Timelike Liouville gravity

Arxiv:
With: Teresa Bautista, Atish Dabholkar

Definition of Euclidean quantum gravity?
$d=4$: conformal factor negative kinetic term

Toy model: $d=2$ timelike Liouville theory
- CFT with $c_L \leq 1$
- $c_L \geq 25$ (spacelike) well understood with ≠ techniques
  (minisuperspace, canonical quantization, path integral, bootstrap)
- conformal bootstrap solved (numerically) for spacelike $c_L \in \mathbb{C}$
  [Ribault-Santachiara, 1503.02067]
- unitary matter in semi-classical limit

But analytic continuation spacelike → timelike non-trivial:
- physical spectrum not known
- identification of the 3-point structure constants problematic
- divergence of the 4-point function $\propto \int_{-\infty}^{\infty} dE \, e^{E^2}$
- fake identity operator (state with $\Delta = 0$ but not the identity)
  [Ikhlef-Jacobsen-Saleur, 1509.03538]
- problems with minisuperspace [Harlow-Maltz-Witten 1108.4417]

Proposal for a definition of timelike Liouville theory:
1. physical states from BRST cohomology + no-ghost theorem
2. generalized Wick rotation for 4-point function
   (same as string field theory [Pius-Sen, 1604.01783])

1. Quantum gravity in 2d

2d gravity with cosmological cst coupled to supercritical matter ($D \geq 25$ scalar fields Y^I)

$$Z = \int \frac{d_g g_{\mu\nu}}{V_{\text{diff}}} \, d_g Y^I \, e^{-S_m[g, Y^I] - S_g[g]}$$

$$S_m[g, Y^I] = \frac{1}{4\pi} \int d^2x \sqrt{g} \, g^{\mu\nu} \partial_\mu Y^I \partial_\nu Y^I$$

$$S_g[g] = \mu_0 \int d^2x \sqrt{g}$$

We will consider mostly the sphere,

Conformal gauge $g_{\mu\nu} = e^{2\chi/q} \bar{g}_{\mu\nu}$ + DDK procedure:

$$Z = \int d_{\bar{g}} Y^I \, e^{-S_m[\bar{g}, Y^I]} \int d_{\bar{g}}(b,c) \, e^{-S_{gh}[\bar{g}, b, c]} \int d_{\bar{g}} \chi \, e^{-S_L[\bar{g}, \chi]}$$

$$S_L[\bar{g}, \chi] = \frac{1}{4\pi} \int d^2x \sqrt{\bar{g}} \left[- \bar{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - q \bar{R} \chi + 4\pi \mu e^{2\beta\chi} \right]$$

$$q = \frac{1}{\beta} - \beta \qquad c_L = 1 - 6q^2 = 26 - D \leq 1$$

Path integral factorizes (for unintegrated operators)
→ focus on Liouville correlation functions for vertex operators

$$V_\alpha(z) = e^{2\alpha\chi(z)} \qquad \alpha = -\frac{q}{2} + iE \in \mathbb{C}$$

Conformal dimensions: $\Delta_\alpha = \alpha(q + \alpha) = -\frac{q^2}{4} - E^2$

Spectrum = range of α (or E)?

2. Spacelike Liouville theory - bootstrap

Follow [Ribault - Santachiara].

$$S_L = \frac{1}{4\pi}\int d^2x \sqrt{g}\left[+ g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + Q\bar{R}\varphi + 4\pi\mu e^{2b\varphi}\right]$$

$$Q = \frac{1}{b} + b \qquad c_L = 1 + 6Q^2 \in \mathbb{C}$$

Vertex operators: $V_a = e^{2a\varphi}$, $a = \frac{Q}{2} + ip$

$$\Delta_a = a(Q-a) = \frac{Q^2}{4} + p^2$$

Bootstrap:

1. OPE spectrum (= internal states in channel factorizations)
   ↳ convergence of 4-point function
   $\Rightarrow p \in \mathbb{R}$, $\Delta_a \geq \frac{Q^2}{4}$ (real part)

2. 3-point structure constants
   ↳ Teschner's recursion relations
   There are 2 solutions:
   - $c_L \notin]-\infty, 1]$: $C(\alpha_1, \alpha_2, \alpha_3)$ given by DOZZ formula [Dorn-Otto-Zamolodchikov²]
   - $c_L \leq 1$: $\hat{C}(\alpha_1, \alpha_2, \alpha_3) \sim C^{-1}$ [Zamolodchikov]

[Ribault - Santachiara]: check crossing symmetry of 4-point function numerically for generic $c_L \in \mathbb{C}$ and $p_i \in \mathbb{C}$ (ext.).

