\pi$

In practice good for numerical computations. stub param: $s_0 = \pi$.

No equivalent prescription for superstring

The propagator is

$$\frac{1}{L_0+\bar{L}_0} \sim \frac{1}{k^2+M^2}$$
$$\frac{X_0}{L_0+\bar{L}_0} \sim \frac{\not{k}+M}{k^2+M^2}$$

$$K^{-1} = - b_0^+ (L_0^+)^{-1} \mathcal{G}$$

It cannot be inverted to find the kinetic term because $\ker X_0 \neq \emptyset$.

Consider the two string fields:

$$\Psi \in \hat{\mathcal{H}}_T \quad , \quad \tilde{\Psi} \in \tilde{\mathcal{H}}_T$$

$$\Psi = \sum_n \psi_n |\varphi_n\rangle = \sum_n \int \frac{d^D k}{(2\pi)^D} \psi_n(k) |n, k\rangle$$

ψ_n → dynamical variables

We impose the constraints

$$b_0^+ |\Psi\rangle = 0 \quad ; \quad b_0^+ |\tilde{\Psi}\rangle = 0$$

Reason: K^{-1} contains b_0^+ and acts on conjugate states. A state ψ can be expanded on a basis of states annihilated by c_0^+ or b_0^+.

$$b_0^+ |\psi\rangle = 0 \implies c_0^+ |\psi\rangle = 0$$

The action is

$$S_{sf} = \frac{1}{g_s^2} \left[-\frac{1}{2} \langle \tilde{\Psi} | c_0^- c_0^+ L_0^+ \mathcal{G} | \tilde{\Psi} \rangle + \langle \tilde{\Psi} | c_0^- c_0^+ L_0^+ | \Psi \rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \{ \Psi^n \} \right]$$

do not include c_0^-: part of the inner product def, not operator

Then

$$K = c_0^+ L_0^+ \begin{pmatrix} +\mathcal{G} & -1 \\ -1 & 0 \end{pmatrix} \implies K^{-1} = -b_0^+ (L_0^+)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & \mathcal{G} \end{pmatrix}$$

It seems that we doubled the # of states but it is not the case: $\tilde{\Psi}$ are free fields.

$$[Q_B, \mathcal{G}] = 0$$

The eom are:

$$c_0^+ L_0^+ |\tilde{\Psi}\rangle + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} [\Psi^{n-1}] = 0$$

$$\{ \Psi^n \} = \langle \Psi | c_0^- | [\Psi^{n-1}] \rangle$$

$$c_0 L_0^+ \left(|\Psi\rangle - \mathcal{G} |\tilde{\Psi}\rangle \right) = 0$$

The first equation can be solved for $|\tilde{\Psi}\rangle$ up to free fields if $|\Psi\rangle$ is known.

$|\Psi\rangle$ eq: $c_0^+ L_0^+ |\Psi\rangle + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \mathcal{G} [\Psi^{n-1}] = 0$

* Aside: type IIB sugra

Bosonic fields:

- NSNS: $g_{\mu\nu}$, $B^{(2)}$, φ
- RR: $C^{(4)}$, $C^{(2)}$, $C^{(0)}$

Field strengths: $H^{(3)} = dB^{(2)}$, $F^{(3)} = dC^{(2)}$

$$\hat{F}^{(5)} = F^{(5)} + B^{(2)} \wedge F^{(3)} \quad ; \quad F^{(5)} = dC^{(4)}$$

Self-dual 5-form: $* \hat{F}^{(5)} = \hat{F}^{(5)}$

↳ should be imposed beside eom

Action with $C^{(4)}$ terms:

$$S = -\frac{1}{2} \int \hat{F}^{(5)} \wedge * \hat{F}^{(5)} + \int F^{(5)} \wedge B^{(2)} \wedge F^{(3)}$$

eom: $d\left(* \hat{F}^{(5)} - B^{(2)} \wedge F^{(3)} \right) = 0$

automatically satisfied

Construct Lorentz covariant local action without additional constraints.

