Topics in string amplitudes

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In collaboration with:

- Corinne de Lacroix
- Juan Maldacena
- Ashoke Sen
- Dimitri Skliros

arXiv: 1810.07197, 1906.06051

Outline: 1. Introduction

Properties of string theory

String theory = theory of extended objects

- consistency? (unitarity, crossing symmetry...)
- differences with local point-particle QFT?
- non-locality?

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1. if possible, direct proof
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Natural framework: string field theory (off-shell, renormalization...)

Plan

Properties of (super)string amplitudes:

1. Tree-level 2-point amplitude
   with: Juan Maldacena, Dimitri Skliros [1906.06051]
2. Analyticity and crossing symmetry at all loops
   with: Corinne de Lacroix, Ashoke Sen [1810.07197]

2-point amplitude

- QFT

$$A_2(k, k') = 2k^0(2\pi)^{D-1}\,\delta^{(D-1)}(\boldsymbol{k} - \boldsymbol{k}')$$

(1-particle state normalization, cluster decomposition)

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BRST point of view: need $N_{\text{gh}} = 6$ but only 2 operators $c\bar{c}V$

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$$\langle V_k(\infty,\infty)V_{k'}(0,0)\rangle_{S^2} \propto \delta(0)\,\delta^{(D-1)}(\boldsymbol{k}-\boldsymbol{k}') = \infty$$

from on-shell + momentum conservation
$\to$ ambiguous, need regularization / better gauge fixing

Analyticity and crossing symmetry

Analyticity of n-point amplitude $A_n(k_1, \ldots, k_n)$

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Why a general proof?

- ensure observed examples not accident of simple amplitudes
- learn about fundamental properties of QFT

Method

Proof idea in QFT [Bros-Epstein-Glaser, '64-65]:

1. prove analyticity of S-matrix in "primitive domain" Δ
2. analytic extension $\mathcal{H}(\Delta)$
3. show that 2) $\Rightarrow$ crossing symmetry

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- 2) and 3) are general statements from theory of several complex variables

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String theory:

- interactions non-locality $\to$ no position space Green functions
- prove 1) perturbatively from Feynman diagrams

Outline: 2. Two-point amplitude

Gauge-fixed amplitude

- 2-point amplitude

$$A_{0,2}(k,k') = \frac{8\pi\alpha'^{-1}}{\mathrm{Vol}\,\mathcal{K}_0}\int \mathrm{d}^2 z \mathrm{d}^2 z' \,\left\langle V_k(z,\bar{z}) V_{k'}(z',\bar{z}')\right\rangle_{S^2}$$

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- evaluate CFT correlation function + regularize zero-modes

$$A_2(k, k') = \lim_{\kappa^0 \to 0} (2\pi)^{D-1} \delta^{(D-1)}(\boldsymbol{k} + \boldsymbol{k}') \, \frac{16\pi^2 \mathrm{i} \, \delta(\kappa^0)}{\alpha' \operatorname{Vol} \mathcal{K}_2}$$

Normalization: $\langle V_k(z, \bar{z}) V_{k'}(z', \bar{z}') \rangle_{S^2} = \dfrac{\mathrm{i} \, (2\pi)^D \delta^{(D)}(k + k')}{|z - z'|^4}.$

numerator = zero-modes $\mathrm{e}^{\mathrm{i}(k+k') \cdot x}$ for Lorentzian target spacetime

Compute CKV volume (1)

- Volume regularization

$$\begin{aligned}\operatorname{Vol}\mathcal{K}_2 &= \int \frac{\mathrm{d}^2 z}{|z|^2} = 2\int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \frac{\mathrm{d}r}{r} \\ &= 4\pi \int_{-\infty}^{\infty} \mathrm{d}\tau = 4\pi \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\varepsilon\tau} \\ \operatorname{Vol}_\varepsilon \mathcal{K}_2 &= 8\pi^2\, \delta(\varepsilon)\end{aligned}$$

- (τ, ε) Euclidean worldsheet (time, energy) on the cylinder (dimensionless)
- problem: Lorentzian spacetime, dimensionful energy
  $\rightarrow$ need Wick rotation and rescaling

Compute CKV volume (2)

Jacobian from mode expansions without oscillators:

