Building string field theory using machine learning

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In collaboration with:

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Massachusetts Institute of Technology

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Outline: 1. Introduction

Talk highlights

- string field theory (SFT)
  - 2nd quantized formulation of string theory
  - amplitude $= 2d$ conformal field theory (CFT) correlation function integrated over moduli space of Riemann surfaces

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  - parametrized by accessory parameters (uniformization)
  - compute mapping radii $\rightarrow$ local coordinates
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  - extract vertex region
- use neural networks to parametrize accessory parameters and vertex region

From worldsheet string theory to string field theory (1)

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  - 1st-quantized (dynamics of a few strings)
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  - constructive, symmetries manifest
  - prove consistency (unitarity, analyticity, finiteness...)
  - study backgrounds (independence, fluxes...)

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  - modern language and tools of field theory
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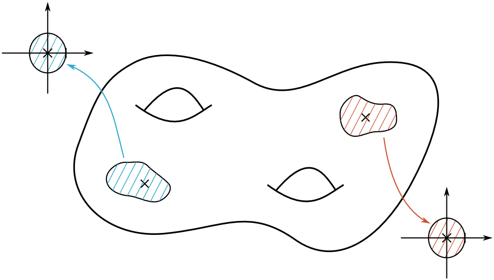
  - prove consistency (unitarity, analyticity, finiteness. . .)
  - study backgrounds (independence, fluxes. . .)
- problems
  - action: non-local, non-polynomial, ∞ number of fields
  - general properties known, but not explicit form

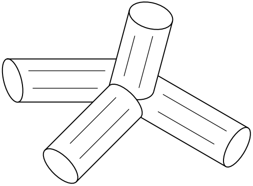
From worldsheet string theory to string field theory (2)

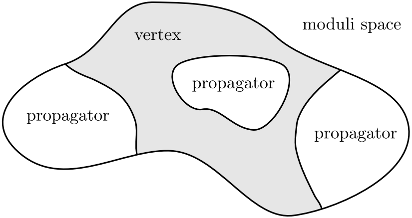
$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Riemann surface}\\ \text{moduli spaces}\end{array} \arrow[r, "2d\ \text{CFT}"] \arrow[d] & \begin{array}{c}\text{on-shell}\\ \text{string amplitudes}\end{array} \arrow[d] \\
\begin{array}{c}\text{Riemann surface}\\ \text{moduli spaces}\\ +\ \text{local coordinates}\end{array} \arrow[r] \arrow[d, "\begin{array}{c}\text{decomposition}\\ \text{by sewing}\end{array}"'] & \begin{array}{c}\text{off-shell}\\ \text{string amplitudes}\end{array} \arrow[d, "\begin{array}{c}\text{decomposition}\\ \text{in Feynman diagrams}\end{array}"'] \\
\begin{array}{c}\text{geometric}\\ \text{BV algebra}\end{array} \arrow[r, "\text{BV morphism}"'] & \begin{array}{c}\text{string field theory}\\ \text{(quantum BV)}\end{array}
\end{tikzcd}
$$

Local coordinates and moduli space decomposition

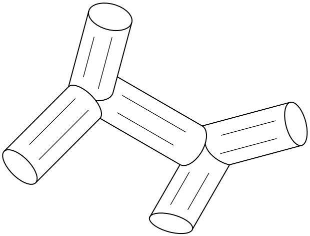


fundamental vertex





graph with propagator



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Objective (physics)

Construct action using machine learning in order to extract numbers from SFT (in particular, closed string tachyon vacuum).

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Objective (math)

Construct functions on and subspaces of moduli space of Riemann surfaces using machine learning.

Tachyon vacuum

- main application: study closed string tachyon vacuum (settle existence or not)
- method
  - perform level-truncation (keep fields up to some mass)
  - compute potential up to some order in g_s
  - integrate out other fields (except dilaton)
  - extrapolate in level and order of interaction

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  - integrate out other fields (except dilaton)
  - extrapolate in level and order of interaction
- truncated tachyon potential

$$V(t) = -t^2 + \sum_{n\geq 3} \frac{v_n}{n!}\, t^n, \qquad v_4 \approx 72.32 \pm 0.15$$

  previous results: $v_4 \approx 72.39$

  [hep-th/9412106, Belopolsky; hep-th/0408067, Moeller]
- other backgrounds: twisted tachyons on $\mathbb{C}/\mathbb{Z}_N \ldots$

  [hep-th/0111004, Dabholkar; hep-th/0403051, Okawa-Zwiebach]