(simplification: consider flat space)

Fields: replace $C^{(4)}$ by the pair $P^{(4)}$, $Q^{(5)} = *Q^{(5)}$

Action:
$$S' = \frac{1}{2} \int dP^{(4)} \wedge * dP^{(4)} - \int dP^{(4)} \wedge Q^{(5)} - \int B^{(2)} \wedge F^{(3)} \wedge Q^{(5)}$$
$$+ \frac{1}{2} \int * \left(B^{(2)} \wedge F^{(3)} \right) \wedge \left(B^{(2)} \wedge F^{(3)} \right)$$

eom: $d\left(* dP^{(4)} - Q^{(5)} \right) = 0$

$$dP^{(4)} + B^{(2)} \wedge F^{(3)} - * \left(dP^{(4)} + B^{(2)} \wedge F^{(3)} \right) = 0$$

Identify: $\hat{F}^{(5)} = \frac{1}{2} \left(Q^{(5)} + B^{(2)} \wedge F^{(3)} + * \left(B^{(2)} \wedge F^{(3)} \right) \right)$

Correct properties:

$$*\hat{F}^{(5)} \neq \hat{F}^{(5)}$$

$$d\hat{F}^{(5)} = d\left(B^{(2)} \wedge F^{(3)}\right) = H^{(3)} \wedge F^{(3)}$$

One can show that $\dfrac{\delta S}{\delta \varphi} = \dfrac{\delta S'}{\delta \varphi}$ for $\varphi \neq C^{(4)}$

A general solution for $P^{(4)}$ is

$$P^{(4)} = C^{(4)} + \hat{P}^{(4)} \quad , \qquad dC^{(4)} = \hat{F}^{(5)} - B^{(2)} \wedge F^{(3)}$$

One gets from the eom:

$$d * d\,\hat{P}^{(4)} = 0 \quad , \quad d\hat{P}^{(4)} = * \, d\hat{P}^{(4)}$$

Gauge invariance: $\delta \hat{P}^{(4)} = d\,\Xi^{(3)}$

$\longrightarrow$ $\hat{P}^{(4)}$ is a free field with self-dual field strength.

This generalizes when including gravity: $\hat{P}^{(4)}$ stays free, it does not even couple to gravity.

note: general coord not manifest

Hence the two theories just differ by free field $\longrightarrow$ does not change the interactions and dynamics.

* Classical action

To construct the classical gauge invariant action:

- relax $b_0^+|\Psi_{cl}\rangle = 0$ (Siegel gauge)
- impose $N_{gh}(\Psi_{cl}) = 2$ (physical states, no ghosts at classical level)
- remark that $Q_B = c_0^+ L_0^+$ + vanishing terms upon constraints

Classical action:

$$S_{cl} = \frac{1}{g_s^2}\left(-\frac{1}{2}\langle\tilde{\Psi}_{cl}|c_0^- Q_B \mathcal{G}|\tilde{\Psi}_{cl}\rangle + \langle\tilde{\Psi}_{cl}|c_0^- Q_B|\Psi_{cl}\rangle + \sum_{n=3}^{\infty}\frac{1}{n!}\{\Psi_{cl}^n\}_0\right).$$

Linearized eom: $Q_B|\Psi_{cl}\rangle = 0$

$$Q_B\left(|\tilde{\Psi}_{cl}\rangle - \mathcal{G}|\tilde{\Psi}_{cl}\rangle\right) = 0$$

Gauge invariance

$$\delta|\Psi_{cl}\rangle = Q_B|\Lambda\rangle + \sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{G}[\Psi_{cl}^n\Lambda]_0$$

$$\delta|\tilde{\Psi}_{cl}\rangle = Q_B|\hat{\Lambda}\rangle + \sum_{n=0}^{\infty}\frac{1}{n!}[\Psi_{cl}^n\Lambda]_0$$

* Quantum action

Relax $N_{gh} = 2$ condition: adds ghosts ($N_{gh} < 2$) and anti-ghosts ($N_{gh} > 2$)