1. worldsheet Wick rotation
$$\tau = \mathrm{i}t, \qquad \varepsilon = -\mathrm{i}E$$
2. Lorentzian regularized volume
$$\mathrm{Vol}_{M,E}\,\mathcal{K}_2 = 8\pi^2\mathrm{i}\,\delta(E)$$
3. Lorentzian mode expansion
$$X^0 = x^0 + \alpha' k^0 t$$
4. scale between spacetime and worldsheet times / energies
$$t = \frac{\xi^0}{\alpha' k^0} \quad \Longrightarrow \quad E = \alpha' k^0 \kappa^0$$
(ξ^0, κ^0) dimensionful worldsheet variables
5. regularized Lorentzian volume
$$\mathrm{Vol}_{M,\kappa^0}\,\mathcal{K}_2 = \frac{8\pi^2\mathrm{i}\,\delta(\kappa^0)}{\alpha' k^0}$$

Result

$$A_2(k,k') = \lim_{\kappa^0\to 0} (2\pi)^{D-1}\delta^{(D-1)}(\boldsymbol{k}+\boldsymbol{k}')\,\frac{16\pi^2 \mathrm{i}\,\delta(\kappa^0)}{\alpha'\,\mathrm{Vol}_{M,\kappa^0}\,\mathcal{K}_2}$$

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Recover QFT result:

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Remarks:

- regularization ambiguous → fixed from unitarity
- Jacobian can be computed from path integral (field shift)
- better approach: gauge fix X^0 [1906.06051]
- operator approach [1909.03672, Seki-Takahashi]
- can always insert 6 ghosts
  example: using $\langle 0|\, c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1\,|0\rangle = 1$

$$A_2(k,k') = \frac{C_{S^2}}{\mathrm{Vol}\,\mathcal{K}_2}\;\langle c\bar{c}V_k(\infty,\infty)c_0\bar{c}_0\;c\bar{c}V_{k'}(0,0)\rangle_{S^2}$$

Zero-point amplitude

Next step

Generalization to 0-point function $\to$ compute on-shell action

- Zero-point amplitude for Minkowski spacetime $\mathcal{M}$:

$$A_0[\mathcal{M}] \sim \frac{\delta^{(D)}(0)}{\mathrm{Vol}\,\mathrm{SL}(2,\mathbb{C})} \stackrel{?}{=} \infty$$

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- curved background X:

$$\mathrm{e}^{-(S_{\mathrm{EH}}[X]-S_{\mathrm{EH}}[\mathcal{M}])} = \frac{A_0[X]}{A_0[\mathcal{M}]} \stackrel{?}{=} \text{finite}$$

(*à la* Gibbons–Hawking–York)

- consider $X =$ black hole?

Outline: 3. Crossing symmetry: QFT

Amplitude and Green functions

4-point scattering process

- $p_a = (E_a, \boldsymbol{p}_a) \in \mathbb{C}$, $a = 1, \ldots, 4$: external momenta
- momentum conservation: $p_1 + \cdots + p_4 = 0$
- on-shell condition: $p_a^2 = -m_a^2$

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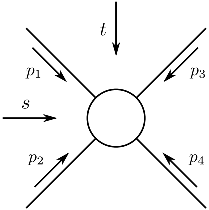
Green functions:

off-shell $\quad G(p_1, \ldots, p_4) =$



truncated $\quad \tilde{G}(p_1, \ldots, p_4) = G(p_1, \ldots, p_4) \prod_{a=1}^{4} (p_a^2 + m_a^2)$

on-shell $\quad A(p_1, \ldots, p_4) = \lim_{p_a^2 \to -m_a^2} \tilde{G}(p_1, \ldots, p_4)$

QFT: G = sum of Feynman diagrams

Physical amplitudes

Mandelstam variables

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 + p_3)^2, \qquad u = -(p_1 + p_4)^2$$

mass-shell: $s + t + u = \sum_a m_a^2$

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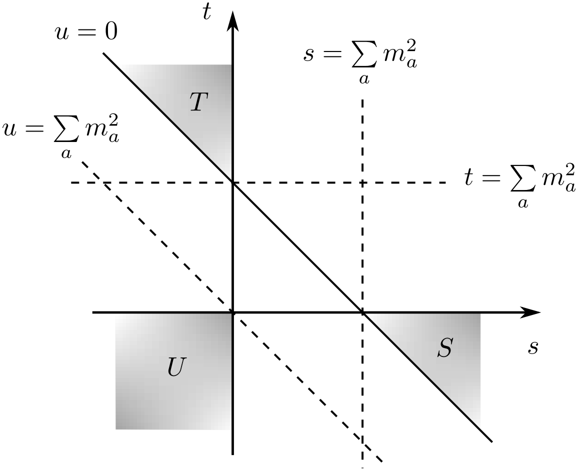
Physical regions