Note: - for Liouville, crossing symmetry
$\Rightarrow$ modular covariance of torus 1-point function
- for $c_L \leq 1$, need $i\varepsilon$-prescription: $p_s \in \mathbb{R} + i\varepsilon$ (CFT def V_g

We distinguish then 2 spectra:

- internal = sum in OPE / internal channels
- external = any states for which correlations functions are well-defined
  (subset of)
  ↳ selected from some other principle

We can have:

- int. $\subseteq$ ext.
- int. $\cap$ ext. $= \emptyset$

In general, one defines timelike $c_L \leq 1$ from analytic continuation of spacelike $c_L \geq 25$.

$\varphi = i\chi$, $Q = iq$, $b = -i\beta$, $a = -i\alpha$, $p = -iE$

The bootstrap tells that it cannot be correct.

→ define timelike $c_L \leq 1$ from ~~analytic~~ cont. of spacelike $c_L \leq 1$.

Note: This is the same as in string theory:
timelike X^0 is defined from a spacelike X^0_E.
(both have $c = 1$).

This amounts to continue the field but not $b \in i\mathbb{R}$.

Note: in QFT, internal states are not observables.
S-matrix = function of ext. states.

3. Spectrum, BRST cohomology, no-ghost theorem.

What is the spectrum?
- minisuperspace quantization
  $\longrightarrow$ $E \in \mathbb{R}$ + ~~some~~ discrete states ~~in~~ $E \in i\mathbb{R}$
- But interpretation difficult $\longrightarrow$ BRST quantization

Fields: - longitudinal: (χ, X, b, c)
- transversal: Y^i
- $Y^I = (X, Y^i)$
- ghost vacuum: $|\downarrow\rangle = c_1|0\rangle$, $N_{gh} = 1$
- describe Liouville field by Coulomb gas, $\mu = 0$
  ↳ see later $\mu \neq 0$

Hamiltonian (focus on holomorphic sector):

$$L_0 = \left(p^2 + \vec{k}^2 - E^2 - \frac{q^2}{4} - 1\right) + \underbrace{\hat{L}_0^{\parallel} + \hat{L}_0^{\perp}}_{\text{level operators}} \qquad \hat{L}_0^x = \sum_{n>0} n N_n^x$$

Focus on relative cohomology: $b_0 = 0$, $L_0 = 0$

$$\Longrightarrow \quad Q_B = \cancel{c_0 L_0} - \cancel{b_0 M} + \hat{Q}_B$$

Light-cone parametrization: $X^{\pm} = \chi \pm X$

$$\Longrightarrow \quad \hat{Q}_B = Q_0 + Q_1 + Q_2, \qquad Q_0^2 = 0$$

Fact: $\mathcal{H}(\hat{Q}_B) = \mathcal{H}(Q_0)$ if $N_{gh} = 1$ $\forall$ states (no ghosts).
Also want to remove ghosts for unitarity.

Result: $\mathcal{H}(Q_0) = \{ L_0^{\parallel} = 0 \} \cup \{ \text{discrete states} \}$
↳ $N_{gh} = 0, 1, 2$
↳ no χ, X, b, c excitations

How to remove the ghost states?

1. Hermiticity of the matter $L_{n\geq 0}$ for Y^I
   $\Rightarrow p, \vec{k} \in \mathbb{R}$
2. Hermiticity of Liouville L_0
   $\Rightarrow E \in \mathbb{R} \cup i\mathbb{R}$
3. Consider states ~~invariant under~~ with linear comb. of $E \to \pm E$:
   (reflection due to cosmological constant)

   $$|\mathcal{E}\rangle = |E\rangle + R(E)\,|-E\rangle \qquad R: \text{reflection coeff.}$$

   not eigenstates of $U(1)$ current excep if $E = 0$

Notes:

- 2) and 3) are compatible
- Hermiticity of $L_{n\geq 0}$ for all fields $\Rightarrow Q_B^\dagger = Q_B$
- " of Coulomb gas $L_{n\geq 0} \Rightarrow E \in \mathbb{R}$ ($\perp$ bootstrap)
- " of Liouville $L_{n\geq 0}$ not clear (due to μ)

<u>Conclusion</u>:

- 1) + 3) remove all ghost states
- on-shell condition $L_0 = 0$ gives states

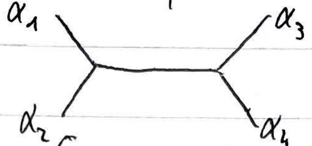
  $$E \in \mathbb{R} \cup \left[-\frac{i}{2}\sqrt{q^2+4}\,,\ \frac{i}{2}\sqrt{q^2+4} \right]$$

  (ground states of Y^i)
- need more work for $Q_B^\dagger = Q_B$.