Outline: 2. String field theory

Classical string field theory action

- string background: $2d$ conformal field theory (CFT)
- string field $\Psi \in \mathcal{H}$ (1st-quantized CFT Hilbert space)
- classical action

$$S = \frac{1}{2} \langle \Psi, Q_B \Psi \rangle + \sum_{n \geq 3} \frac{g^{n-2}}{n!} \mathcal{V}_n(\Psi^n)$$

- 1st-quantized BRST operator $Q_B : \mathcal{H} \to \mathcal{H}$
- string vertices $\mathcal{V}_n : \mathcal{H}^{\otimes n} \to \mathbb{C}$ (contact interactions)

Example: ϕ^4 scalar field

- action

$$
\begin{aligned}
S = \frac{1}{2} \int \mathrm{d}^d k \, \phi(-k)(k^2 + m^2)\phi(k) \\
\quad + \frac{\lambda}{4!} \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_4 \, \delta^{(d)}(k_1 + \cdots + k_4)\, \phi(k_1) \cdots \phi(k_4) \\
= \frac{1}{2} \, \langle \phi, K\phi \rangle + \frac{\lambda}{4!} \, \mathcal{V}_4(\phi^4)
\end{aligned}
$$

- 1st-quantized momentum state basis $\{|k\rangle\}$

$$
|\phi\rangle = \int \mathrm{d}^d k \, \phi(k) \, |k\rangle \, , \qquad \langle k, k' \rangle = \delta^{(d)}(k + k')
$$

- Klein–Gordon operator $K = (p^2 + m^2)$
- quartic vertex

$$
\mathcal{V}_4(\phi^4) = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_4 \, V_4(k_1, \ldots, k_4)\, \phi(k_1) \cdots \phi(k_4)
$$

$$
V_4(k_1, \ldots, k_4) = \delta^{(d)}(k_1 + \cdots + k_4)
$$

String amplitudes

- off-shell n-point string amplitude with external states $A_i \in \mathcal{H}$

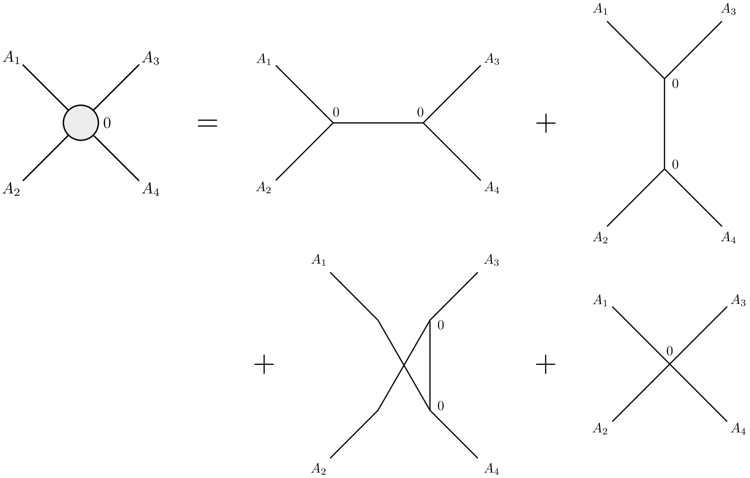
$$\mathcal{A}_n(A_1, \dots, A_n) = \int_{\mathcal{M}_n} \mathrm{d}^{n-3}\xi \, \left\langle \text{ghosts} \times \prod_i f_{n,i} \circ A_i(0) \right\rangle_{\Sigma_n}$$

  - $\langle \cdots \rangle$ CFT correlation function
  - sum over topologically inequivalent spheres Σ_n with n punctures at $(\xi_1, \dots, \xi_n)$
  - can fix 3 points $(\xi_{n-2}, \xi_{n-1}, \xi_n) = (0, 1, \infty)$
  - $\xi_\lambda \in \mathcal{M}_n \sim \mathbb{C}^{n-3}$ $(\lambda = 1, \dots, n-3)$ moduli space
  - ghosts: 1) measure over $\mathcal{M}_n$, 2) needed for BRST invariance
  - $f_{n,i}(w_i; \xi_\lambda)$ local coordinates = conformal maps

$$f_{n,i}(0; \xi_\lambda) := \xi_i, \qquad f_{n,i} \circ A_i(0) := |f'_{n,i}(0)|^{2h_i} A_i\big(f_{n,i}(0)\big)$$

  if A_i is primary with weight (h_i, h_i)

