# Crossing symmetry in superstring theory

arxiv: 1810.07197
with: Corinne de Lacroix, Ashoke Sen

Outline:

1. Motivations
2. Crossing symmetry in QFT
3. String field theory (SFT)
4. Analyticity of SFT
5. Conclusion

## 1. Motivations

50 years ago (Sept. 1968): birth of string theory

Veneziano amplitude: model for hadronic interactions

assumptions: crossing symmetry, linear Regge behaviour

= relations between amplitudes with exchange of particles/anti-particles in initial/final states

Crossing symmetry: often assumed or observed

What is the interest of a general proof?

- be sure that observed examples is not an accident of simple amplitudes
- learn about properties of QFT (related to causality and locality)

Goal: derive crossing symmetry of superstring amplitudes at all loops

Restriction: 4-point amplitudes of stable states (massless, BPS, stable non-BPS)

Method (Bros-Epstein-Glaser '64-65): prove analyticity + use BEG

1. prove analyticity in primitive domain $\Delta$
2. analytic extension $\mathcal{H}(\Delta)$
3. show that crossing is implied by analyticity in $\mathcal{H}(\Delta)$

By-product (from analyticity in $\Delta$):

- analyticity in Jost-Lehman-Dyson domain
- " of elastic forward amplitude ($t=0$, $s \in \mathbb{C}$)
- dispersion relations (?)
- bounds on amplitudes (?)
- CPT invariance (?)

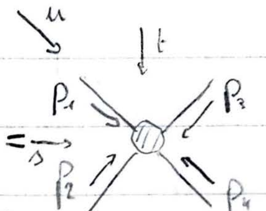
## 2. Crossing symmetry in QFT

Consider scattering 4-point amplitude

$p_a = (E_a, \vec{p}_a)$, $a = 1, \dots, 4$ : external momenta

momentum conservation : $p_1 + \dots + p_4 = 0$ (always imposed)

mass-shell : $p_a^2 = -m_a^2$

Off-shell Green function: $G(p_1, \dots, p_4) =$ 


$p_a \in \mathbb{C}$

$p_a = p_{aR} + i p_{aI}$

calculated as sum of Feynman graphs

Truncated Green function: $\tilde{G}(p_1, \dots, p_4) = G(p_1, \dots, p_4) \prod_{a=1}^{4} (p_a^2 + m_a^2)$

On-shell: $A(p_1, \dots, p_4) = \lim_{p_a^2 \to -m_a^2} \tilde{G}(p_1, \dots, p_4)$ (LSZ)

Mandelstam variables:

$$s = -(p_1 + p_2)^2 \quad , \quad t = -(p_1 + p_3)^2 \quad , \quad u = -(p_1 + p_4)^2$$

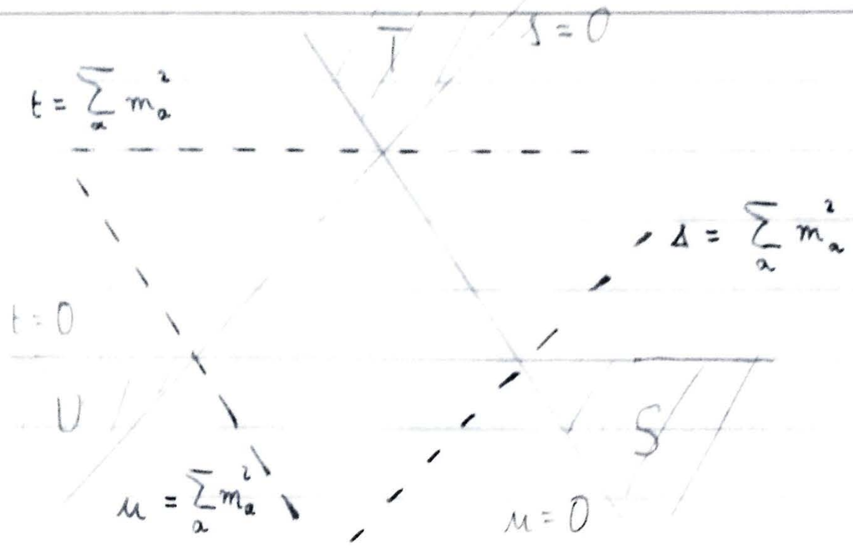
mass-shell $\Rightarrow s + t + u = \sum_a m_a^2$