- S (s-channel): $s \geq \sum_a m_a^2, \quad t, u \leq 0$
- T (t-channel): $t \geq \sum_a m_a^2, \quad s, u \leq 0$
- U (u-channel): $u \geq \sum_a m_a^2, \quad s, t \leq 0$

Physical amplitudes

$$A_{S,T,U}(p_1, \ldots, p_4) = \lim_{p_a \in S,T,U} A(p_1, \ldots, p_4)$$

Mandelstam plane



$p_a \in \mathbb{R}$ on-shell

Statement of crossing symmetry

Crossing symmetry

The processes
$$\begin{array}{l} S : 1 + 2 \to 3 + 4 \\ T : 1 + \bar{3} \to \bar{2} + 4 \\ U : 1 + \bar{4} \to 3 + \bar{2} \end{array}$$
(and CPT conjugates) are equivalent under analytic continuation on the complex mass-shell

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$$A_S(s,t) = A_T(t,s), \qquad A_S(s,u) = A_U(u,s)$$

- looks natural from LSZ: $A_{S,T,U}$ all come from a single function A
- but: not guaranteed that A is analytic in a domain with paths between S, T, U

QFT proof (1)

Outline of proof [Bros-Epstein-Glaser '64-65][Bros '86]:

1. assumptions: $m_a^2 > 0$, asymptotic states = stable particles
2. define the "primitive domains"

$$\Delta_k = \bigcap_{A_\alpha} \Big[\{ \operatorname{Im} P_{(\alpha)} \neq 0, (\operatorname{Im} P_{(\alpha)})^2 \leq 0 \} \\ \cup \{ \operatorname{Im} P_{(\alpha)} = 0, -P_{(\alpha)}^2 < M_\alpha^2 \} \\ \cap \{ \operatorname{Im} p_a^i = 0, i = k, \ldots, D-1 \} \Big]$$

$$A_\alpha \subset \{1, \ldots, n\}, \quad P_{(\alpha)} = \sum_{a \in A_\alpha} p_a, \quad M_\alpha\text{: production threshold}$$

In words: p_a with k possible complex components s.t. all P_α have: 1) non-zero imaginary timelike part, or 2) real momentum squared below multi-particle threshold in channel A_α

QFT proof (2)

3. prove analyticity inside Δ_D of S-matrix from micro-causality (fields commute at spacelike separations) [Araki, Burgoyne, Ruelle, Steimann, '60-61]
   problem: $\Delta_D \cap \text{mass-shell} = \emptyset$
4. compute the "envelope of holomorphy" $\mathcal{H}(\Delta_2)$ (= analytic extension)
   $\rightarrow \mathcal{H}(\Delta_2) \cap \text{mass-shell} \neq \emptyset$
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Notes:

- only $\mathcal{H}(\Delta_2)$ is necessary
- 4) and 5) $\Leftarrow$ theory of several complex variables only
- work with the complete S-matrix

Primitive domain and mass-shell

Proof that $\Delta_D \cap \text{mass-shell} = \emptyset$:

1. complex mass-shell:

$$\mathrm{Re}\, p_a \cdot \mathrm{Im}\, p_a = 0, \qquad (\mathrm{Re}\, p_a)^2 - (\mathrm{Im}\, p_a)^2 + m_a^2 = 0$$

2. if $\mathrm{Im}\, p_a$ timelike, $(\mathrm{Im}\, p_a)^2 \leq 0$, then need $\mathrm{Re}\, p_a$ timelike, $(\mathrm{Re}\, p_a)^2 < 0$, for 2nd condition, but violates 1st condition

3. if $\mathrm{Im}\, p_a = 0$, then $-P_{(\alpha)}^2 \geq M_\alpha^2$

Envelope of holomorphy

More on the envelope of holomorphy:

- consider $f(z_1, \ldots, z_n)$ analytic in Δ
- analyticity in several variables $\Rightarrow$ constrain shape of Δ
- if shape not arbitrary: analyticity in Δ $\Rightarrow$ analyticity in $\mathcal{H}(\Delta)$
- given Δ, $\mathcal{H}(\Delta)$ is independent of f
- typically: use edge-of-the-wedge theorem (Bogoliubov)