Amplitude and Feynman diagrams



$$\mathcal{A}_4 \qquad = \qquad \mathcal{F}_4^{(s)} + \mathcal{F}_4^{(t)} + \mathcal{F}_4^{(u)} \quad + \qquad \mathcal{V}_4$$

String vertices

- string vertex

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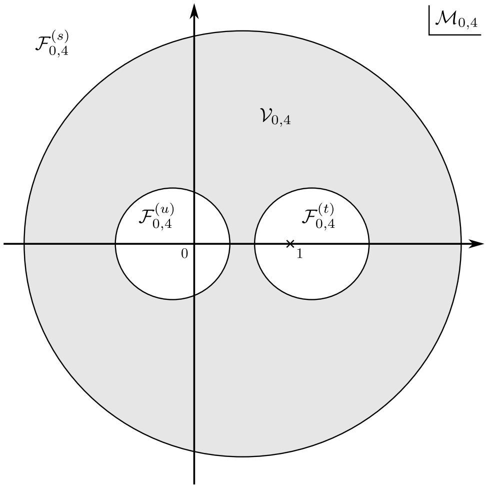
- defined such that

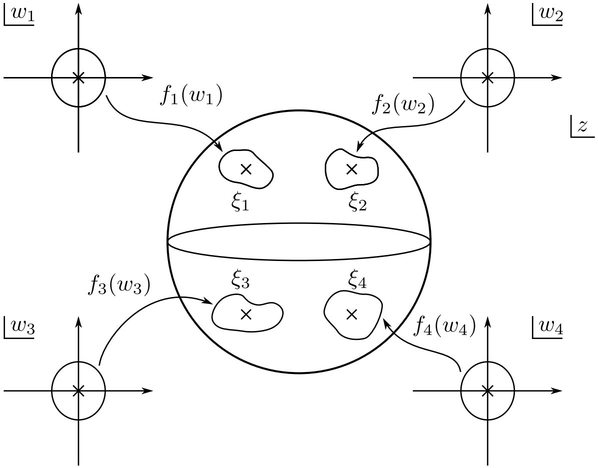
$$\mathcal{A}_n(A_1, \dots, A_n) = \mathcal{F}_n(A_1, \dots, A_n) + \mathcal{V}_n(A_1, \dots, A_n)$$

  $\mathcal{F}_n$ contributions from Feynman diagrams (Riemann surfaces) containing propagators (long tubes) and $\mathcal{V}_{n'}$ with $n' < n$
- $\mathcal{V}_n \subset \mathcal{M}_n$: vertex region $\subset$ moduli space
- constraints between all $\{f_{n,i}\}$ (“gluing compatibility”)

Moduli space covering



Local coordinates



Motivation: restore $\mathrm{SL}(2,\mathbb{C})$ invariance, broken by punctures (transformation between patches)

How to build vertices

- $\mathrm{SL}(2,\mathbb{C})$ vertices
  - $n = 3$ sphere: simplest vertex
  - $n = 4$ sphere: analytical $\mathcal{V}_4$ boundary, no explicit coordinates [HE, in progress]
  - $n = 1$ torus: analytical vertex boundary, no explicit coordinates [1704.01210, Erler-Konopka-Sachs]
- hyperbolic vertices [1706.07366, Moosavian-Pius; 1909.00033, Costello-Zwiebach; 2102.03936, Fırat]
- minimal area string vertices: optimal representation [Zwiebach '91; hep-th/9206084, Zwiebach]

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- minimal area string vertices: optimal representation [Zwiebach '91; hep-th/9206084, Zwiebach]
- note: superstring vertices can be obtained by dressing bosonic vertices [hep-th/0409018, Berkovits-Okawa-Zwiebach; 1403.0940, Erler-Konopka-Sachs]

Outline: 3. Minimal area vertices

Minimal area vertex

- vertex constructed from minimal area metric with bounds on length of shortest closed geodesic (systole) and heights of internal foliation [Zwiebach '90]
- *n*-punctured sphere vertex: construct metric from Strebel quadratic differential [Saadi-Swiebach '89]
  - fixed up to moduli-dependent parameters
  - lead to contact interactions (internal foliation height $= 0$)

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  - lead to contact interactions (internal foliation height $= 0$)
- goal: $\forall n \geq 3$ obtain $f_{n,i}$ and $\mathcal{V}_n$
- state-of-the-art:
  - analytic solution for $n = 3$, numerical for $n = 4, 5$ [Moeller, hep-th/0408067, hep-th/0609209]
  - convex program for any genus and n, but not implemented and not restricted to vertex region [1806.00449, Headrick-Zwiebach]