Physical regions $(p_a \in \mathbb{R})$:

- S (s-channel) : $s \geqslant \sum_a m_a^2$ , $t, u \leqslant 0$
- T (t-channel) : $t \geqslant \sum_a m_a^2$ , $s, u \leqslant 0$
- U (u-channel) : $u \geqslant \sum_a m_a^2$ , $s, t \leqslant 0$

Physical amplitude : $A_S, A_T, A_U$

$$A_S(p_1, \dots, p_4) = \lim_{p_a \in S} A(p_1, \dots, p_4)$$

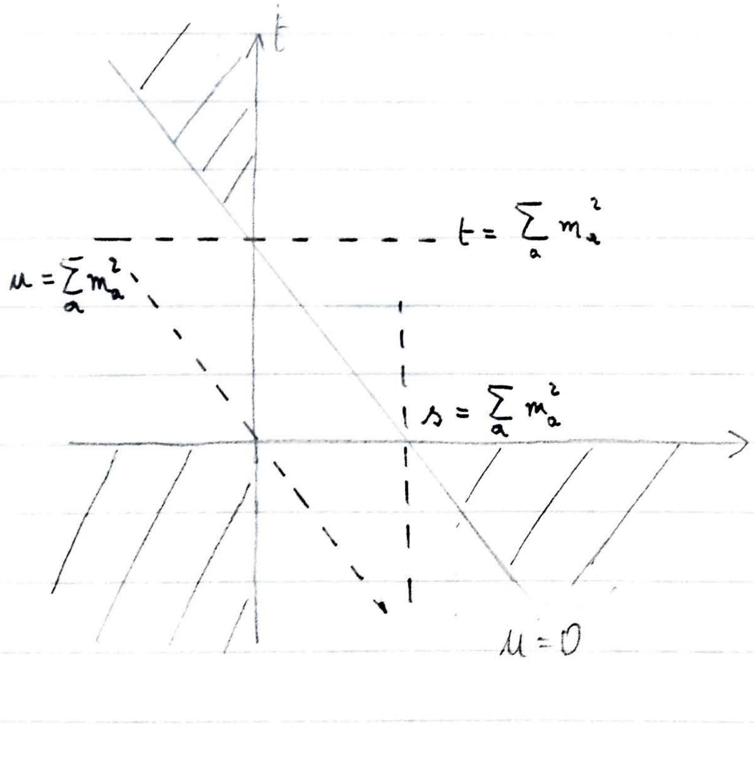



Identical masses:

$$E_i^2 = \vec{p}_i^{\,2} + m^2$$

centre of mass: $\vec{p}_1 = -\vec{p}_2$, $E_1 = E_2$

$$s = -(p_1+p_2)^2 = (E_1+E_2)^2 = 4E_1^2 = 4(\vec{p}_1^{\,2} + m^2) \geq 4m^2$$



## Crossing symmetry

The processes $1+2 \to 3+4$
$1+\bar{3} \to \bar{2}+4$ $\qquad A_S(s,t) = A_T(t,s)$
$1+\bar{4} \to \bar{2}+3$ $\qquad A_S(s,u) = A_U(u,s)$
are equivalent under analytic continuation on the complex mass-shell.

This looks natural from the LSZ prescription : all $A_S, A_T, A_U$ arise from a single function $A$.
However, it is not guaranteed that $A$ is analytic on a domain which admits paths between the physical regions $S, T, U$.

Note: for neutral particles, crossing relates $\neq$ channels and gives some invariance for the amplitude.

Proof in QFT: Epstein - Bros - Glaser '64-65 (see also Bros '86)
1. assumption : only $m_a^2 > 0$, stable particles as asymptotic states
2. define the primitive domains

$$\Delta_k = \bigcap_{A_\alpha} \{ \mathrm{Im}\, P_{(\alpha)} \neq 0, (\mathrm{Im}\, P_{(\alpha)})^2 \leq 0 \} \cup \{ \mathrm{Im}\, P_{(\alpha)} = 0, -P_{(\alpha)}^2 < M_\alpha \}$$
$$\cap \{ \mathrm{Im}\, p_a^i = 0, \; i = k, \dots, D-1 \}$$

where $A_\alpha$ = any subset of $\{p_a\}$, $P_{(\alpha)} = \sum_{a \in A_\alpha} p_a$