$\longrightarrow$ needs BV formalism

Action:

$$S = \frac{1}{g_s^2}\left[-\frac{1}{2}\langle\tilde{\Psi}|c_0^- Q_B|\tilde{\Psi}\rangle + \langle\tilde{\Psi}|c_0^- Q_B|\Psi\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\{\Psi^n\}\right].$$

It satisfies quantum BV master equantion — S is real

$$\frac{1}{2}\underbrace{\{S,S\}}_{\text{antibracket}} + \Delta S = 0$$

Interaction terms at all loops $\longleftrightarrow$ path integral measure not gauge invariant.

In practice: work with 1PI action (in perturbation theory).

5. Momentum representation

* Action

Fields $\{\varphi_A\}$ (include auxiliary, pure gauge and ghosts) (Lorentzian signature, Siegel gauge)

$$S = \frac{1}{2}\sum_{A,B}\int d^D k\, \varphi_A(-k)\, K_{AB}(k)\, \varphi_B(k)$$
$$+ \sum_{n=1}^{\infty}\sum_{\{A_1,\dots,A_n\}}\int d^D k_1 \cdots d^D k_n\, \delta^{(D)}(k_1+\dots+k_n)\, V^{(n)}_{A_1,\dots,A_n}(k_1,\dots,k_n)\, \times$$
$$\times\, \varphi_{A_1}(k_1)\cdots\varphi_{A_n}(k_n)$$

Propagator (mass eigenstates)

$$K_{AB}(k)^{-1} \sim \frac{-i}{k^2+m_A^2}\,\Delta_{AB}\,Q(k) =$$

↳ polynomial

Vertices :

↱ parameters for the $\bar{\mathcal{R}}_{g,m,n}$

$$-i\,V^{(n)}_{A_1,\dots,A_n}(k_1,\dots,k_n) = -i\int [dy]\, e^{-\sum_{i,j}^{\{A_i\}} g_{ij}(y)\, k_i\cdot k_j}\, P_{\{A_i\}}(k_1,\dots,k_n;y)$$

↳ polynomial

$=$

Note : $V^{(n)}$ includes all loop orders.

Adding stubs = rescaling local coordinates

- multiply vertices by $\exp\left(-\sum_i \Lambda_i(y)\,(k_i^2+m_{A_i}^2)\right)$
- choose Λ_i to make $g_{ij}+\Lambda_i\delta_{ij}$ positive definite
- $e^{-\Lambda_i m_{A_i}^2}$ damps the contribution from the infinite # of fields $e^{c\, m_{A_i}}$

The action is real with appropriate conditions on the fields and

$$V^{(n)}_{A_1,\dots,A_n}(k_1,\dots,k_n)^* = V^{(n)}_{\bar{A}_1,\dots,\bar{A}_n}(-k_1^*,\dots,-k_n^*)$$

$$K_{AB}(k)^* = K_{\bar{A}\bar{B}}(-k^*)$$

* Correlation functions

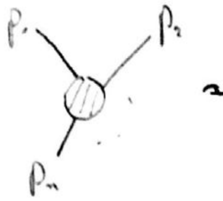
forget labels A_i

Momenta: external $\{p_\alpha\}$, internal $\{k_i\}$, loop $\{l_r\}$

↳ linear combination of p_α and l_r

Feynman diagram

$$I^{(n)}(p_1, \dots, p_n) =$$



$$\sim \int [dY] \prod_s d l_s^0 \, e^{-G_{rs}(Y) l_r l_s - 2 H_{r\alpha}(Y) l_r \cdot p_\alpha - F_{\alpha\beta}(Y) p_\alpha \cdot p_\beta} \times \prod_i \frac{1}{k_i^2 + m_i^2} \, P(l_r, p_\alpha ; Y)$$

G_{rs} is positive definite $\longrightarrow$ the integrals
- over spatial momenta $\vec{l}_s$ converge
- over energies l_s^0 diverge