Quadratic differential

- quadratic differential $\varphi = \phi(z)\mathrm{d}z^2$ [Strebel, '84]

$$\phi(z) = \sum_{i=1}^{n} \left[\frac{-1}{(z-\xi_i)^2} + \frac{c_i}{z-\xi_i} \right]$$

$$0 = \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} (-1 + c_i \xi_i) = \sum_{i=1}^{n} (-2\xi_i + c_i \xi_i^2)$$

- $c_i(\xi_i, \bar{\xi}_i)$ accessory parameters
  (limit from Liouville accessory parameters)
- constraints: regularity at $z = \infty$

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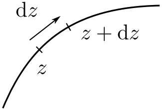
- $c_i(\xi_i, \bar{\xi}_i)$ accessory parameters
  (limit from Liouville accessory parameters)
- constraints: regularity at $z = \infty$
- φ induces metric with semi-infinite flat cylinders around punctures (= external strings)

$$\mathrm{d}s^2 = |\phi(z)|^2 |\mathrm{d}z|^2, \qquad \mathrm{d}s^2|_{w_i} = \frac{|\mathrm{d}w_i|^2}{|w_i|^2}$$

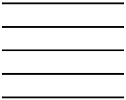
Critical trajectory



Definitions:

- $\{z_i(c_i, \xi_i)\}$ zeros of $\phi(z)$
- horizontal trajectory = path with $\varphi = \phi(z)\mathrm{d}z^2 > 0$

$\mathrm{d}z^2$



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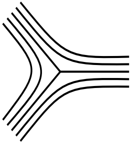

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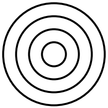
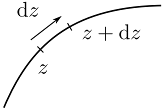


Critical trajectory



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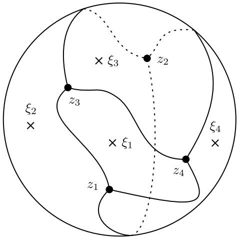
- critical trajectory = horizontal trajectory with ends at $\phi(z) = 0$
- critical graph = {critical trajectories}



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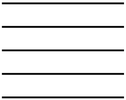

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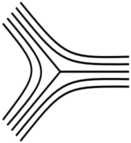

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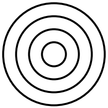

Strebel quadratic differential

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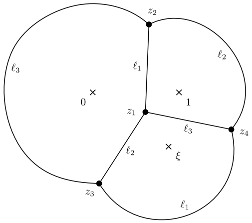
Quadratic differential such that its critical graph is:

1. a polyhedron (measure zero):
   - vertices = zeros
   - edges = critical trajectories
   - faces = punctures
2. connected (no propagator = long tube)



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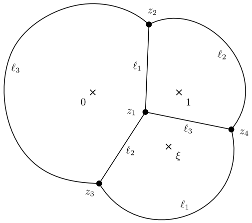
- unique given ξ_λ
- define minimal area metric
- defines string vertices
  - provide local coordinates
  - allow determining vertex region



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- hard mathematical problem (related to Fuchsian uniformization, Liouville theory...)
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$$\forall (z_i, z_j) : \quad \operatorname{Im} \ell(z_i, z_j) = 0$$

- for fixed ξ_i, give equations on c_i
- [Moeller, hep-th/0408067, hep-th/0609209]: solve point by point using Newton method for $n = 4, 5$ (and fit for $n = 4$)

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- series expansion

$$z = f_{n,i}(w_i) = \xi_i + \rho_i w_i + \sum_{k \geq 2} d_{i,k-1} (\rho_i w_i)^k$$

$$\varphi \sim_{\xi_i} \left(-\frac{1}{(z-\xi_i)^2} + \sum_{k \geq -1} b_{i,k} (z - \xi_i)^k \right) \mathrm{d}z^2 = -\frac{\mathrm{d}w_i^2}{w_i^2}$$

  where $b_{i,k} = b_{i,k}(c_i, \xi_i)$, e.g. $b_{i,-1} = c_i$
- match coefficients

$$d_{i,1} = \frac{b_{i,-1}}{2}, \qquad d_{i,2} = \frac{1}{16}\left(7 b_{i,-1}^2 + 4 b_{i,0}\right), \qquad \ldots$$

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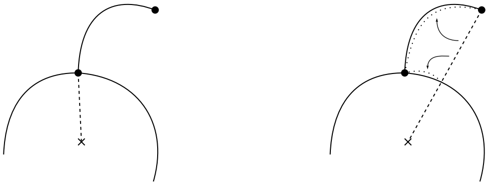
- remaining unknown: mapping radii $\rho_i \in \mathbb{R}$