This means that, $p_a$ are s.t. all $P_{(\alpha)}$ have either:
- non-zero timelike imaginary part
- have squared below the lowest multi-particle threshold $M_\alpha$ in the channel described by $A_\alpha$.
- only the first $k$ components of the momentum can have an imaginary part

3. Prove analyticity in $\Delta_{0**}$ for the truncated off-shell Green functions
This follows from microcausality (= field can infuence only inside the light-cone).
Ref: Steinmann, Ruelle, Araki, Burgoyne '60-61

The domain does not intersect the mass-shell (even complex):

$$p_a^2 = -m_a^2 \implies p_{aI} \cdot p_{aR} = 0 \;,\; p_{aR}^2 - p_{aI}^2 + m_a^2 = 0$$

If $p_{aI}$ is timelike, $p_{aI}^2 \leq 0$, then one must have $p_{aR}^2 < 0$ to satisfy the second condition, but this contradicts the first.
If $p_{aI} = 0$, then any pair of $p_a$ will have $P_{(\alpha)}^2 > M_\alpha^2$.

Hence $\Delta_0$ is not sufficient to prove crossing symmetry.

4. Compute the "envelope of holomorphy" $\mathcal{H}(\Delta_2)$ (= analytic extension of domains).
mass-shell $\cap \; \mathcal{H}(\Delta_2) \neq 0$

5. Show that there is a path in $\mathcal{H}(\Delta_2)$ interpolating between all pairs of physical regions, on the mass-shell.

Note: - only $\Delta_2$ is necessary ($\operatorname{Im} p_a$ lies in a 2d plane)
- the computations of $\mathcal{H}(\Delta_2)$ and of the crossing paths only use general facts of the theory of several complex variables.

Note: due to energy-momentum conservation, only sets $A_\alpha$ with one or two momenta are useful.

## 3. String field theory

Standard QFT s.t:
- infinite # of fields
- non-local interactions $\propto e^{k^2}$
- reproduce worldsheet amplitudes (if well defined)

Non-locality:
- cannot use position space representation
- " " assumptions from local QFT (microcausality ...)
- " derive analyticity like in QFT

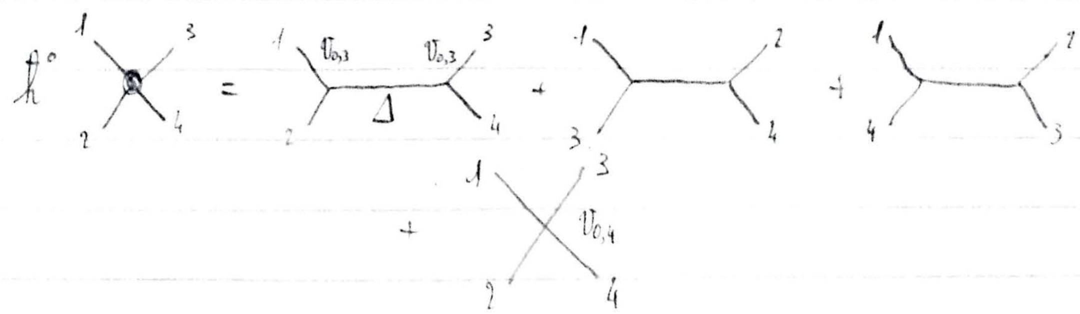
→ study the singularities of the Green functions from the Feynman diagrams in momentum space.

SFT action (bosonic or heterotic NS):

$$S = \frac{1}{2} \langle \Psi | c_0^- Q_B | \Psi \rangle + \sum_{g,n \geq 0} \frac{\hbar^g g_s^{2g-2+n}}{n!} V_{g,n}(\Psi^n)$$

$V_{g,n}$: fundamental $g$-loop $n$-point vertex
- defined by some regions of the moduli space $\mathcal{M}_{g,n}$
  + $n$ holomorphic functions with some constraints (local coord.)
- gluing of lower-order vertices with propagators + $V_{g,n}$ should give a single cover of $\mathcal{M}_{g,n}$

Ex:



Ex: scalar field with $\varphi^4$

$$S = \frac{1}{2} \langle \varphi | Q_B | \varphi \rangle + \frac{\lambda}{4!} V_4(\varphi^4)$$