$\longrightarrow$ cannot use standard Wick rotation and analyticity properties

Prescription
- multiply external energies with $u \in \mathbb{C}$: $(E_\alpha, \vec{p}_\alpha) \longrightarrow (u E_\alpha, \vec{p}_\alpha)$
- define the Green function for Euclidean momenta : $u = i$, $l_r^0 \in \mathbb{R}$
  ↳ the poles ~~are complex~~ of the propagators are complex, the integration is well-defined
- analytic continuation towards $u \longrightarrow 1$ and of l_r^0, deforming the contour to keep the poles on the same side but with end points kept at $\pm i\infty$.

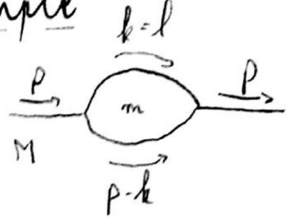
Yields analytic Green functions for

$$\mathrm{Re}\, u \geqslant 0 \;, \quad \mathrm{Im}\, u > 0$$

Note: Equivalent to $i\varepsilon$ prescription from Berera '94, Witten '13

$$\frac{1}{k^2 + m^2} = \int_0^{i\infty} dt \; e^{-t(k^2 + m^2 - i\varepsilon)}$$

* Example

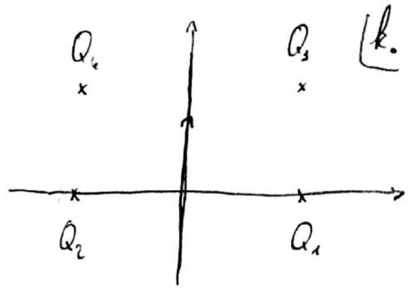
$A =$ 

$$\sim \int d^D k \; e^{-s_0(k^2+m^2)-s_0((k-p)^2+m^2)} \frac{1}{[k^2+m^2][(k-p)^2+m^2]}$$

Poles in k^0 plane, take $\vec{p}=0$ and $E = uM$

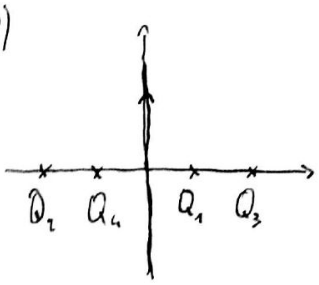
$$Q_1 = \sqrt{\vec{k}^2+m^2} \qquad Q_3 = p^0 + \sqrt{\vec{k}^2+m^2}$$

$$Q_2 = -\sqrt{\vec{k}^2+m^2} \qquad Q_4 = p^0 - \sqrt{\vec{k}^2+m^2}$$

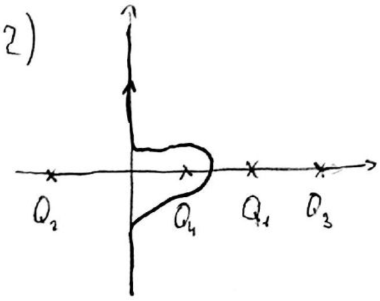


Three cases are possible for $u \to 1$:

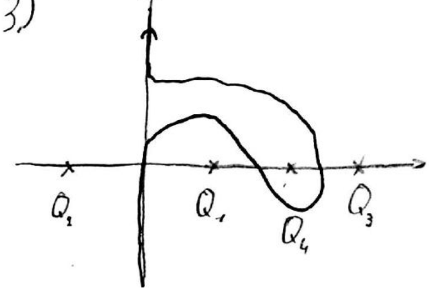
1)



2)



3)



Cases 2) and 3): the integral is the sum of two contours

- $k^0 \in i\mathbb{R}$
- residue around Q_4

Note: standard Wick

6. Applications

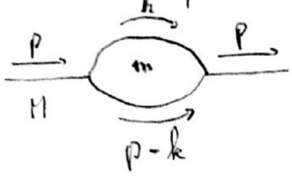
* Unstable particles

Compute 1-loop mass renormalization of massive particle.