Outline: 4. Crossing symmetry: string theory

String field theory

- field theory (second-quantization)
- rigorous, constructive formulation [hep-th/9206084, Zwiebach]
- make gauge invariance explicit (L_∞ algebras et al.)
- use standard QFT techniques (renormalization, analyticity...) $\rightarrow$ prove consistency (Cutkosky rules, unitarity, soft theorems, background independence...) [Sen '14-19]
- help to compute worldsheet scattering amplitudes [Sen '14-19] and effective actions [1912.05463, HE-Maccaferri-Vošmera]
- study backgrounds (= classical solutions), marginal and RR fluxes deformations [1811.00032, Cho-Collier-Yin; 1902.00263, Sen]
- access collective, non-perturbative, thermal, dynamical effects
- worldvolume theory ill-defined for ($p > 1$)-branes

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To appear: "String Field Theory – A Modern Introduction", Lecture Notes in Physics, Springer

SFT in a nutshell

SFT $=$ standard QFT s.t.:

- infinite number of fields (of all spins)
- infinite number of interactions
- non-local interactions $\propto \mathrm{e}^{-\# k^2}$
- reproduce worldsheet amplitudes (if well-defined)

[1703.06410, De Lacroix-HE-Kashyap-Sen-Verma]

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→ study Green function singularities from Feynman diagrams in momentum space

Action and Feynman diagrams

- gauge-fixed action

$$S = \frac{1}{2} \langle\Psi| \, c_0^- c_0^+ L_0^+ \, |\Psi\rangle + \sum_{g,n\geq 0} \frac{\hbar^g \, g_s^{2g-2+n}}{n!} \, \mathcal{V}_{g,n}(\Psi^n)$$

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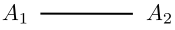
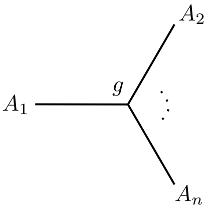
- propagator

$$\langle A_1 | \, \frac{b_0^+}{L_0^+} \, b_0^- \, | A_2 \rangle \quad = $$



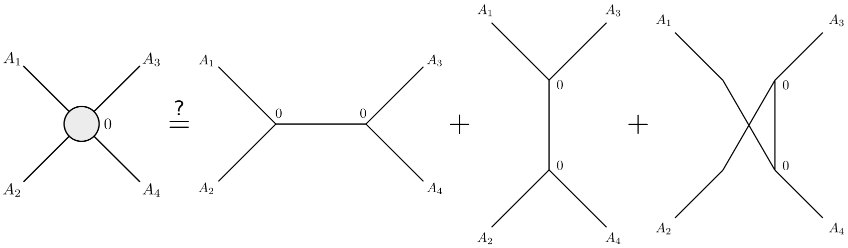
- fundamental g-loop n-point vertex

$$\mathcal{V}_{g,n}(A_1, \ldots, A_n) \quad = $$



defined s.t. sum of all graphs ⇒ recover worldsheet amplitudes

Example



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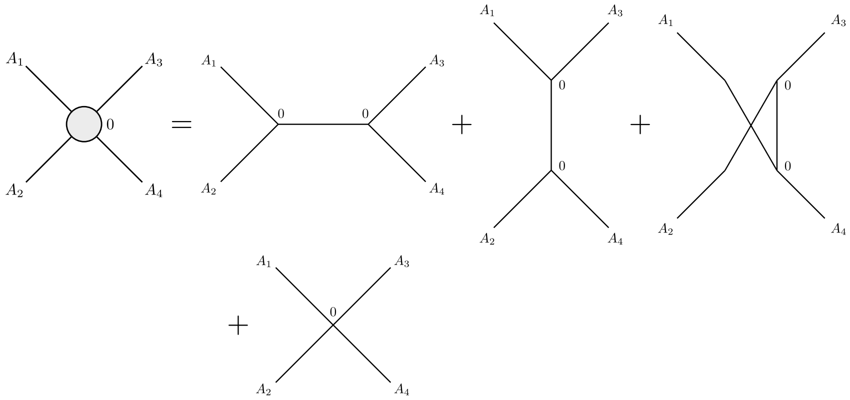

Momentum representation (1)

- string field Fourier expansion

$$|\Psi\rangle = \sum_A \int \frac{\mathrm{d}^D k}{(2\pi)^D} \, \phi_A(k) \, |A, k\rangle$$

k: D-dimensional momentum
A: discrete labels (Lorentz indices, group repr., KK modes...)