Position representation:

$$|\varphi\rangle = \int d^D x \, \varphi(x) \, |x\rangle \otimes |\downarrow\rangle \qquad c|\downarrow\rangle = |\uparrow\rangle, \quad \langle \uparrow | \downarrow \rangle = 1$$

Operators:

$$Q_B = cH = c(p^2 + m^2)$$

$$V_4(\varphi^4) = \int d^D x_1 \cdots d^D x_4 \, \underbrace{V_4(x_1, \dots, x_4)}_{= \, \delta^{(D)}(x_1 - x_2)\, \delta^{(D)}(x_1 - x_3)\, \delta^{(D)}(x_1 - x_4)} \varphi(x_1) \cdots \varphi(x_4)$$

---

String field Fourier expansion:

$$|\Psi\rangle = \sum_A \int d^D k \, \varphi_A(k) \, |A, k\rangle$$

$k$: D-dim. (cont.) momentum
A: discrete labels (Lorentz indices, group repr., KK modes, ...)

1PI action + gauge fixing:

$$S = \int d^D k \, \varphi_A(-k) K_{AB}(k) \varphi_B(k) + \sum_n \int d^D k_1 \cdots d^D k_n \, V^{(n)}_{A_1 \cdots A_n}(k_1, \dots, k_n) \times \varphi_{A_1}(k_1) \cdots \varphi_{A_n}(k_n)$$

Propagator:

$$K_{AB}(k)^{-1} \sim \frac{-i}{k^2 + m_A^2} M_{AB} \, \underbrace{Q(k)}_{\text{polynomial}} \qquad \text{(no sum)}$$

Vertices:

$$-i V^{(n)}_{A_1, \dots, A_n}(k_1, \dots, k_n) = -i \underbrace{\int dt}_{\text{moduli}} \, e^{-g_{ij}^{\{A_a\}}(t) k_i \cdot k_j - \overbrace{c \sum_{a=1}^{n} m_a^2}^{\text{damping } c>0}} \, \underbrace{P_{\{A_a\}}(k_1, \dots, k_n)}_{\text{polynomial}}$$

no sing. $\forall k_i \in \mathbb{C}$

- Truncated Green function = sum of Feynman diagrams

General form:

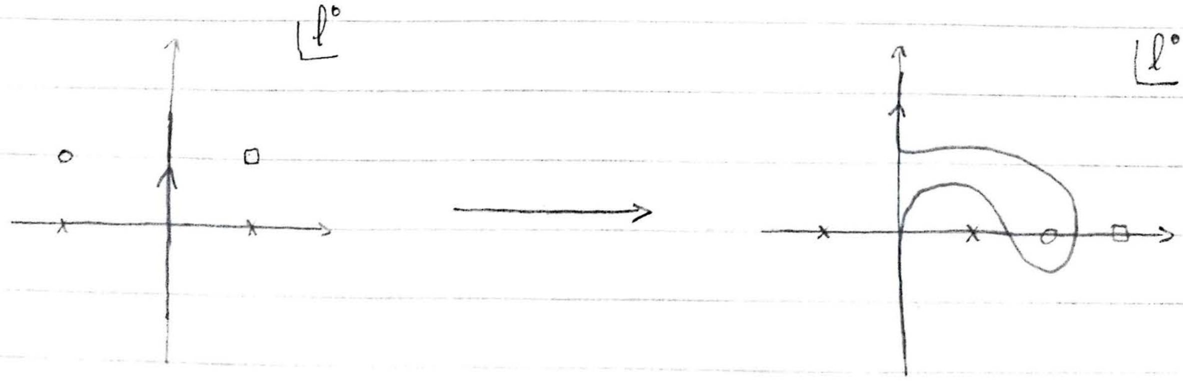
$$\mathcal{F}(p_1, \dots, p_n) = \int dT \prod_s d^D l_s \, e^{-G_{rs}(T)\, l_r \cdot l_s - 2 H_{ra}(T)\, l_r \cdot p_a - H_{ab}(T)\, p_a \cdot p_b}$$
$$\times \prod_i \frac{1}{k_i^2 + m_i^2} \, \mathcal{P}(l_r, p_a; T)$$