Mapping radii

- mapping radius for ξ_i (conformal invariant)

$$\ln \rho_i = \ln \left| \frac{\mathrm{d}f_i}{\mathrm{d}w_i} \right|_{w_i=0} = \lim_{\epsilon \to 0} \left[\operatorname{Im} \int_{\xi_i+\epsilon}^{z_c} \mathrm{d}z \sqrt{\phi(z)} + \ln \epsilon \right]$$

- z_c is any point on critical graph (path after crossing closest trajectory does not contribute to imaginary part)
  $\rightarrow$ compute $z_c = z_i \, \forall i$, then average



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- indicator function

$$\int_{\mathcal{V}_n} \cdots = \int_{\mathcal{M}_n} \Theta(\xi) \cdots, \qquad \Theta(\xi) := \begin{cases} 1 & \text{if } \xi \in \mathcal{V}_n \\ 0 & \text{if } \xi \notin \mathcal{V}_n \end{cases}$$

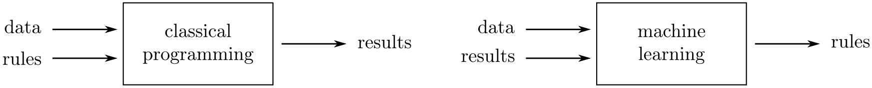
Outline: 4. Machine learning

Machine learning

Definition (Samuel)

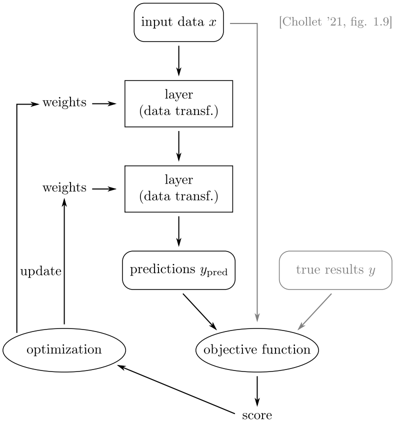
The field of study that gives computers the ability to learn without being explicitly programmed.

- approximate function $y = F(x)$ by some structure (neural network, decision tree. . .)
- agreement measured by some metric (distance, constraint. . .)
- tune structure parameters to improve approximation



Neural network

- neural network
  $=$ sequence of layers
  implementing computations
- layer
  - output $=$ different data representation
  - transformation parametrized by weights
- goal: find weights such that the network reproduces the target fonction $y = F(x)$
- comparison: objective function
- optimization by gradient descent
- general architecture defined by hyperparameters (number of layers...)



Why neural networks?

- generically outperform other machine learning approaches
- flexible inputs (complex numbers, graphs. . .)
- neural network = differentiable function
  - solve for the full function, not points one by one
  - better expressivity than fit
  - may extrapolate outside training region
  - classication task provides (probabilistic) measure
- transfer learning
- compact representation of the result, easily reused and shared

Learning the accessory parameter

- idea : $c_\lambda(\xi_\lambda) =$ complex neural network $C_\lambda(\xi_\lambda; \boldsymbol{W}, \boldsymbol{b})$
- $\boldsymbol{W}$ weights (complex matrices), $\boldsymbol{b}$ biases (complex vectors)

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- unsupervised training with loss

$$\mathcal{L}(C_\lambda, \xi_\lambda) = \binom{2n-4}{2}^{-1} \sum_{i \geq j} \Big(\operatorname{Im} \ell(z_i, z_j) \Big)^2 \Big|_{c_\lambda = C_\lambda}$$

  $\rightarrow$ minimize with gradient descent
- for fixed ξ_λ, global minimum for any n with c_λ given by Strebel differential