$$M \longrightarrow M_R + i\Gamma \in \mathbb{C}$$

String theory ($SO(32)$ heterotic): first massive on leading Regge $M=2$, $m=0$. Reinterpret worldsheet integral as $I_1 + I_2 \longrightarrow$ regularize $\operatorname{Im}(\delta M^2) = -\dfrac{47}{264\pi}\Omega_8 g_s^2$

Consider previous example:

$\delta M^2 \approx$ 

$$p = (M, \vec{0}), \quad p^2 = -M^2$$

Incorrect computation: introduce Schwinger parameters

$$\frac{e^{-s_0(k^2+m^2)}}{k^2+m^2} = \int_{s_0}^{\infty} dt_1\, e^{-t_1(k^2+m^2)} \; ; \quad \frac{e^{-s_0[(k-p)^2+m^2]}}{(k-p)^2+m^2} = \int_{s_0}^{\infty} dt_2\, e^{-t_2[(k-p)^2+m^2]}$$

yields after integrating over k:

$$\delta M^2 \sim \int_{s_0}^{\infty} dt_1 \int_{s_0}^{\infty} dt_2 \frac{1}{(t_1+t_2)^{D/2}} \exp\left(\frac{t_1 t_2}{t_1+t_2} M^2 - (t_1+t_2)\, m^2\right)$$

This is UV finite due to the lower cut-off, but diverges for $t_1, t_2 \longrightarrow \infty$ if $M > 2m$.
Can be fixed by taking the upper limit to be $i\infty$.

Correct computation: write $\delta M^2 = I_1 + I_2$

- contour $k^0 \equiv iK \in i\mathbb{R}$

$$I_1 \sim \int d^{D-1}\vec{k} \int_{-\infty}^{\infty} dK\, e^{-s_0(K^2+\vec{k}^2+m^2) - s_0[(K+iM)^2+\vec{k}^2+m^2]} \frac{1}{[K^2+\vec{k}^2+m^2][(K+iM)^2+\vec{k}^2+m^2]}$$

- pole if $Q_L > 0$:

$$I_2 \sim \int d^{D-1}\vec{k}\, e^{-s_0(M-\sqrt{\vec{k}^2+m^2}) - s_0(\vec{k}^2+m^2)} \frac{1}{2M\sqrt{\vec{k}^2+m^2}\,(2\sqrt{\vec{k}^2+m^2} - M - i\varepsilon)}\, \Theta\left(M - \sqrt{\vec{k}^2+m^2}\right)$$

$\hookrightarrow$ contour deformation

Can be evaluated up to integral over $v = |\vec{k}|$ and u, both are finite. Moreover this result is consistent with Cutkosky rules. One has $I_1 \in \mathbb{R}$, $I_2 \in \mathbb{C}$.

× Unitarity and cutting rules

Note: old proof suffers from ambiguity
(from light cone + equivalence with covariant)

S-matrix: $S = 1 - iT$

Unitarity:

$$SS^\dagger = 1 \implies i(T - T^\dagger) = T^\dagger T = \sum_n T^\dagger |n\rangle\langle n| T$$

where $\{|n\rangle\}$: complete set of asymptotic physical states

Method: 't Hooft - Veltman '74

- prove Cutkosky cutting rules:

$$i(T-T^\dagger) = \sum_N T^\dagger |N\rangle\langle N| T$$

where $\{|N\rangle\}$: all physical, unphysical and pure gauge states
- use Ward identities to show decoupling of unphysical and pure gauge

Note: the action must be real.

Cutkosky rule: separate diagram in two pieces by cutting
- replace cut propagators by $\delta(k^2+m^2)\,\theta(k^0)$
- evaluate LHS using usual Feynman rules
- evaluate RHS by complex conjugating the energies

This yields the imaginary part of the diagram.