Momenta: external $\{p_a\}$, internal $\{k_i\}$, loop $\{l_r\}$
↳ linear comb. of $(p_a, l_r)$

- The matrix $G_{rs}$ is positive definite:
  - integrations over spatial momenta $\vec{l}_r$ converge
  - " " energies $l_r^0$ diverge

Prescription to define the amplitudes (Pius - Sen, 1604.01783)
1. define for Euclidean ext. and int. momenta
2. analytic continuation of external momenta to Lorentzian energy + of integration contour s.t.:
   a) keep the poles on the same side
   b) keep ends at $\pm i\infty$

Implies: Cutkosky rules, unitarity, moduli space $i\varepsilon$-prescription (review: 1703.06410)



## 4. Primitive domain for SFT

First: regularize massless fields, 2 solutions:

1. introduce projector $P$ onto massless states, split each propagator as $P + (1-P)$ and consider the part where all prop. have $(1-P)$
   note: not gauge inv., but $[Q_B, P] = 0 \Rightarrow \exists$ BV master eq. for the subtracted Green functions (encode gauge inv.)
2. add mass term (break BRST inv. and decoupling of unphys. states)

This places us in the same setting as the original QFT proof.
(massless fields $\Rightarrow$ IR problems ~~ex~~, non-analytic amplitude)

Method to study analyticity:

1. start with $p_a = 0$ (with $l_n^0 \in i\mathbb{R}$, $\vec{l}_n \in \mathbb{R}$; no singularity)
2. find a path between $p_a = 0$ and the desired value $p_a \in \mathbb{C}$
3. deform the integral contour as the poles move
4. if two poles from opposite sides of the contour collide, then it cannot be deformed $\longrightarrow$ (pinched) singularity of the Green function

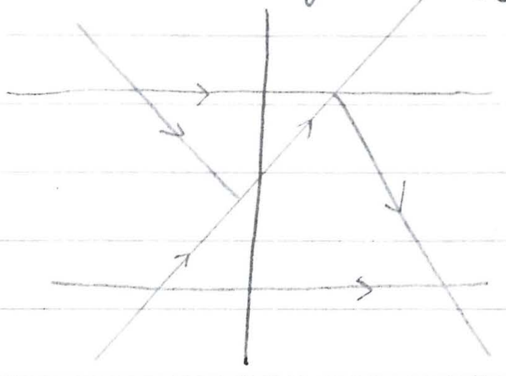
Pius-Sen: analyticity for $\operatorname{Im} p_a^0 > 0$, $\operatorname{Re} p_a^0 \geqslant 0$
($i\epsilon$-prescription $\Rightarrow \operatorname{Im} p_a^0 \geqslant 0$)

Here: investigate the primitive domains $\Delta_k$. Two steps:

1. analyticity in $\Delta_1$: deform from the origin $p_a = 0$ to the desired values of $\operatorname{Re} p_a$ and $\operatorname{Im} p_a^0$ (keep $\operatorname{Im} \vec{p}_a = 0$) s.t. $p_a \in \Delta_1$
2. analyticity in $\Delta_2$: deform from $p_a \in \Delta_1$ to the desired value of $\operatorname{Im} p_a^1$ (keep $\operatorname{Im} p_a^i = 0 \ \forall i \geqslant 2$) s.t. $p_a \in \Delta_2$

Step 1: consider a straight line from $p_a = 0$ to $p_a \in \Delta_1$.

- Assume there is a pinch singularity
  = some of the internal propagators are on-shell
- Associated to reduced diagram (all off-shell prop. contracted to points)
- Assign an arrow according to the sign of energy $k^0 = \sqrt{\vec{k}^2 + m^2}$
- Can show: cannot have closed loops
  $\longrightarrow$ partial ordering, energy flows from left to right (arrow)



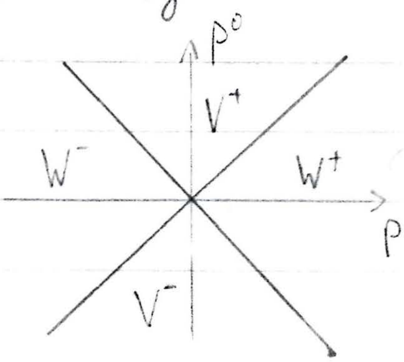
- But: intersects on-shell particles $\mathcal{B}$ with total momentum given by one $P_{(\alpha)}$

$$P^0_{(\alpha)} = \sum_{i \in \mathcal{B}} \sqrt{\vec{k}_i^2 + m_i^2} \qquad \vec{P}_{(\alpha)} = \sum_{i \in \mathcal{B}} \vec{k}_i$$