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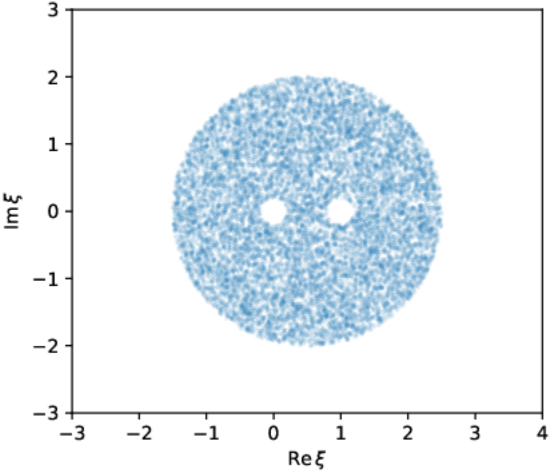
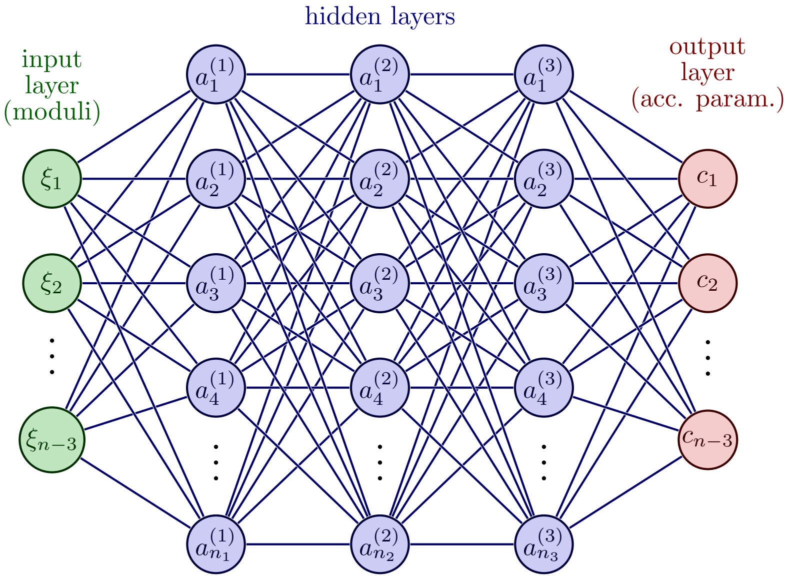
- for fixed ξ_λ, global minimum for any n with c_λ given by Strebel differential
- training set: uniform sampling in $\mathcal{M}_n$ minus disks around fixed punctures $(\xi_{n-2}, \xi_{n-1}, \xi_n) = (0, 1, \infty)$

Data



Neural network architecture



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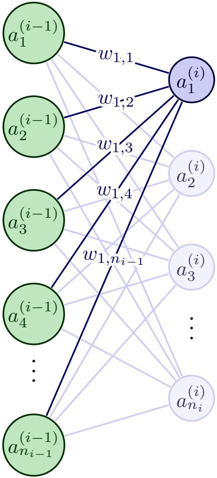

$$= \sigma\left(w_{1,1}a_1^{(i-1)} + w_{1,2}a_2^{(i-1)} + \ldots + w_{1,n_{i-1}}a_{n_{i-1}}^{(i-1)} + b_1^{(i)}\right)$$

$$= \sigma\left(\sum_{k=1}^{n_{i-1}} w_{1,k}a_k^{(i-1)} + b_1^{(i)}\right)$$

$$\begin{pmatrix} a_1^{(i)} \\ a_2^{(i)} \\ \vdots \\ a_{n_i}^{(i)} \end{pmatrix} = \sigma\left[\begin{pmatrix} w_{1,0} & w_{1,1} & \ldots & w_{1,n_{i-1}} \\ w_{2,0} & w_{2,1} & \ldots & w_{2,n_{i-1}} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n_i,0} & w_{n_i,1} & \ldots & w_{n_i,n_{i-1}} \end{pmatrix}\begin{pmatrix} a_1^{(i-1)} \\ a_2^{(i-1)} \\ \vdots \\ a_{n_{i-1}}^{(i-1)} \end{pmatrix} + \begin{pmatrix} b_1^{(i)} \\ b_2^{(i)} \\ \vdots \\ b_{n_i}^{(i)} \end{pmatrix}\right]$$

$$\implies a^{(i)} = \sigma\left(W^{(i)}a^{(i-1)} + b^{(i)}\right)$$

4-punctured sphere

- notations for $n = 4$

$$\xi_1 := \xi \in \mathbb{C}, \qquad c_1 := a \in \mathbb{C},$$
$$\ell(z_1, z_2) := \ell_1, \qquad \ell(z_1, z_3) := \ell_2, \qquad \ell(z_1, z_4) := \ell_3$$

- analytic solutions

$$a(1/2) = 2, \qquad a\left(Q = \frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}\right) = 2 + \mathrm{i}\,\frac{2}{\sqrt{3}} \approx 2 + 1.1547\mathrm{i}$$
$$a(\xi \in \mathbb{R}) = \begin{cases} 0 & \xi \leq 0 \\ 4\xi & 0 \leq \xi \leq 1 \\ 4 & \xi \geq 1 \end{cases}$$

Results: 4-punctured sphere (1)