Note: with the previous example one finds

$$\mathcal{M}^2 - (\mathcal{M}^2)^* \simeq i \int\limits_{\hookrightarrow \mathbb{R}^D} d^D k \; e^{-s_0(k^2+m^2) - s_0[(k-p)^2+m^2]}\, \delta(k^2+m^2)\,\theta(k^0)\, \delta((p-k)^2+m^2)\,\theta(p^0-k^0)$$

* Soft theorems

Idea: split metric field in 3 parts

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu} + 2S_{\mu\nu}$$

background ← $\eta_{\mu\nu}$; $h_{\mu\nu}$ ↳ finite energy fluctuations; $S_{\mu\nu}$ ↳ soft graviton

Procedure:

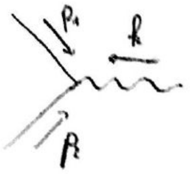
1. Set $S_{\mu\nu}=0$ and get action for $h_{\mu\nu}$ and other fields on background $\eta_{\mu\nu}$
2. Make Lorentz covariant gauge fixing
3. Replace $\eta_{\mu\nu} \longrightarrow \eta_{\mu\nu} + 2S_{\mu\nu}$ and covariantize, expanding to first order in $\partial S_{\mu\nu}$.
   ↳ subleading theorem
   This gives the coupling of $S_{\mu\nu}$ to the other fields.
   Note: $S_{\mu\nu}$ comes from $\eta_{\mu\nu}$, $\partial S_{\mu\nu}$ from $\Gamma^{\rho}_{\mu\nu}$ and $\partial^2 S_{\mu\nu}$ from $R_{\mu\nu\rho\sigma}$
   ↳ coupling non universa
   All coupling done from the kinetic term.
4. Replace $S_{\mu\nu} \longrightarrow \varepsilon_{\mu\nu}\, e^{ikx}$ to find the vertex, where

$$\eta^{\mu\nu}\varepsilon_{\mu\nu} = 0\,; \quad \varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}\,; \quad k^{\mu}\varepsilon_{\mu\nu} = 0$$

$\eta^{\mu\nu} \longrightarrow \eta^{\mu\nu} - 2S^{\mu\nu}$

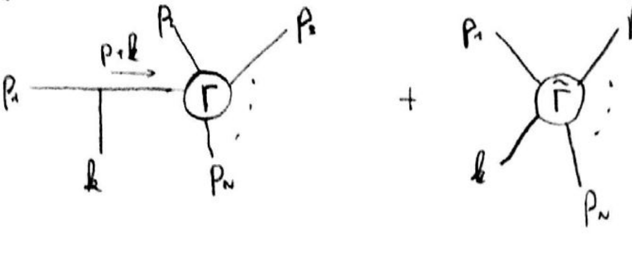
Example: scalar field

$$S_0 = -\frac{1}{2}\int d^Dx \left(\eta^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + M^2\varphi^2\right)$$

vertex: 

$= -2i\,\varepsilon^{\mu\nu} p_{1\mu} p_{2\nu} = 2i\,\varepsilon^{\mu\nu} p_{1\mu}(p_{1\nu} + \cancel{k_\nu})$

Amplitude for one soft graviton:

$\Gamma^{(N+1)}(\varepsilon, k\,; p_1, \dots, p_N) =$ 

First contribution:

$$2i\,\varepsilon^{\mu\nu} p_{i\mu} p_{i\nu} \times \frac{-i}{(p_i+k)^2+M^2} \times \Gamma^{(N)}(p_1,\dots,p_i+k,\dots,p_N)$$

$$\simeq \varepsilon^{\mu\nu} p_{i\mu} p_{i\nu} \times \frac{1}{p_i\cdot k}\left(\Gamma^{(N)}(p_1,\dots,p_N) + k_\rho \frac{\partial}{\partial p_{i\rho}} \Gamma^{(N)}(p_1,\dots,p_N)\right)$$

then sum over i.

Second graph: soft graviton is attached to an hard internal line, finite $\lim_{k\to 0}$.