- $\vec{k}_i \in \mathbb{R} \implies P_{(\alpha)} = (P^0_{(\alpha)}, \vec{P}_{(\alpha)}) \in \mathbb{R}$
  since the ext. states are on-shell, one has $-P^2_{(\alpha)} \geqslant M^2_\alpha$
- This contradicts the condition $-P^2_{(\alpha)} < M^2_\alpha$
  $\implies$ no singularity, the Green functions are analytic in $p_a \in \Delta_1$

Step 2: deform also $\operatorname{Im} p^1_a$ to reach $p_a \in \Delta_2$.

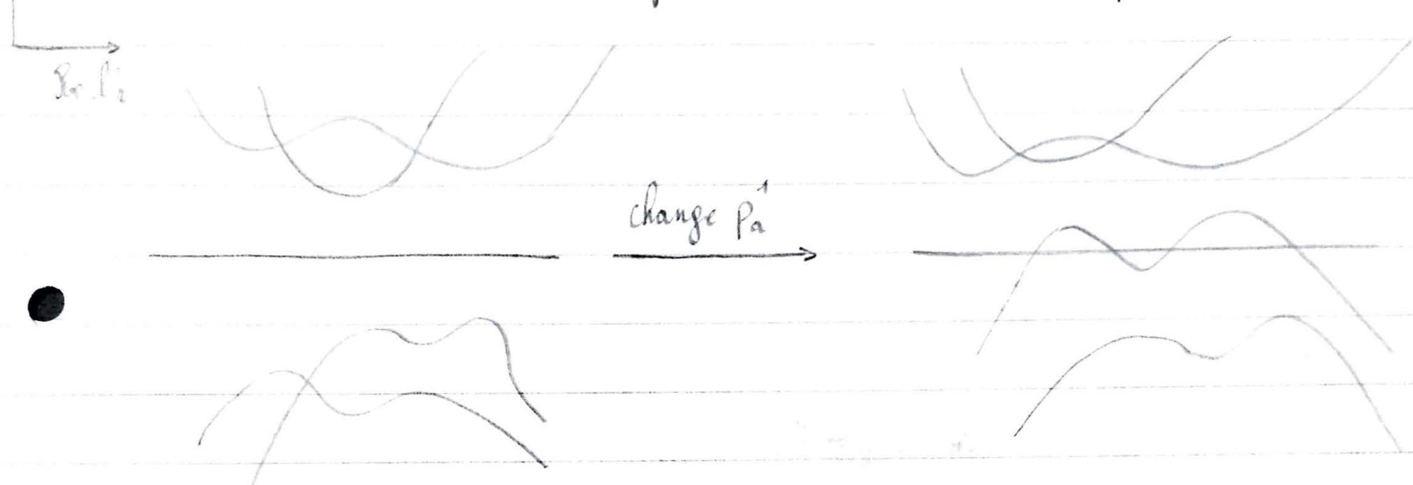
- Write $\quad p_\parallel = (p^0, p^1)\,, \quad p_\perp = (p^2, \dots, p^{D-1})$
  $$p = p_R + i p_I = (p_{\parallel R}, + p_{\perp R}) + i\,(p_{\parallel I}, 0)$$
- Denote by $V^\pm$ the past/future interiors of the 2d light-cone
  $W^\pm$ the complements




- $p_a \in \Delta_2$ is equivalent to $\operatorname{Im} p_{a\perp} = 0$ and:
  $P_{(\alpha)\parallel I} \in V^\pm \cup (\partial V^\pm - 0)$
  or $P_{(\alpha)\parallel I} = 0\,, \; -P^2_{(\alpha)} < M^2_\alpha$

- Deform Im $p_a^1$ along a straight line towards the desired value.
  note : this can be done within $\Delta_2$.