- neural network (Jax)
  - fully connected, 3 layers $(512, 128, 1028)$, $\mathbb{C}$ReLU activation

$$\mathbb{C}\mathrm{ReLU}(z) := \mathrm{ReLU}(\operatorname{Re} z) + \mathrm{i}\,\mathrm{ReLU}(\operatorname{Im} z)$$

  - training: 10^5 points, Adam, ℓ_2 regularization, weight decay, early stopping (~ 1000 epochs)

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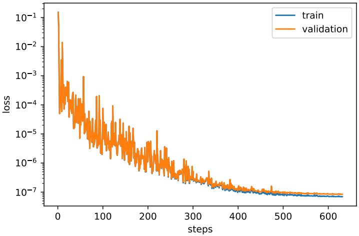
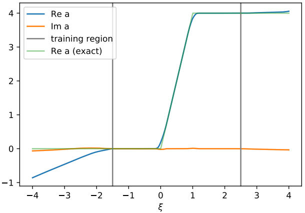
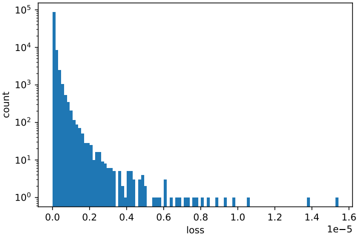
  - training: 10^5 points, Adam, ℓ_2 regularization, weight decay, early stopping (~ 1000 epochs)
- loss statistics (exact solution $\sim 10^{-12}$)
  - mean: $8.9 \cdot 10^{-8}$
  - median: $3.8 \cdot 10^{-8}$
  - min: $1.3 \cdot 10^{-11}$
  - max: $1.5 \cdot 10^{-5}$

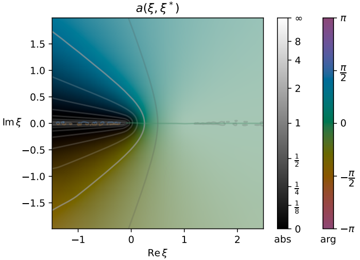
  note: already good performance with 10^3 points, 100 epochs (e.g. mean loss $= 2.7 \cdot 10^{-5}$)
- mean error compared to Moeller's fit: $5.5 \cdot 10^{-3}$

Results: 4-punctured sphere (2)





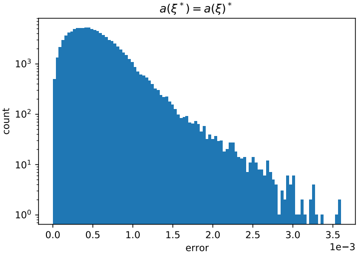


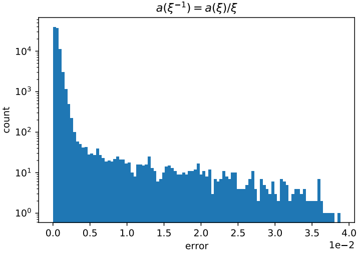


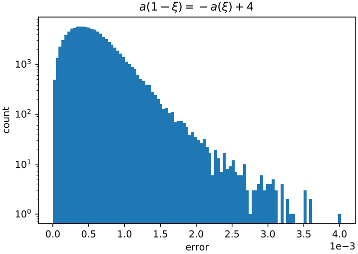
$$a(1/2) = 1.9995 + 0.0001\mathrm{i}, \qquad a(Q) = 1.9997 + 1.1548\mathrm{i}$$

$$\mathcal{L}(1/2) = 6.6 \cdot 10^{-7} \qquad \mathcal{L}(Q) = 6.1 \cdot 10^{-8}$$

Results: symmetries



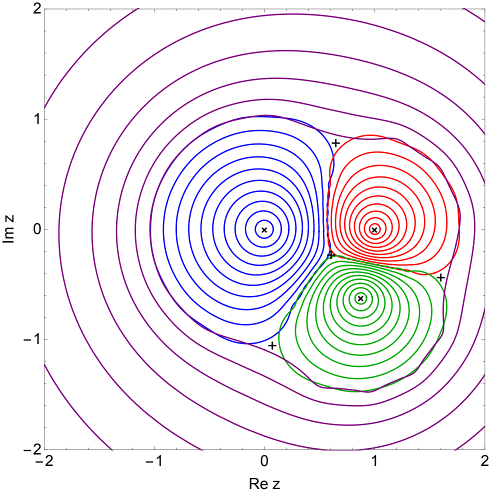




complex conjugation and permutation of fixed punctures

$$a(\xi^*) = a(\xi)^*$$
$$a(1-\xi) = 4 - a(\xi)$$
$$a(\xi^{-1}) = \frac{a(\xi)}{\xi}$$