Then $\tilde{\Gamma}^{(N+1)}(\varepsilon,k;p_1,\dots,p_N)$ can be interpreted as a deformation of $\Gamma^{(N)}(p_1,\dots,p_N)$ under a change of background $\eta^{\mu\nu} \to \eta^{\mu\nu} - 2\varepsilon^{\mu\nu}$ (leading term in k).

$\tilde{\Gamma}^{(N+1)}(\varepsilon,k;p_1,\dots,p_N)$ depends on the metric only through

$$g^{\mu\nu} p_{i\mu} p_{j\nu} = (\eta^{\mu\nu} - 2\varepsilon^{\mu\nu}) p_{i\mu} p_{j\nu} = \eta^{\mu\nu}(p_{i\mu} - \varepsilon_\mu{}^\rho p_{i\rho})(p_{j\nu} - \varepsilon_\nu{}^\rho p_{j\rho})$$

such that

$$\hat{\Gamma}^{(N+1)}(\varepsilon,k;p_1,\dots,p_N) \simeq \Gamma^{(N)}(p_1 - \varepsilon\cdot p_1,\dots,p_N - \varepsilon\cdot p_N)$$

$$\simeq -\sum_{i=1}^{N} \varepsilon_\mu{}^\nu p_{i\nu} \frac{\partial}{\partial p_{i\mu}} \Gamma^{(N)}(p_1,\dots,p_N)$$

Sum of both contributions: subleading soft theorem (one graviton)

$$\Gamma^{(N+1)}(\varepsilon,k;p_1,\dots,p_N) = \sum_{i=1}^{N} \varepsilon^{\mu\nu} p_{i\mu} p_{i\nu} \frac{1}{p_i\cdot k} \Gamma^{(N)}(p_1,\dots,p_N)$$

$$+ \sum_{i=1}^{N}\left(\varepsilon^{\mu\nu} p_{i\mu} p_{i\nu} \frac{1}{p_i\cdot k} k^\rho - \varepsilon_\rho{}^\nu p_{i\nu}\right) \frac{\partial}{\partial p_{i\rho}} \Gamma^{(N)}(p_1,\dots,p_N) + O(k)$$

Similarly: leading theorem (m gravitons)

$$\Gamma^{(N+m)}(\varepsilon_1,k_1,\dots,\varepsilon_m,k_m;p_1,\dots,p_N) = \prod_{s=1}^{m}\left(\sum_{i=1}^{N} \frac{1}{p_i\cdot k_s} \varepsilon^{(s)}_{\mu\nu} p_i^\mu p_i^\nu\right) \Gamma^{(N)}(p_1,\dots,p_N)$$

$$+ O(1)$$

These theorems can be extended to field theories:

- with higher-spin bosonic and fermionic fields
- any loop order

Subleading theorem with one graviton:

$$\Gamma^{(N+1)}(\varepsilon, k; \varepsilon_1, p_1; \dots; \varepsilon_N, p_N) = \sum_{i=1}^{N} \frac{1}{p_i \cdot k} \varepsilon_{\mu\nu} p_i^\mu p_i^\nu \, \Gamma^{(N)}(p_1, \dots, p_N)$$

$$+ \sum_{i=1}^{N} \left(\frac{1}{p_i \cdot k} \varepsilon_{\mu\nu} p_i^\mu p_i^\nu k_\rho - \varepsilon_{\rho\sigma} p_i^\sigma \right) \varepsilon_{i,M} \frac{\partial}{\partial p_{i\rho}} \Gamma^M_{(i)}(\{p_i\})$$

$$+ \sum_{i=1}^{N} \frac{1}{p_i \cdot k} k_\mu \varepsilon_{\nu\rho} p_i^\rho \, \varepsilon_{i,M} (J^{\mu\nu})^M{}_N \, \Gamma^N_{(i)}(\{p_i\})$$

where: ε_M : polarization tensor; $\varphi_M = \varepsilon_M e^{ikx}$

$$\Gamma^{(N)}(p_1, \dots, p_N) \equiv \sum_i \varepsilon_{i,M} \Gamma^M_{(i)}(p_1, \dots, p_N)$$