- Deform the contour, but consider variation of Im $l_r^1$.
  (= if set Im $l_r^1 = 0$, recover previous contour)




For Im $p_a^1 = 0$, the singular loci (poles of propagators) are always far from the axis Re $l_r^1$.
After changing Im $p_a^1$, the singular loci are translated vertically (in opposite directions for above/below).
If a curve crosses the real axis, need to deform the contour of $l_r^1$ to Im $l_r^1 \neq 0$. If the gap closes $\longrightarrow$ singularity.

Diagram: write $k_i = l_r + L_i(\underbrace{l_s, p_a}_{\text{fixed}})$ prop. r in loop s

$$\text{poles}: \quad \tilde{k}_i^1 = \pm i \sqrt{(k_{i\perp})^2 - (k_i^0)^2 + m_i^2}$$
$$= \pm i \sqrt{(l_{s\perp} + L_{i\perp})^2 - (l_s^0 + L_i^0)^2 + m_i^2}$$
$$l_s^0 \sim \pm i\infty \Rightarrow \tilde{k}_i^1 \sim \pm i\infty$$

relation above gives pole locations in $l_r^1$-plane.
step 1 $\Rightarrow$ Im $\tilde{l}_r^1 \neq 0$ for poles
$\longrightarrow$ draw only poles for which have Re $\tilde{l}_r^1$ = Re $l_r^1$
(given a point $l_r^1$ of the contour, display only poles for this Re $l_r^1$)

- Assume there is a singularity.
- Write reduced diagram.
- Assign a (double) arrow for the sign of $\mathrm{Im}\, k_n^1$.
- Can show: cannot have closed loops
  $\longrightarrow$ partial ordering
- Consider an internal (on-shell) propagator:

$$k^2 + m^2 = 0$$

$$\Longrightarrow k_{\parallel R} \cdot k_{\parallel I} = 0 \quad , \quad k_{\parallel R}^2 + k_{\perp R}^2 - k_{\parallel I}^2 + m^2 = 0$$

Three possibilities:

a) $k_{\parallel R} \in V^{\pm} \cup (\partial V^{\pm} - 0) \Longrightarrow k_{\parallel I} \in W^{\pm} \cup \partial W^{\pm}$

b) $k_{\parallel R} \in W^{\pm} \cup (\partial W^{\pm} - 0) \Longrightarrow k_{\parallel I} \in V^{\pm} \cup \partial V^{\pm}$

c) $k_{\parallel R} = 0$ and $k_{\parallel I} \in W^{\pm}$, or $k_{\parallel I} = 0$ and $k_{\parallel R} \in V^{\pm}$

Cases a) and b) imply that $k_{\parallel R}^2$ and $k_{\parallel I}^2$ have opposite signs, and the second equation has a solution if $k_{\parallel R}^2 \leq 0$, $k_{\parallel I}^2 \geq 0$ since $k_{\perp R}^2 \geq 0$.
If $k_{\parallel R}^2 = 0$, then also $k_{\parallel I}^2 = 0$ (for them to be both orthogonal, they must be null)
This is not compatible with the second eq.

$$\Longrightarrow k_{\parallel R} \in V^{\pm} \cup 0 \, , \quad k_{\parallel I} \in W^{\pm} \cup 0 \qquad \text{but } k_{\parallel} \neq (0,0)$$

- Vertical cut: on-shell particles $\mathcal{C}$, flow of $\mathrm{Im}\, k_i^1$ from left to right

$$k_{iI}^1 > 0 \Longrightarrow k_{i\parallel I} \in W^+ \Longrightarrow \sum_{i \in \mathcal{C}} k_{i\parallel I} \in W^+$$

By momentum conservation:

$$\sum_{i \in \mathcal{C}} k_i = P_{(\alpha)} \quad \text{for some } P_{(\alpha)}$$

$\Longrightarrow P_{(\alpha) I} \in W^+$, which contradicts initial assumption
$\rightarrow$ no singularity

## 5. Conclusion

Results:

- proof of crossing symmetry at the same level for string theory and QFT
- show that, in some sense, string theory behaves like a well-behaved local QFT
- adds to the list of consistency properties of string theory
- analyticity in $\Delta_2$ = starting point for other properties
- alternative proof valid for more general QFTs

Todo:

- extend to general $\Delta = \Delta_D$
- envelope of holomorphy: proof based on Feynman diagrams?
- most consistency properties of string theory now established via SFT
  - $\longrightarrow$ move to more advanced computations = find explicit action
    - cubic closed SFT
    - ML construction