Strebel differential



$\xi = 0.87 - 0.62\mathrm{i}$

Learning the vertex region

- idea: $\Theta(\xi) =$ neural network $\theta(\xi)$
  - $\theta(\xi)$ becomes probability distribution
  - useful for Monte Carlo integration
  - easily find boundary, e.g. $\theta(\xi) \in [0.2, 0.8]$

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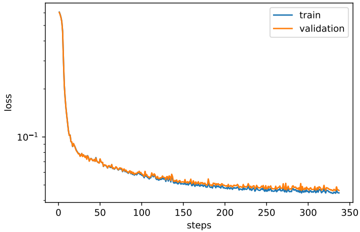
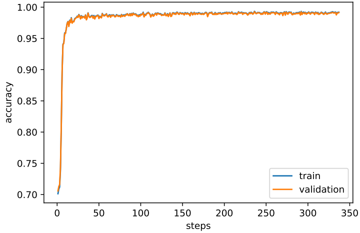
  - easily find boundary, e.g. $\theta(\xi) \in [0.2, 0.8]$
- supervised classification, binary cross-entropy loss

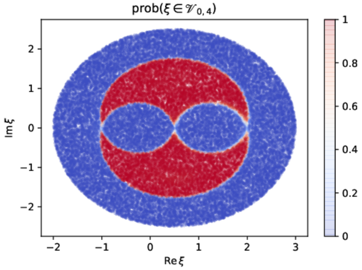
$$\mathcal{L}(\xi) = -\Theta(\xi) \ln \theta(\xi) - \big(1 - \Theta(\xi)\big) \ln \big(1 - \theta(\xi)\big)$$

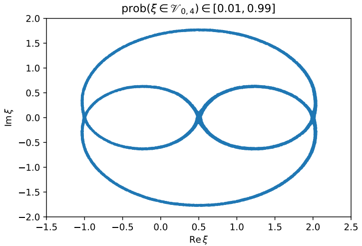
- neural network (Jax)
  - fully connected, 4 layers $(512, 32, 8, 8)$, ELU activation
  - training: 10^5 points, Adam, ℓ_2 regularization, weight decay, early stopping (~ 800 epochs)

Results: vertex region









Accuracy: 99.34 % (train set), 99.27 % (validation set), 99.68 % (test set)

Tachyon potential

Truncated tachyon potential (ignore other fields)

$$V(t) = -t^2 + \frac{v_3}{3!}\, t^3 - \frac{v_4}{4!}\, t^4 + \cdots$$

$$v_n := \mathcal{V}_n(T^n) = (-1)^n \,\frac{2}{\pi^{n-3}} \int_{\mathcal{V}_n} \mathrm{d}^{n-3}\xi \, \prod_{i=1}^{n} \frac{1}{\rho_i^2}$$

- mapping radii

$$\rho_i := \left| \frac{\mathrm{d}f_i}{\mathrm{d}w_i}(0) \right|$$

- $v_3 = -3^9/2^{11} \approx -9.61$
  [hep-th/9409015, Belopolsky-Zwiebach]

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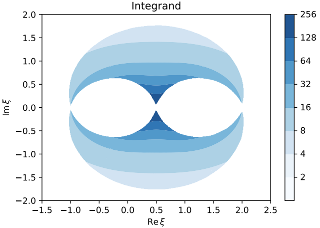

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Results

method	v_4
[hep-th/9412106, Belopolsky]	72.39
[hep-th/0408067, Moeller]	72.390
[hep-th/0506077, Yang-Zwiebach]	72.414
trapezoid (mean)	72.320 ± 0.146
trapezoid (best)	72.396
Monte Carlo (best)	72.366 ± 0.096

- ML statistics: train 10 neural networks, keep the ones (4) extrapolating well
- error in potential coefficient: $\sim 10^{-3}$
  $\rightarrow$ expect sufficiently precise for determining vacuum
- full pipeline: $\sim$ 4 hours

Outline: 5. Conclusion

Results and outlook

Results:

- new method to construct n-point string vertices
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Outlook:

- increase precision (note: difficult and non-standard ML problem!)
- generalize to $n \geq 5$
- compute closed string tachyon vacuum
- compute quadratic differentials for Feynman regions
- generalize to hyperbolic vertices
- generalize higher-genus surfaces (loop corrections) (compute mass renormalization and vacuum shift)