- 1PI action

$$S = \int \mathrm{d}^D k \, \phi_A(k) K_{AB}(k) \phi_B(-k) + \sum_n \int \mathrm{d}^D k_1 \cdots \mathrm{d}^D k_n \, V^{(n)}_{A_1,\ldots,A_n}(k_1, \ldots, k_n) \phi_{A_1}(k_1) \cdots \phi_{A_n}(k_n)$$

Momentum representation (2)

Propagator

$$K_{AB}(k)^{-1} = \frac{-\mathrm{i}\, M_{AB}}{k^2 + m_A^2}\, Q_A(k)$$

- M_{AB} mixing matrix for states of equal mass
- Q_A polynomial

Momentum representation (3)

Vertices

$$-\,\mathrm{i}V^{(n)}_{A_1,\dots,A_n}(k_1,\dots,k_n) = -\mathrm{i}\int \mathrm{d}t\,\mathrm{e}^{-g^{\{A_a\}}_{ij}(t)\,k_i\cdot k_j - c\sum_{a=1}^{n} m_a^2} \times P_{A_1,\dots,A_n}(k_1,\dots,k_n;t)$$

- t moduli parameters
- $P_{\{A_a\}}$ polynomial
- $c > 0 \to$ damping in sum over states
- g_{ij} positive definite
- no singularity for $k_i \in \mathbb{C}$ (finite)
- $\lim_{k^0\to\pm\mathrm{i}\infty} V^{(n)} = 0$
- $\lim_{k^0\to\pm\infty} V^{(n)} = \infty$

Green function

Truncated Green function = sum of Feynman diagrams of the form

$$\mathcal{F}(p_1, \ldots, p_n) \sim \int \mathrm{d}T \prod_s \mathrm{d}^D \ell_s \, \mathrm{e}^{-G_{rs}(T)\, \ell_r \cdot \ell_s - 2H_{ra}(T)\, \ell_r \cdot p_a - F_{ab}(T)\, p_a \cdot p_b}$$

$$\times \prod_i \frac{1}{k_i^2 + m_i^2} \, \mathcal{P}(p_a, \ell_r; T)$$

T, moduli parameters, $\mathcal{P}$, polynomial in (p_a, ℓ_r)

- momenta:
  - external $\{p_a\}$
  - internal $\{k_i\}$
  - loop $\{\ell_s\}$

  k_i = linear combination of $\{p_a, \ell_s\}$
- G_{rs} positive definite
  - integrations over spatial loop momenta $\boldsymbol{\ell}_r$ converge
  - integrations over loop energies ℓ_r^0 diverge

Momentum integration

Prescription = generalized Wick rotation [1604.01783, Pius-Sen]:

1. define Green function for Euclidean internal/external momenta
2. analytic continuation of external energies + integration contour s.t.
   - keep poles on the same side
   - keep ends at $\pm i\infty$

$\to$ analyticity for $\boldsymbol{p}_a \in \mathbb{R}$, p_a^0 in first quadrant $\operatorname{Im} p_a^0 > 0, \operatorname{Re} p_a^0 \geq 0$

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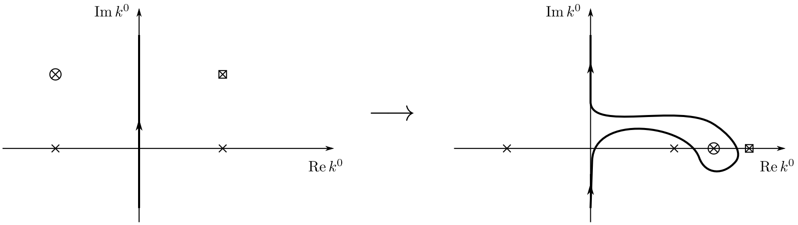

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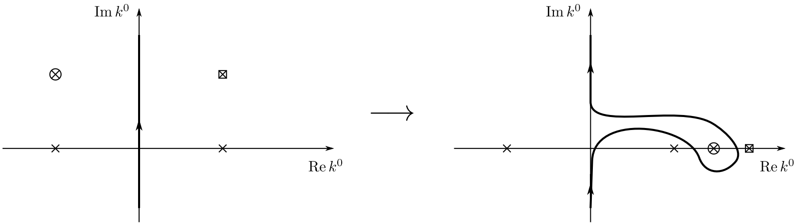

$\Rightarrow$ Cutkosky rules, unitarity, spacetime and moduli space iϵ-prescriptions [Pius, Sen]
Timelike Liouville theory [1905.12689, Bautista-Dabholkar-HE]

Analyticity for string theory (1)