Note: more general case (X: Coulomb gas, transverse: any CFT) is in preparation.

4. Correlation functions

4-point function (spacelike $c_L \ll 1$):

$C_4(\alpha_i, z_i) =$ 

$$= \int_{\text{OPE}} d\alpha_s \; \hat{C}(\alpha_1, \alpha_2, \alpha_s)\, \hat{C}(-q-\alpha_s, \alpha_3, \alpha_4) \left| \mathcal{F}^{(s)}_{\alpha_s}(\alpha_i, z_i) \right|^2$$

s-channel conf. block ←

Behavior of integrand:

$$\hat{C}\hat{C}\,|\mathcal{F}^{(s)}|^2 \underset{|p_s|\to\infty}{\sim} |q|^{2p_s^2} \propto |q|^{L_0+\bar{L}_0} \qquad |q|<1$$

$$\xrightarrow[p_s \to \pm\infty]{} 0$$

↳ same as SFT plumbing fixture

$$\xrightarrow[p_s \to \pm i\infty]{} \infty$$

$\Rightarrow$ OPE spectrum $= p \in \mathbb{R}$
(argument valid $\forall c_L \in \mathbb{C}$ since integrand has same asympt.)

Cannot use standard Wick rotation:
(– 2 types of states in BRST cohomology)
– don't converge if $E_s = ip_s \in \mathbb{R}$

The same divergence (due to $|q|^{L_0+\bar{L}_0}$) appears in SFT loop amplitudes in Siegel gauge.

Prescription:

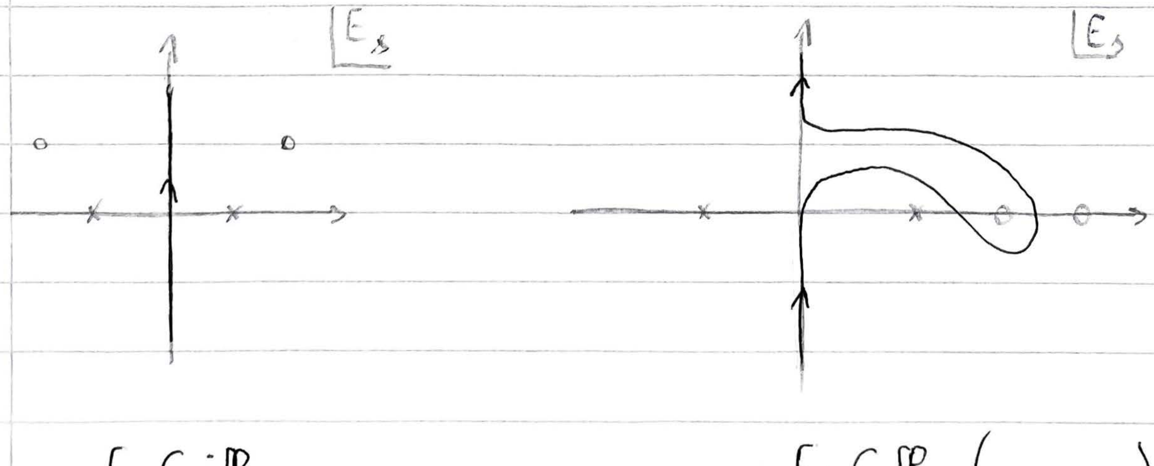
1. Define the correlation functions with external states $E_i \in i\mathbb{R}$ and integration contour $E_s \in i\mathbb{R}$.
2. Perform an analytic continuation of the external states ~~such th~~ to $E_i \in \mathbb{R}$ s.t.:
   - deform the contour to avoid poles crossing it as they move
   - keep contour ends at $\pm i\infty$.

Generalizes standard QFT results (including to SFT):
Cutkosky rules, unitarity, $i\varepsilon$-prescription, analyticity, crossing symmetry
[Sen; + Pius; + De Lacroix - Erbin]

Liouville $c_L \ll 1$: magic happens because integrand poles in E_s-plane are independent of external momenta E_i.
→ same contour as [Ribault - Santachiara].

not true for $c_L \notin (-\infty, 1]$

With this definition, the 4-pt function is crossing symmetric.
The prescription should generalize to higher n-point functions (at least on the sphere).



$E_i \in i\mathbb{R}$ $\qquad$ $E_i \in \mathbb{R}$ (one case)

5. Conclusion

First proposal for a complete and consistent definition of the timelike Liouville theory.

Future directions:

- more consistency checks
- integrated correlation functions, higher-genus
- boundaries and susy esctensions
- non-critical string (time-dependent background).