Result

Analyticity inside Δ_2 of n-point superstring Green functions at all loop orders:

- implies crossing symmetry for $n = 4$
- identical analyticity properties for QFT and string theory

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Analyticity inside Δ_2 of n-point superstring Green functions at all loop orders:

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- identical analyticity properties for QFT and string theory

Comments:

- Feynman graphs $\rightarrow$ perturbative computations
- valid for states with any spin
- technical assumptions: mass gap, stable external states
- regularization of massless states: removes IR non-analyticity (identical to QFT)

Analyticity for string theory (2)

Method to study singularity:

1. start with some $p_a = p_a^{(1)}$, $\ell_r^0 \in \mathrm{i}\mathbb{R}$, $\boldsymbol{\ell}_r \in \mathbb{R}$ s.t. no singularity
2. find a path $p_a = p_a^{(1)} \to$ desired $p_a = p_a^{(2)}$
3. deform the integral contour as the poles move
4. assume $\exists$ singularity $=$ on-shell internal propagator
   pinching $=$ collision of two poles from opposite sides
5. analyze reduced diagram, display an inconsistency

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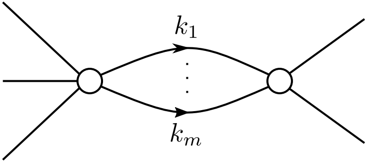
Proceed by steps:

1. analyticity in Δ_1: go from $p_a = 0$ to desired $\operatorname{Re} p_a$ and $\operatorname{Im} p_a^0$ (keep $\operatorname{Im} \boldsymbol{p}_a = 0$)
2. analyticity in Δ_2: go from $p_a \in \Delta_1$ to desired $\operatorname{Im} p_a^1$ (keep $\operatorname{Im} p_a^i = 0 \ \forall i \geq 2$)

First step

- $p_a^0 \in \mathbb{C}$, $\boldsymbol{p}_a \in \mathbb{R}$
- pinching implies reduced graph:



$k_i^2 = -m_i^2$, arrow = sign of k_i^0

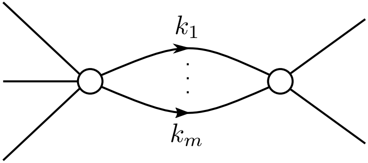
- $\boldsymbol{p}_a, \boldsymbol{\ell}_r \in \mathbb{R} \Rightarrow \boldsymbol{k}_i \in \mathbb{R}$, then $k_i^2 = -m_i^2 \Rightarrow k_i \in \mathbb{R}$
- one can prove $\forall i : k_i^0 > 0$
- implies

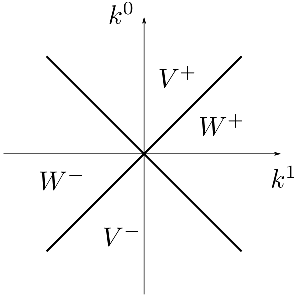
$$P_{(\alpha)} = \sum_i k_i \in \mathbb{R}, \qquad p_a^2 = -m_a^2 \quad \Rightarrow \quad -P_{(\alpha)}^2 \geq M_\alpha^2$$

$\rightarrow$ contradiction – one must have $-P_{(\alpha)}^2 < M_\alpha^2$

Second step

- $p_a^{\parallel} = (p_a^0, p_a^1) \in \mathbb{C}$, $p_a^{\perp} \in \mathbb{R}$
- pinching implies reduced graph:





arrow = sign of $\text{Im}\, k_i^1$

- one can prove $\forall i : \text{Im}\, k_i^1 > 0$, and

$$k_i^2 = -m_i^2 \;\Rightarrow\; \text{Im}\, k_i^{\parallel} \in W^+ \;\Rightarrow\; \text{Im}\, P_{(\alpha)} = \sum_i \text{Im}\, k_i \in W^+$$

$\rightarrow$ contradiction – one must have $\text{Im}\, P_{(\alpha)}$ timelike

Outline: 5. Conclusion

Conclusion

Results:

- tree-level 2-point amplitude computation consistent with QFT
- analyticity of superstring n-point amplitudes in Δ_2
- proof of crossing symmetry for 4-point superstring amplitudes at the same level as in QFT
- show that, in some sense, string theory behaves like local QFT
- new proof of analyticity valid for more general QFTs

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Outlook:

- tree-level 0-point function for generic background
- CPT theorem
- analyticity in Δ_D
- analyticity in $\mathcal{H}(\Delta_2)$ just from Feynman diagrams